Assignments for Week 13

Part A13. No reading assignment this week.

Part B13, discussed on 30.1.2026.

1) Let k be an algebraically closed field of characteristic $\neq 2, 3$. Determine the singular points on the following varieties:

- $V(y^2 - x^3) \subset k^2$;
- $V(x^2y^3 + y^2z^3 + z^2x^3) \subset k\mathbb{P}^2$;
- $V(x^2 + y^2 + z^2) \subset k\mathbb{P}^3$.

2) Let V be an algebraic variety (over an algebraically closed field). Prove that, if V is connected and reducible, then V is not smooth. Deduce that a smooth projective hypersurface $V = V_p(F) \subset k\mathbb{P}^n$ must be irreducible. Give an example of an affine algebraic variety that is reducible **and** smooth.

3) Let V be an affine algebraic variety over an algebraically closed field k and let $x \in V$.

- Denote by m_x the unique maximal ideal of the local ring $\mathcal{O}_{V,x}$. Prove that T_xV is isomorphic to the dual of the vector space m_x/m_x^2.
- As a first application of this fact, show that the following inequalities hold:

$$\dim T_xV \geq \dim_{\text{Krull}} \mathcal{O}_{V,x} \geq \dim_x V.$$

- As a second application of this fact, prove that $\dim T_x$ is the minimal number of a set of generators for the maximal ideal m_x in $\mathcal{O}_{V,x}$.
- Conclude that, if V is a curve, the point $x \in V$ is smooth if and only if the local ring $\mathcal{O}_{V,x}$ is a principal ideal domain.

4) Let k be an algebraically closed field. Let S be a cubic surface in $k\mathbb{P}^3$ that contains two distinct singular points $a, b \in S$. Prove that the line joining a and b is contained in S.

5) Let F be an irreducible homogeneous polynomial of degree d in $k[x, y, z]$. Denoting by μ_P the multiplicity of a point $P \in V(F)$, prove that

$$\sum_{P \in V(F)} \mu_P(\mu_P - 1) \leq (d-1)(d-2).$$

Note: You may first prove that the left-hand-side is well-defined.

What happens if the polynomial F is reducible?

Part C13, due on 5.2.2026.

- Let V and W be two algebraic varieties. Let $v \in V$ and $w \in W$. Prove that $V \times W$ is smooth at (v, w) if and only if V is smooth at v and W is smooth at w.

- (the nodal cubic) Let V be a 3-dimensional vector space over an algebraically closed field k. Let C be an irreducible cubic in $\mathbb{P}(V)$. Assume that C has a *node* p, that is a singular point of multiplicity 2 at which C admits two distinct tangent lines. Prove that there is a choice of coordinates $[x : y : z]$ on $\mathbb{P}(V)$ in which one can write:

$$C = V(x^3 + y^3 - xyz).$$

  Note: The tangent lines to a plane curve at a given point are defined in the textbook by Perrin (Page 97, Paragraph 4.8).