Assignments for Week 8

Part A8, due on 17.12.2025. No reading assignment this week. Write down any question and or comment you have on the material covered in class this week. Was anything interesting, confusing?

Part B8, discussed on 12.12.2025.

1) Let $F, G \in k[x, y, z]$ be two homogeneous polynomials without any common factor. Let ℓ_1 and ℓ_2 be two projective lines in $k\mathbb{P}^2$, both disjoint from the zero set $V(F, G)$, and let F_{ℓ_i} and G_{ℓ_i} denote the dehomogenizations of F and G with respect to the line ℓ_i. Prove that, for any point $p \in V(F, G)$, the multiplicities

$$\dim_k \mathcal{O}_{k^2,p}/(F_{\ell_i}, G_{\ell_i})$$

computed in the affine charts $k\mathbb{P}^2 \setminus \ell_i \simeq k^2$ are the same for $i = 1$ and for $i = 2$.

2) Let $(X, \mathcal{O}_X)$ be a scheme over a field k.

- Prove that the space of global sections of the structure sheaf $\mathcal{O}_X$ is non-zero.
- Let $\mathcal{F}$ be a sheaf of abelian groups on X that admits an injective morphism of sheaves $\mathcal{O}_X \to \mathcal{F}$. Show that $\mathcal{F}$ has a non-zero global section.

3) Let k be a field. Let $(Z, \mathcal{O}_Z)$ be a scheme with Z finite discrete and with $\mathcal{O}_Z$ a sheaf of k-algebras whose stalks all have finite dimension over k. Prove that the space of global sections $A = \mathcal{O}_Z(Z)$ has finite dimension as a k-vector space. Using [Atiyah-McDonald, Theorem 8.7] and results from the lectures, prove that there is an isomorphism of schemes:

$$(Z, \mathcal{O}_Z) \simeq (\operatorname{Spec} A, \mathcal{O}_A).$$

4) Construct an example of a finite morphism of schemes that is not surjective.

5) Let $f : X \to Y$ be a finite morphism of schemes.

- Prove that, for every $y \in Y$, the preimage $f^{-1}(\{y\})$ is a finite set.
- Prove that the image $f(X)$ is a closed subset of Y. Is f a closed map between the underlying topological spaces?

Part C8, due on 17.12.2025.

1) (Hartshorne, II Exercise 3.4) Let $f : X \to Y$ be a morphism of schemes. Show that f is finite if and only if, for every open set $V \subset Y$ isomorphic to an affine scheme $V \simeq \operatorname{Spec} B$, the preimage $f^{-1}(V)$ is isomorphic to an affine scheme $\operatorname{Spec} A$, and A is a finite dimensional B-module.