Assignments for Week 14

Part A14, due before your oral exam. Any non-empty submission on this part of the assignment will automatically obtain the maximal grade. These few questions are primarily intended to give you an opportunity to reflect on the contents of the class. Your answers also provide valuable feedback: they will certainly help us improve this lecture for future students!

1) Make a short list (between four and ten entries) of the most important concepts and objects that you learned about in this class. Mark the few notions that were the most challenging to you and, for each of them, mention an example, an exercise or a property that helped your understanding. If you want, you can also mark your most/least favourite concepts and mention why.

2) What do you consider to be the most important theorem of the class? Can you name a concept or result that was crucial in its proof? Can you mention at least one application of it?

3) Do you have any feelings on the speed of the class and on the amount of material that was covered?

4) Do you have any feelings on the exercises sheets and on the exercise sessions?

5) Do you have any other comments or remarks to improve this class?

Part B14, discussed on 6.2.2026.

1) Let k be an algebraically closed field. Let $Q \subset k\mathbb{P}^3$ be an irreducible quadric.

- Prove that every point of Q lies on a line contained in Q.

  Hint: Use the same strategy as in the lecture, when building ten lines from one line on a smooth cubic surface.
- Let $P \in Q$ be a smooth point and let ℓ be a line in $k\mathbb{P}^3$ that passes through P and is not contained in Q. Prove that ℓ intersects Q at exactly one point distinct from P.
- Assume that Q is singular. Prove that Q is a cone over a smooth conic and describe all of the lines on Q.
- Assume that Q is smooth. Prove that every point of Q lies on exactly two lines contained in Q.
- Keep assuming that Q is smooth. Construct an isomorphism of algebraic varieties

  $$f : Q \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$$

  that, for every point $q \in Q$, sends the two lines through q to the two fibers $\mathbb{P}^1 \times \{q_2\}$ and $\{q_1\} \times \mathbb{P}^1$, where we denote $f(q) = (q_1, q_2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.
- Prove that $\mathbb{P}^1 \times \mathbb{P}^1$ is rational.

2) Let k be an algebraically closed field of characteristic $\neq 2, 3$. Let C be a smooth cubic curve in $k\mathbb{P}^2$ and let $O \in C$ be an inflection point.

- Prove that the group $(C, +)$ defined in the lecture using the point O as a neutral element contains exactly four 2-torsion points.

  Definition: For $k \in \mathbb{N}$, a k-torsion point of $(C, +)$ is a point $A \in C$ such that $kA = O$.
- Assume that C admits another inflection point O'. Prove that O' is a 3-torsion point for $(C, +)$.

Part C14, due on 11.2.2026.

- In the proof of the rationality of a smooth cubic surface in [Hulek, Proposition 5.17, Pages 161-162], the following result is used: For any two disjoint lines $\ell, m \subset \mathbb{P}^3$ and any point $Q \notin \ell \cup m$, there exists a unique line through Q that intersects both ℓ and m. Prove it.
- Read the rest of the proof in [Hulek, Proposition 5.17, Pages 161-162].
- **(optional)** For two disjoint lines ℓ and m in S, what is the exact number of lines on S that intersect both ℓ and m?