Assignments for Week 2

Part A1, due on 05.11.2025. No reading assignment this week. Write down how you feel about the video lectures. Was anything interesting, confusing?

Part B1, discussed on 31.10.2025.

1) Let R be a non-zero commutative ring with a unit. Prove that the topological space $\mathrm{Max}(R)$ has the T1-property. Prove that $\mathrm{Spec}(R)$ has the T1-property if and only if R is a field.

2) Explain why the usual identification $\mathbb{R}^2 \simeq \mathbb{C}$ does NOT induce an isomorphism between $\mathbb{RP}^2$ and $\mathbb{CP}^1$.

3) Let k be a finite field of cardinality p and $n \in \mathbb{N}$. Prove that the projective space $k\mathbb{P}^n$ is a finite set of cardinality:

$$|k\mathbb{P}^n| = \frac{p^{n+1}-1}{p-1}.$$

4) Consider the homogeneous polynomial $Q(x, y, z) = x^2 - y^2 - z^2$. What are the points at infinity of $V_p(Q)$ in the real projective plane $\mathbb{RP}^2$? What are the points at infinity of $V_p(Q)$ in the complex projective plane $\mathbb{CP}^2$?

5) Let k be a field and $n \in \mathbb{N}$. Prove that, for any projective hyperplane $H \subset k\mathbb{P}^n$, the complement $k\mathbb{P}^n \setminus H$ is isomorphic to k^n.

Part C2, due on 05.11.2025.

1) Let R be a non-zero commutative ring with a unit. Recall (from a commutative algebra course) that the set of nilpotent elements of R is a prime ideal of R, and denote it by $\mathrm{Nil}(R)$. Prove that

$$\overline{\{\mathrm{Nil}(R)\}} = \mathrm{Spec}(R),$$

where $\overline{S}$ stands for the Zarsiki closure of the subset S.

2) Let k be a finite field of cardinality p and $n \in \mathbb{N}$. Prove that the set of projective lines in $k\mathbb{P}^n$ is finite, of cardinality:

$$\frac{(p^{n+1}-1)(p^n-1)}{(p^2-1)(p-1)}.$$

How many points are contained in one projective line? How many lines go through the same point? What do you notice when $n = 2$? Do you have an explanation for that?

Hint: To interpret the denominator, you can prove that $(p^2-1)(p^2-p)$ is the number of pairs of non-colinear vectors in k^2.