Assignments for Week 12

Part A12. No reading assignment this week.

Part B12, discussed on 23.1.2026.

1) (another result on locally free sheaves)

- Let R be a commutative ring with a unit and M be a finitely generated R-module. Prove that any surjective morphism $\phi : M \to M$ is an isomorphism.
- Let $r \in \mathbb{N}$. Let X be a scheme and $\mathcal{F}$ and $\mathcal{G}$ be two locally free sheaves of $\mathcal{O}_X$-modules, both of rank r. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a surjective morphism. Prove that every point $x \in X$ admits a neighborhood $U \subset X$ on which the induced map on global sections $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. Deduce that ϕ is an isomorphism.

2) Let k be an algebraically closed field. Consider the Grassmannian $\mathrm{Gr}(2,4)$ with its Plücker embedding and its Plücker coordinates $x_{i,j}$ for $1 \leq i < j \leq 4$. Prove that

$$\mathrm{Gr}(2,4) = V_p(x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3})$$

in the appropriate projective space. Deduce that $\mathrm{Gr}(2,4)$ is an irreducible algebraic variety.

3) (affine charts of Grassmannians) Let $d, n \in \mathbb{N}$. Let k be an algebraically closed field.

- Prove that a matrix $A \in \mathcal{M}_{d,n}(k)$ can be written in the form

  $$A = (I_d \mid M) \text{ for some } M \in \mathcal{M}_{d,n-d}(k)$$

  if and only if all of its $(d-1)$-th minors and its unique d-th minor indexed among the first d rows and d columns all equal 1.
- Prove that the following subset of the Grassmannian $\mathrm{Gr}(d,n)$ is a Zariski-open subset:

  $$\{[W] \in \mathrm{Gr}(d,n) \mid W \text{ spanned by the rows of } (I_d \mid M) \text{ for some } M \in \mathcal{M}_{d,n-d}(k)\}.$$

- Describe a **finite** covering of $\mathrm{Gr}(d,n)$ by Zariski-open subsets $(U_i)_{i\in I}$, where each U_i can be written as above in **some** basis of k^n.

4) (irreducibility of Grassmannians) Let $d, n \in \mathbb{N}$. Let k be an algebraically closed field. Prove that the algebraic variety $\mathrm{Gr}(d,n)$ is irreducible.

Hint: Try to imitate the proof of the irreducibility of the projective space using its affine charts.

Part C12, due on 29.1.2026.

- Let $n \in \mathbb{N}$ and k be an algebraically closed field. Let $\mathrm{O}(n,k)$ denote the orthogonal linear subgroup of $\mathrm{GL}(n,k)$. Prove that $\mathrm{O}(n,k)$ is an affine algebraic variety and compute its dimension.

  Hint: It can be useful to construct a morphism of algebraic varieties $f : \mathrm{O}(n,k) \to \mathbb{S}_k{}^{n-1}$ where $\mathbb{S}_k{}^{n-1}$ denotes the unit sphere in k^n.