Assignments for Week 1

Part A1, due on 28.10.2025. No reading assignment this week. Write down how you feel about what we covered in class this week. Was anything interesting, confusing?

Part B1, discussed on 24.10.2025.

1) Let k be a field and $n, m \in \mathbb{N}$. Consider two affine algebraic sets $V \subset k^n$ and $W \subset k^m$. Prove that the product $V \times W$ also is an affine algebraic set.

2) Is $\{(t, \sin(t)) \mid t \in \mathbb{R}\}$ an affine algebraic set in $\mathbb{R}^2$?

3) Let k be an infinite field and $n \in \mathbb{N}$. Show that any two non-empty Zariski-open sets of k^n have a non-empty intersection.

4) Let k be a field. Consider

$$C := \{(t, t^2, t^3) \mid t \in k\} \subset k^3.$$

Prove that C is an affine algebraic set and compute the ideal $I(C)$ in $k[X, Y, Z]$. Is C irreducible?

5) Consider the ideal $I = (X^2Y, (X+1)(Y-1)^2)$ in $\mathbb{R}[X, Y]$. Draw and then describe rigorously the affine algebraic set $V(I)$ in $\mathbb{R}^2$. What are its irreducible components? Denoting by A the quotient ring $\mathbb{R}[X, Y]/I$, prove that the sets $\mathrm{Max}(A)$ and $\mathrm{Spec}(A)$ are both finite, list their respective elements and describe their topology.

Part C1, due on 28.10.2025.

1) Let k be an infinite field and $U \subsetneq k$ be a Zariski-open set. Prove that $I(U) = (0)$.

2) Let k be a field of characteristic $\neq 2$. Consider the set

$$C := \left\{ \left(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1} \right) \mid t \in k \text{ with } t^2 + 1 \neq 0 \right\} \subset k^2.$$

Prove that C is an affine algebraic set and compute the ideal $I(C)$ in $k[X, Y]$. Is C irreducible?

Hint: You can start by drawing and describing C when $k = \mathbb{R}$.