Assignments for Week 3

Part A3, due on 11.11.2025. Read the section titled "7. Appendix: Graded Rings" on Page 33-34 in the textbook *Algebraic Geometry* by Daniel Perrin. Answer the following questions:

1) In Definition 7.1, what does the author mean by the "subspaces" R_n? Are they k-linear subspaces, subrings, ideals of R?

2) Explain how to grade the polynomial ring $R = k[X, Y]$ with respect to the degree: What are the R_n?

3) Give two examples of $k[X]$-graded modules.

4) Any question and/or comment on this text? Was anything interesting, confusing?

5) Write down how you feel about what we covered in class this week. Was anything interesting, confusing?

6) How much time did you spend reading and answering these questions?

Part B3, discussed on 11.11.2025.

1) Check that considering the projective algebraic subsets of $k\mathbb{P}^n$ as closed subsets indeed defines a topology on $k\mathbb{P}^n$.

2) Prove that the map

$$f : (t_1, \ldots, t_n) \in k^n \longrightarrow [1 : t_1 : \ldots : t_n] \in k\mathbb{P}^n$$

is injective, continuous, and open (with respect to the Zariski topologies).

3) Consider the homogeneous polynomial $Q(x, y, z) = x^2 + y^2 - z^2$ and its zero-locus $V_p(Q)$ in $\mathbb{RP}^2$. Find three affine charts $U_{\text{cir}}, U_{\text{par}}, U_{\text{hyp}}$ of $\mathbb{RP}^2$ such that the intersection with $V_p(Q)$ is

- a circle in U_{cir},
- a parabola in U_{par},
- and a hyperbola in U_{hyp}.

Is there any affine chart whose intersection with $V_p(Q)$ is a union of two lines?

4) Let k be a field of characteristic other than 2. The zero-locus of a homogeneous polynomial of degree 2 in $k\mathbb{P}^2$ is called a *conic*. Prove that, up to the action of $\mathrm{PGL}(3, k)$, every conic is of the form:

- $V_p(x^2 + y^2 + z^2)$,
- $V_p(x^2 + y^2 - z^2)$,
- the union of the two projective lines $V_p(xy)$,
- or the double line $V_p(x^2)$.

When are these four $\mathrm{PGL}(3, k)$-equivalence classes of conics disjoint? What happens if $k = \mathbb{C}$?

Part C3, due on 11.11.2025.

1) Let $P \in k[x_0, \ldots, x_{n-1}]$ be a non-constant polynomial. Prove that the affine algebraic set $V(P)$ is irreducible if and only if the projective algebraic set $V_p(P_{\text{hom}})$ is irreducible.