Assignments for Week 7

Part A7, due on 09.12.2025. You can choose to complete **either of** the following two reading assignments.

Warm-up on Artinian rings. The reference for this assignment is the book "Introduction to Commutative Algebra" by Atiyah-MacDonald.

1) Read Page 74 (until Example 2) included). On Page 76, find and read the definition of an *Artinian ring.*

2) Is the ring $R_n := k[x]/(x^n)$ Artinian? If it is, please write down which of the results from [Atiyah-MacDonald, Chapter 6, Pages 74-76] are used to prove this fact. No need to write a proof yourself.

3) Read Propositions 8.1 and 8.3, as well as their proofs. How do they apply (concretely) to the Artinian ring $\mathbb{Z}/n\mathbb{Z}$?

4) Within [Atiyah-MacDonald, Chapter 8], find a theorem that allows to decompose any Artinian ring into elementary building blocks. How does it apply (concretely) to the Artinian ring $\mathbb{Z}/n\mathbb{Z}$?

5) Write down any question and or comment you have on this text, as well as on the material covered in class this week. Was anything interesting, confusing?

6) Write down how much time you spent on this reading assignment.

Intersection multiplicities algorithmically. This is, in many ways, an open-ended assignment: Its goal is to prompt you to think and read about a few topics that we just briefly cover in class. You do not need to entirely answer every question to do well and learn from your work on this assignment. The main reference here is the book "Algebraic Geometry" by Perrin.

1) Read [Perrin, Page 227, Problem VII,Part 1)]. **NB:** The sentence starting by "Moreover" in Item 7) should be separate from the list of axioms: It is a consequence of all seven axioms.

2) Which of the axioms are clearly implied by the formula $\mu_P(F,G) = \dim_k \mathcal{O}_{k^2,P}/(F,G)$?

3) Explain in your own words (in natural language) how to use Perrin's axioms to compute intersection multiplicities *algorithmically* for explicit polynomials F and G at a given point P.

4) Over the field $k = \mathbb{Q}$ and for $n \geq 2$ in general, do you think that an algorithm exists that, for any n polynomials $F_1, \ldots, F_n$, can compute the set of points $V(F_1, \ldots, F_n)$? You may use any resources at your disposal to answer this question, as long as you cite your sources adequately.

5) Write down any question and or comment you have on this text, as well as on the material covered in class this week. Was anything interesting, confusing?

6) Write down how much time you spent on this assignment.

Part B7, discussed on 05.12.2025.

1) Let $\mathcal{G}$ be a sheaf on a topological space X. Asume that $\mathcal{G}$ admits a global section s. Let S be a non-empty set. Prove that the following data defines a presheaf $\mathcal{F}$ on X:

- $\mathcal{F}(U) = \mathcal{G}(U)$ on every proper open set $U \subsetneq X$ and $\mathcal{F}(X) = \mathcal{G}(X) \sqcup S$;
- the restriction map $r_{V,U}$ for open sets $V \subset U \subsetneq X$ is the restriction map of $\mathcal{G}$ and the restriction map $r_{U,X}$ is induced by the restriction map $\mathcal{G}(X) \to \mathcal{G}(U)$, together with the constant map $S \to \mathcal{G}(U)$ of value $s|_U$.

Is the presheaf $\mathcal{F}$ separated? What is the sheafification of $\mathcal{F}$?

2) Consider the affine scheme $(\operatorname{Spec}\mathbb{Z}, \mathcal{O}_{\mathbb{Z}})$.

- List all points of the space $\operatorname{Spec}\mathbb{Z}$ and describe their respective closures. Does $\operatorname{Spec}\mathbb{Z}$ have the T1-property?
- At every point $x \in \operatorname{Spec}\mathbb{Z}$, describe the stalk $\mathcal{O}_{\mathbb{Z},x}$. When is it a field?
- What are the automorphisms of the scheme $(\operatorname{Spec}\mathbb{Z}, \mathcal{O}_{\mathbb{Z}})$?

3) Consider the polynomial $P(X,Y) = (X^2+Y^2)^2 + 3X^2Y - Y^3$, whose zero-locus in $\mathbb{R}^2$ is depicted in [Perrin, Page 3, Figure 4]. Prove that the affine schemes associated to the following rings are finite discrete spaces, with finite-dimensional $\mathbb{C}$-vector spaces as stalks at each point. Compute the number of points of the scheme and the dimensions of the stalks at each point. Deduce the dimension of the space of global sections of the structure sheaf.

- $R_1 = \mathbb{C}[X,Y]/(X,P)$;
- $R_2 = \mathbb{C}[X,Y]/(Y,P)$;
- $R_3 = \mathbb{C}[X,Y]/(Y-X^2,P)$;
- $R_4 = \mathbb{C}[X,Y]/(Y^3+X^2,P)$;
- $R_5 = \mathbb{C}[X,Y]/(Y^2-X^3,P)$.

Hint: Interpret the points of $\operatorname{Spec} R_5$ as a certain zero-locus in $\mathbb{C}^2$, and use a parametrization of the cuspidal cubic by $t \mapsto (t^2,t^3)$ to count them (without multiplicities). To compute multiplicities, you may use the properties given by [Perrin, Problem VII] without proof; see also [Perrin, Chapter V, Definition 4.4] for a relevant definition.

4) (Hartshorne, Exercise II.2.7) Let $(X, \mathcal{O}_X)$ be a scheme and K be a field. Show that a morphism of schemes from $\operatorname{Spec} K$ to X is equivalent to giving yourself a point $x \in X$ together with an embedding of the residue field of X at x into K. The set of such points are called the *K-points of X*. What are the $\mathbb{Q}$-points of $\operatorname{Spec}(\mathbb{Z})$?

Part C7, due on 09.12.2025.

1) Consider the affine schemes associated to the following two rings:

- k^2, where k is a field, endowed with the product ring structure;
- $\mathbb{Z}_p$, the ring of p-adic integers, where p is a prime number.

In both cases, prove that Spec is a set with two elements, describe its topology, and compute the stalks of the structure sheaf.