Assignments for Week 11

Part A11. No reading assignment this week.

Part B11, discussed on 16.1.2026 and 21.12026.

1) (finishing the proof of Krull's Hauptidealsatz) Let R be an integral domain. Let S be a subring of R such that R is a finitely generated S-module. Consider an element $f \in R$ that generates a prime ideal.

- Prove that the kernel of the composition $S \subset R \to R/(f)$ is a prime ideal in S.
- Prove that every element $r \in R$ is the root of a monic polynomial in $S[X]$. Construct a map $r \in R \mapsto P_r \in S[X]$ such that each P_r is a monic polynomial with $P_r(r) = 0$ and

$$N_{R/S} : r \in R \mapsto P_r(0_S) \in S$$

  is a multiplicative group homomorphism (up to removing the zero on both sides).
- Prove that $f_0 := N_{R/S}(f) \in R$ is contained in the ideal (f).
- Show that the kernel of the composition $S \subset R \to R/(f)$ is the ideal generated by f_0 in S.

2) (proving the inequality in the dimension theorem)

- Let $\phi : V \to W$ be a morphism of irreducible affine algebraic varieties. Denote by d_W the dimension of W. Let $w \in W$. Prove that the fiber $\phi^{-1}(w)$ can be written as a finite union of irreducible components of a Zariski-closed set of the form $V(g_1, \ldots, g_{d_W})$ for some $g_1, \ldots, g_{d_W} \in \Gamma(V)$. Deduce that the inequality in the dimension theorem holds when both varieties involved are affine.
- Explain how to deduce the inequality in the dimension theorem for **any** dominant morphism $\phi : X \to Y$ between irreducible varieties from the inequality in the dimension theorem for **only those** morphisms **to an affine variety** Y.
- Reduce the proof of the inequality in the dimension theorem to the case when X and Y are affine.
- Conclude the proof.

3) Let k be a field and $d \in \mathbb{N}$ be a prime number. Consider the image C_d of the Veronese embedding of $k\mathbb{P}^1$ into $k\mathbb{P}^d$. Prove that C_d cannot be written in the form $V_p(f_1, \ldots, f_{d-1})$ using homogeneous polynomials $f_1, \ldots, f_{d-1}$ in $k[x_0, \ldots, x_n]$.

Remark: You may admit and use one of the exercise statements in Part C10.

4) Let k be an algebraically closed field. Let $V \subset k\mathbb{P}^n$ be a projective variety of dimension d.

- Prove that for any homogeneous polynomials $g_1, \ldots, g_d$ in $k[x_0, \ldots, x_n]$ such that the zero-locus $V_p(g_1, \ldots, g_d)$ is non-empty, the intersection $V \cap V_p(g_1, \ldots, g_d)$ is non-empty.
- Prove the existence of homogeneous polynomials $f_1, \ldots, f_{d+1}$ in $k[x_0, \ldots, x_n]$ such that the intersection $V \cap V_p(f_1, \ldots, f_{d+1})$ is empty.

5) Let k be a field and n be an integer. Let V be a k-vector space of dimension n. Consider the *complete flag variety*, i.e., the set

$$F := \{(V_i)_{1 \leq i \leq n} \mid V_i \text{ linear subspace of } V, \dim V_i = i, V_i \subset V_{i+1} \text{ for every } 1 \leq i \leq n\}.$$

Prove that F is an algebraic variety and compute its dimension in terms of n.

6) (dimension under a finite morphism)

(a) Let A and B be rings. Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a finite morphism between the corresponding affine schemes.

- Prove that f is closed.
- Prove that f is surjective if and only if $\Gamma(f^\sharp) : A \to B$ is injective.

(b) Let T be a reduced irreducible affine scheme over a field k of dimension d. Prove the existence of a finite surjective morphism of schemes from T to the affine space $\mathbb{A}^d_k$.

(c) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a finite morphism of reduced irreducible schemes. Prove that $\dim X \leq \dim Y$ and that the equality holds if and only if f is surjective.

Part C11, due on 26.1.2026.

- (more on Bézout theorem) Let $f_1, \dots f_n$ be homogeneous polynomials in $k[x_0, \dots, x_n]$. Suppose that the scheme $V_p(f_1, \dots, f_n)$ is finite. Show that its number of points, counted with multiplicity, is the product of the degrees of the polynomials f_i.

- Let k be an algebraically closed field. Let $f : k\mathbb{P}^n \to k\mathbb{P}^m$ be a morphism of algebraic varieties. Prove that the image of f is either a point, or a closed subset of dimension n.