Assignments for Week 6

Part A6. No reading assignment this week. Write down how you feel about what we covered in class this week. Was anything interesting, confusing?

Part B6, discussed on 28.11.2025.

1) Let $f : X \to Y$ be a continuous map between two topological spaces. Let $\mathcal{F}$ be a presheaf (of sets, abelian groups, or rings) on X and $x \in X$ be a point. Prove that f induces is a morphism between the stalks

$$[f_*\mathcal{F}]_{f(x)} \to \mathcal{F}_x.$$

Prove moreover that if f is injective **and open**, this morphism is an isomorphism between the stalks.

2) **(just sheafify once at the end)** Fix a topological space X and consider the category $\mathrm{PrSheaf}(X)$ of presheaves on X. Consider a functor

$$H\colon \mathrm{PrSheaf}(X) \to \mathrm{PrSheaf}(X)$$

that is non-decreasing with respect to inclusion and commutes to taking stalks, i.e., there is a functor

$$H'\colon \mathrm{Set} \to \mathrm{Set}$$

such that, for every $x \in X$, the equality $H(\mathcal{F})_x = H'(\mathcal{F}_x)$ holds.

- Prove that, for any presheaf $\mathcal{F}$ on X, the sheaves obtained by sheafifying once at the end $H(\mathcal{F})^+$ and twice (at each step) $H(\mathcal{F}^+)^+$ are naturally isomorphic.
- Give an example of a space X, a functor H and a presheaf $\mathcal{F}$ such that the presheaf $H(\mathcal{F}^+)$ obtained by sheafifying once at the start is NOT a sheaf.
- Modify the statement of the exercise (by working with appropriate categories and non-decreasing functors between them) to explain why the cokernel of a morphism of sheaves of abelian groups $\phi : \mathcal{F} \to \mathcal{G}$ is isomorphic to the quotient sheaf $\mathcal{F}/\mathrm{im}(\phi)$.

3) **(algebraic Hartogs' extension theorem)** Let k be an algebraically closed field. Compute the space of sections $\mathcal{O}_{k^2}(k^2 \setminus \{(0,0\})$.

4) Let k be an algebraically closed field. Consider the projective space $k\mathbb{P}^2$ and denote by h the line at infinity. What is the ring of sections $\mathcal{O}_{k\mathbb{P}^2}(k\mathbb{P}^2 \setminus h)$?

Part C6, due on 03.12.2025.

1) Let k be an algebraically closed field. Compute the space of sections $\mathcal{O}_{k\mathbb{P}^2}(k\mathbb{P}^2 \setminus \{[0:0:1]\})$.

2) Let k be an algebraically closed field and $n \in \mathbb{N}$. Prove that the structure sheaf $\mathcal{O}_{k^n}$ is invariant under the $\mathrm{GL}(n,k)$-action on k^n, i.e., any linear isomorphism $\gamma : k^n \to k^n$ induces a natural isomorphism of sheaves of rings

$$\gamma^{\#}\colon \mathcal{O}_{k^n} \to \gamma_*\mathcal{O}_{k^n}.$$

3) (optional) Using **2)**, compute the ring of sections $\mathcal{O}_{k^n}(k^n \setminus h)$ for an arbitrary linear subspace $h \subset k^n$.