Assignments for Week 5

Part A5. No reading assignment this week. Write down how you feel about what we covered in class this week. Was anything interesting, confusing?

Part B5, discussed on 20.11.2025. In what follows, assume that the field k is infinite.

1) Prove that every isomorphism $f : k\mathbb{P}^1 \to k\mathbb{P}^1$ is induced by an element $M_f \in \mathrm{PGL}(2,k)$. Deduce that any automorphism of $k\mathbb{P}^1$ fixing three points is the identity.

2) Let V, W be two algebraic sets and consider a rational map $f \colon V \dashrightarrow W$. Prove that the following statements are equivalent:

- f is birational;
- there exist open sets $V_0 \subset V$ and $W_0 \subset W$ such that f induces an isomorphism

$$f|_{V_0} \colon V_0 \xrightarrow{\sim} W_0.$$

3) Let X be a topological space and let S be a set. Prove that the assignment of

$$\mathcal{F}(U) := \{f \colon U \to S \mid f \text{ is constant}\}$$

to any open set $U \subset X$ defines a presheaf on X. Show that $\mathcal{F}(X) \simeq S^I$, where I denotes the set of all connected components of X. Determine whether $\mathcal{F}$ is a sheaf in the following cases:

- $X = \mathbb{R}$ endowed with the Euclidean topology;
- $X = k^n$ endowed with the Zariski topology;
- X is discrete.

4) Consider the rational map $\sigma : k\mathbb{P}^2_{x,y,z} \dashrightarrow k\mathbb{P}^2_{x,y,z}$ defined on the complement of $V_p(xz)$ by

$$\sigma([x : y : z]) = \left[\frac{y}{z} : \frac{y+z}{x} : 1\right].$$

What is the maximal open set of definition of σ? of σ^2? Prove that σ is a birational map and compute its inverse. Finally, show that $\sigma^5 = \mathrm{id}_{k\mathbb{P}^2}$.

Hint: Construct a set S of four points and six lines in $k\mathbb{P}^2$ that is preserved by σ. Describe the action of σ^5 on S. Prove that σ^5 is defined everywhere on $k\mathbb{P}^2$. From there, you may try to apply **1)** to conclude.

Part C5, due on 25.11.2025.

1) Let X be the real line endowed with the Euclidean topology. Let $\mathcal{C}$ be the presheaf of $\mathbb{R}$-valued infinitely differentiable functions on X and let $\mathcal{P}$ be its subpresheaf of $\mathbb{R}$-valued polynomial functions.

- Are they sheaves of $\mathbb{R}$-vector spaces?
- How would you name the sheafification of the presheaf $\mathcal{P}$? Can you describe it geometrically?
- Prove that, for any two points $a, b \in X$, the stalks at a and b are isomorphic $\mathbb{R}$-vector spaces:

$$\mathcal{C}_a \simeq \mathcal{C}_b \quad \text{and } \mathcal{P}_a \simeq \mathcal{P}_b?$$

- Describe the stalk of $\mathcal{P}$ as an $\mathbb{R}$-vector space. **Hint:** You may write down an isomorphism of the stalk to a known vector space; alternatively, you may find a basis of the stalk.
- Does the inclusion $\mathcal{P} \subset \mathcal{C}$ induce an isomorphism at the level of stalks?