Assignments for Week 9 and 10

Part A9, due on 14.1.2026. No reading assignment this week. Write down any question and or comment you have on the material covered in class this week. Was anything interesting, confusing?

Part B9, discussed on 9.1.2026 and 16.12026 if necessary. Exercises 1) - 3) concern topics of Week 9. Exercises 4) - 7) concern topics of Week 10.

1) Let C be the nodal cubic curve defined in Lecture 1. We denote its nodal point by p. For a line L through the point p, determine the number of points of intersection of L and C in $k\mathbb{P}^2$, counted **without** multiplicities.

2) Let k be an algebraically closed field. For $d \in \mathbb{N}$ and $Z \subset k\mathbb{P}^2$ a Zariski-closed subset, we define $W_d(Z)$ as the set of homogeneous polynomials of degree d that vanish at every point of Z.

- Check that, for two Zariski-closed subsets $Z \subset Z' \subset k\mathbb{P}^2$, we have $W_d(Z') \subset W_d(Z)$.
- From Lecture 20, recall a sheaf of $\mathcal{O}_{k\mathbb{P}^n}$-modules whose global sections are the elements of $W_d(\varnothing)$. What is the dimension of $W_d(\varnothing)$?
- Let $Z \subset k\mathbb{P}^2$ be a Zariski-closed subset. Construct a sheaf of $\mathcal{O}_{k\mathbb{P}^n}$-modules whose global sections are precisely the elements of $W_d(Z)$.
- Let $p \in k\mathbb{P}^2$ be a point. Prove that the k-linear subspace $W_d(p)$ is a hyperplane in $W_d(\varnothing)$.
- Let $p_1, \ldots, p_r$ be a set of distinct points all contained in a common line ℓ in $k\mathbb{P}^2$. Prove that the k-vector space $W_d(p_1, \ldots, p_r)$ is of dimension:

$$\dim_k W_d(p_1, \ldots, p_r) = \begin{cases} \dim_k W_{d-1}(\varnothing) & \text{if } d < r; \\ \dim_k W_d(\varnothing) - r & \text{if } d \geq r. \end{cases}$$

- Let p_1, p_2, p_3, p_4, p_5 be a set of points of $k\mathbb{P}^2$ such that no three of them lie on a line. Prove that there is a unique and irreducible curve of degree 2 — a so-called **conic** — passing through all of them.

3) Let $n \in \mathbb{N}$ and k be an algebraically closed field.

- Fix $d \in \mathbb{Z}$. Prove that the dual sheaf $(\mathcal{O}_{k\mathbb{P}^n}(d))^\vee$ is naturally isomorphic to the sheaf $\mathcal{O}_{k\mathbb{P}^n}(-d)$.
- Prove that the set of the sheaves $\mathcal{O}_{k\mathbb{P}^n}(d)$, for $d \in \mathbb{Z}$, forms a group under the tensor product.

4) Let p be a prime number. Denote by $\mathbb{Z}_p$ the ring of p-adic integers. Compute the dimension of $\operatorname{Spec} \mathbb{Z}_p$ in two different ways (topologically and algebraically).

5) Let R be a principal ideal domain. Prove that the dimension of $\operatorname{Spec} R$ is zero or one. Provide a necessary and sufficient condition on R for this dimension to be zero.

6) Let $k \subset K_1 \subset K_2$ be three fields. Prove that

$$\operatorname{trdeg}_k K_2 = \operatorname{trdeg}_k K_1 + \operatorname{trdeg}_{K_1} K_2.$$

7) (some spaces of dimension zero)

- Let V be an algebraic variety. Prove that V is of dimension zero if and only if V is finite.
- Give an example of a topological space of dimension zero that is not finite.
- Can you construct a scheme whose underlying topological vector space is of dimension zero and yet not finite? What about an affine scheme? What about an affine noetherian scheme?

Part C9, due on 14.1.2026. You may choose to solve any two of these three exercises (or of course all three of them, if you so wish).

1) Let X be a scheme. Prove that the set of (isomorphism classes of) locally free sheaves of $\mathcal{O}_X$-modules of rank one on X is stable under the tensor product, and is endowed thereby with a group structure.

Note: This group is called the *Picard group* of X and denoted $\mathrm{Pic}(X)$. An important theorem states that, under several assumptions on X — notably if X is a projective algebraic variety, its Picard group admits a natural scheme structure that enjoys some compatibility with the group structure. That makes the Picard scheme into an important example of a *group scheme*.

2) **(Pascal's theorem)** Using Bézout's theorem, prove the following result: Let $C \subset k\mathbb{P}^2$ be a conic. Let $p_1, p_2, p_3, p_4, p_5, p_6$ be six distinct points on C and let us also set $p_7 := p_1$. We denote by e_i the line joining p_i and p_{i+1}, for $1 \leq i \leq 6$. Then, the three points

$$e_1 \cap e_4, \quad e_2 \cap e_5, \quad e_3 \cap e_6$$

lie on a line. Does this result still hold when C is the union of two distinct lines?

3) Let $n \in \mathbb{N}$. Determine the subset of $\mathbb{N} \cup \{0, \infty\}$ consisting of the dimensions of all possible topological structures on the set $\{1, \ldots, n\}$.