Summer term 2026

Seminar: Topics in Algebraic Geometry

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Time and place: Fridays, 10:15 - 11:45, Room IA 1/181.

Further information:

- Length of a talk: ca 90 minutes (including questions).
- About 1 week prior to your talk, please discuss its content with one of us. You should write us an email (2 weeks before your talk) to schedule a meeting.
- Credits: As a standalone seminar (1 talk for Bachelor students, 2 talks for Master students) OR as 2 SWS to cumulate with the Algebraic Geometry II lecture (2 SWS + 2SWS + 2 SWS Übungen) for a standard course.

Short description: We study various topics on curves and surfaces within algebraic geometry. We discuss the following topics:

- Divisors on curves
- Positivity notions for divisors on curves, embeddings of curves into projective space.
- Elliptic curves over $\mathbb{C}, \mathbb{F}_q, \mathbb{Q}$
- Divisors on surfaces
- K3 surfaces

Talk 1. (April 24th, Leon) Divisors and linear systems on curves. The Riemann–Roch formula.

a) Define a divisor on a curve. Define the notion of a principal divisor and the degree of a divisor. Define the pullback of a divisor by a morphism between curves.

b) Define the notion of a linear system on a curve. Explain how a linear system gives rise to a morphism to a projective space. Give some examples. Define the base locus of a linear system. State the Riemann–Roch theorem, both in its historical form and in its modern form (with Serre duality). Present at least one application.

c) *(optional)* For a finite morphism between curves, define the ramification divisor and state the Riemann–Hurwitz formula. Use the Riemann–Hurwitz formula to prove that $\mathbb{P}^1_{\mathbb{C}}$ admits no étale cover.

References: [Hul03, Sections 6.1, 6.2, 6.4] and [Har77, Part IV, Chapters 1, 2].

Talk 2. (May 8th, Hannah) Embeddings curves into $\mathbb{P}^3$.

Define the notions of ample and very ample divisors (on a curve). Prove the sufficient criterion of very ampleness [Har77, Chapter IV, Corollary 3.2]. Deduce a characterization of very ampleness in low genus

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($g = 0, 1, 2$). Using secant varieties, prove that any curve can be embedded in $\mathbb{P}^3$ with at most nodal singularities.

References: [Hul03, Section 6.5] and [Har77, Part IV, Chapter 3].

Talk 3. (May 15th, Martin) Elliptic curves over $\mathbb{C}$
Prove that any curve of genus 1 (over an algebraically closed field) admits a Weierstraßequation [Sil09, Chap. III, Proposition 3.1]. Define elliptic functions attached to a lattice $\Lambda \subseteq \mathbb{C}$. Show that $\mathbb{C}/\Lambda$ equals $E(\mathbb{C})$ for an elliptic curve $E/\mathbb{C}$ [Sil09, Chap. VI, Proposition 3.6]. State the converse result (unformization of elliptic curves) without proof [Sil09, Chap. VI, Theorem 5.1, Corollary 5.1.1].

Talk 4. (May 22th, Leon) Elliptic curves over finite fields
Introduce the q-Frobenius map on an elliptic curve over $\mathbb{F}_q$ [Sil09, III, Example 4.6]. Prove Hasse's bound for elliptic curves [Sil09, V, Theorem 1.1]. For this, you will need to introduce isogenies of elliptic curves, their degrees and some properties of them, see [Sil09, III,§4]. Illustrate isogenies via elliptic curves over $\mathbb{C}$ (given by lattices); it might be helpful to mention [Sil09, VI, §4].
If time permits, explain as much as possible on the Weil conjectures for elliptic curves over finite fields [Sil09, V, Theorem 2.3.1].

Talk 5. (June 5th, Hannah) Supersingular elliptic curves Study the endomorphism ring of an elliptic curve over $\mathbb{F}_q$, introduce supersingular elliptic curves and the Hasse invariant [Sil09, V,§3] (you will probably have to omit mentioning formal group law in [Sil09, V, Proposition 3.1]). Discuss then as much as possible of [Sil09, V,§4].

Talk 6. (June 12th, Martin) Picard group and intersection theory on a surface.
Explain the correspondence between line bundles (up to isomorphism) and divisors (up to linear equivalence) on a smooth surface S [Bea96, I.1]. Define the intersection product and prove that it is symmetric and bilinear [Bea96, Theorem I.5]. State the Hodge Index Theorem, which gives the index of the quadratic form induced by the intersection product [BHPVdV04, IV.2.13 and IV.2.15]. Define numerical equivalence of divisors on a surface and define the Néron–Severi lattice $\mathrm{NS}(S)$ of a surface S as the set of divisors on S modulo numerical equivalence. State/prove that $\mathrm{NS}(S)$ is a free abelian group of finite rank, and define the Picard number $\rho(S)$ of S to be that rank. Compute some examples [Bea96, Proposition I.8 and Example I.9]. Define the canonical divisor K_S of a surface S. Prove the Riemann–Roch formula [Bea96, Theorem I.13] and derive the genus formula [Bea96, I.15]. Give a word of caution about the genus of a singular curve [Bea96, Remark I.16 (i)].

References: [Bea96, Chapter I], [BHPVdV04, II.9, II.10, II.11], [Deb11, Page 25], [Kon20, Section 3.1].

Talk 7. (June 19th, Benji) Blow-up of a smooth point in a surface and Castelnuovo's contraction theorem.

a) Define the blow-up $\hat{S}$ of a smooth point p in a surface S. Define the exceptional divisor E, define the strict transform of any curve C under the blow-up [Bea96, II.1]. Explain how to pullback a divisor by a morphism and relate the pullback and strict transform of a curve C in S, both when $p \in C$ and when $p \notin C$ [Bea96, II.2].

b) Compute the intersection product on the blow-up $\hat{S}$ [Bea96, II.3]: in particular, show that the exceptional divisor satisfies $E \cdot E = -K_S \cdot E = -1$.

c) State the universal property of the blow-up (without proof) [Bea96, II.8]. Prove the factorization theorem [Bea96, II.11] and present the consequences given in [Bea96, II.13].

d) Present one of the examples among [Bea96, II.14].

e) To conclude, state Castelnuovo's contraction theorem [Bea96, II.17]. Say as much as you can about the proof. If time is short, you may want to focus on Part (a) of the proof, that is constructing a candidate for the blow-down map ϕ. This theorem should be viewed as a *numerical criterion* to detect that a surface S is the blow-up of another surface S_0 at a smooth point.

f) If time allows, present the following application of Castelnuovo's contraction theorem: consider the surface S obtained by blowing up two distinct points P, Q in $\mathbb{P}^2$, and prove that the strict transform ℓ of the unique line through P and Q is an exceptional curve in S.

References: [Bea96, Chapter II]. Note that you may skip many statements, such as [Bea96, II.4, II.5, II.6, II.7, II.9, II.10, II.12, II.15, II.16].

Talk 8. (July 3rd, Alex L.) Introducing K3 surfaces
Define the notion of a $K3$ surface. Prove that for a K3 surface S, the lattice $\mathrm{NS}(S)$ embeds in a lattice $H^2(S;\mathbb{Z})$ of rank 22, endowed with a quadratic form of signature $(3, 19)$. Deduce that the Picard ranks of K3 surfaces range between 1 and 20.
Give plenty of examples, for instance: prove that a general quartic is a K3 surface of Picard rank 1; prove that the Fermat quartic $V({x_0}^4 + {x_1}^4 + {x_2}^4 + {x_3}^4) \subset \mathbb{P}^3$ is a K3 surface of Picard rank 20; prove that the Kummer construction provides, for any abelian surface A, a K3 surface $\mathrm{Kum}(A)$ of Picard rank $16 + \rho(A)$.

References: [Kon20, Sections 4.1, 4.3] and recorded lectures by Alessandra Sarti at CIRM in January 2019:
Lecture 1 at `https://www.youtube.com/watch?v=0xTiabWkCd0`,
Lecture 2 at `https://www.youtube.com/watch?v=z_G22Btt_nw`.

Talk 9. (July 10th, Alex I.) Curves of negative and trivial self-intersection on K3 surfaces.
Using the Riemann–Roch formula, note that a curve C on a K3 surface must either be smooth, isomorphic to $\mathbb{P}^1$ and satisfy $C \cdot C = -2$, or satisfy $C \cdot C \geq 0$. As an example, exhibit a few such (-2)-curves on the Fermat quartic surface, on any Kummer surface, and prove that a general quartic surface contains no (-2)-curve.
For the case of trivial self-intersection, state and prove [Huy16, Proposition 3.10], as well as the necessary prerequisites.

References: Mostly [Huy16, Chapter 2].

Talk 10. (July 17th, Cécile) Elliptic (K3) surfaces.

References: [Mir89], [Huy16, Chapter 11, Sections 1 and 2].

Talk 11. (July 24th, canceled) Nikulin's characterization of Kummer surfaces.
Define the Kummer lattice K and prove that a K3 surface S is a Kummer surface if and only if there is a primitive embedding of K into $\mathrm{NS}(S)$ [Huy16, Chapter 14, Section 3].

Reference: [Huy16, Chapter 14, Section 3].

References

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[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, 2nd enlarged ed. ed., Ergeb. Math. Grenzgeb., 3. Folge, vol. 4, Berlin: Springer, 2004 (English).

[Deb11] O. Debarre, *Introduction to Mori theory*, M2 lecture notes, Université Paris-Diderot. Available at `https://www.math.ens.psl.eu/~debarre/M2.pdf.`, 2011.

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[Hul03] Klaus Hulek, *Elementary algebraic geometry*, Stud. Math. Libr., vol. 20, Providence, RI: American Mathematical Society (AMS), 2003 (English).

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[Kon20] Shigeyuki Kondō, $K3$ *surfaces. Translated from the Japanese by the author*, EMS Tracts Math., vol. 32, Berlin: European Mathematical Society (EMS), 2020 (English).

[Mir89] Rick Miranda, *The basic theory of elliptic surfaces*, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989.

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, 2nd ed., Graduate Texts in Mathematics, vol. 106, Springer, New York, 2009.