Assignments for Week 4

Part A4. No reading assignment this week. Write down how you feel about what we covered in class this week. Was anything interesting, confusing?

Part B4, discussed on 14.11.2025. In what follows, assume that the field k is infinite.

1) Prove that any invertible linear map $\phi \in \mathrm{GL}(n,k)$ from $k^n \to k^n$ is an isomorphism. For any $n \geq 2$, construct an isomorphism from k^n to k^n that is NOT linear. What happens when $n = 1$?

2) Let V be an affine algebraic set. Prove that the set of isomorphisms from V to V is a group. It is called the *automorphism group* of V.

3) Let V, W be algebraic sets and let $f : V \dashrightarrow W$ be a rational map. Prove that there is a Zariski-open subset $V_{f,\max}$ of V on which f is defined that is maximal under inclusion.

4) We consider the cuspidal cubic $C = V(Y^2 - X^3)$ over the field $k = \mathbb{C}$.

- Recall that there is a morphism $f : t \in \mathbb{C} \mapsto (t^2, t^3) \in C$. Is there a rational map $g : C \dashrightarrow \mathbb{C}$ such that f and g are inverses of each other?
- Prove that the field of rational functions on C is isomorphic to the fraction field $\mathbb{C}(t)$.
- Prove that every automorphism of C fixes the point $(0;0)$. **Hint:** Prove that $\mathfrak{m} := (X, Y)$ is the only maximal ideal of $\mathbb{C}[X, Y]$ such that the ideal $(Y^2 - X^3)$ is contained in both $\mathfrak{m}$ and $\mathfrak{m}^2$.
- Prove that the automorphism group of C is isomorphic to $\mathbb{C}^*$. Describe all orbits of this automorphism group acting on C.

Part C4, due on 20.11.2025.

1) Composition of rational maps. Consider two rational maps $f : V \dashrightarrow W$ and $g : W \dashrightarrow X$, respectively defined on two Zariski-open sets V_f and W_g. Construct an explicit Zariski-open set of V on which the composition $g \circ f : V \dashrightarrow X$ is well defined.

2) Let C be the nodal cubic of the form $V(X^3 - Y^3 - XY)$ in k^2.

- Consider the rational map
$$f : t \in k \mapsto \left(\frac{t}{1-t^3}, \frac{t^2}{1-t^3}\right) \in C.$$
Can you define a rational map $C \dashrightarrow k$ that is the inverse of f?
- What is the function field of C?
- Prove that C is not isomorphic to the affine line k.