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Positivité du faisceau (co)tangent et des classes de Chern

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RÉSUMÉ

Cette thèse participe à la description de certaines variétés complexes projectives à diviseur anticanonique numériquement effectif (nef).

Dans la première partie, nous montrons que les faisceaux tangent et cotangent réflexivisé d'une variété normale projective de Calabi-Yau ou irréductible holomorphe symplectique à singularités canoniques ne sont pas pseudoeffectifs, ce qui généralise des résultats de Höring et Peternell en retranchant une hypothèse de lissité en codimension 2. La positivité de la seconde classe de Chern orbifold joue un rôle important dans la preuve, notamment dans un théorème technique faisant le lien entre la pseudoeffectivité d'un faisceau réflexif suffisamment stable de déterminant trivial et l'annulation de sa seconde classe de Chern orbifold. Nous présentons également des exemples de variétés de Calabi-Yau de petite dimension ayant des singularités en codimension 2.

Dans la deuxième partie, nous exposons plusieurs résultats liés à la classification des quotients de variétés abéliennes par des groupes finis agissant librement en codimension 2 qui admettent une variété de Calabi-Yau comme résolution. Il est équivalent de classifier les variétés de Calabi-Yau admettant une annulation partielle de la seconde classe de Chern. Tandis qu'Oguiso construit deux exemples en dimension 3, nous prouvons qu'il n'y en a pas en dimension 4. Nous montrons également qu'à dimension fixée et à isogénie près, il y a seulement deux variétés abéliennes susceptibles d'admettre de tels quotients, à savoir $\left(E_{\frac{-1+i\sqrt{3}}{2}}\right)^n$ et $\left(E_{\frac{-1+i\sqrt{7}}{2}}\right)^n$. Quant au groupe fini agissant, nous montrons qu'il est engendré par ses éléments admettant des points fixes, et nous classifions ses sous-groupes de la forme $\mathrm{PStab}(a)$, fixant un point commun a de la variété abélienne étudiée : ces sous-groupes sont des 3-groupes ou des 7-groupes abéliens élémentaires. Finalement, nos résultats impliquent qu'aucun quotient de variété abélienne par un groupe agissant librement en codimension 3 n'admet de résolution crépante simplement connexe.

Le but de la troisième partie est d'établir la conjecture du cone pour les paires de Schoen (une terminologie que nous introduirons), généralisant l'article de Grassi et Morrison sur les variétés de Calabi-Yau de dimension 3 introduites par Schoen. Pour prouver cette conjecture dans ce cas particulier, nous décrivons complètement le cone nef des variétés de Schoen, en utilisant leur description en tant que produits fibrés au dessus de $\mathbb{P}^1$. Ce travail est une collaboration avec Hsueh-Yung Lin et Long Wang.

Dans la quatrième partie, nous prouvons qu'une variété X projective lisse de dimension $n \geq 4$, respectivement $n \geq 5$, dont la troisième, respectivement quatrième, puissance extérieure du fibré tangent est strictement nef est une variété de Fano. Nous classifions également une telle variété X sous l'hypothèse additionnelle $\rho(X) \neq 1$. Enfin, nous prouvons que si la $(n-1)$-ième puissance extérieure du fibré tangent est nef et X est rationnellement connexe, alors X est une variété de Fano.

Mots clés : variétés de Calabi-Yau, variétés irréductibles holomorphes symplectiques, variétés abéliennes, positivité du faisceau (co)tangent, classes de Chern, correspondence de McKay, résolution crépante, cône nef, conjecture du cône de Kawamata-Morrison, variétés rationnellement connexes, variétés de Fano.

ABSTRACT

This thesis contributes to the description of some complex projective varieties with nef anticanonical divisor.

In the first part, we prove that the tangent and the reflexivized cotangent sheaves of any normal projective Calabi-Yau or irreducible holomorphic symplectic variety with canonical singularities are not pseudoeffective, generalizing results of Höring and Peternell by removing an assumption of smoothness in codimension 2. Positivity of the second orbifold Chern class plays a key role in the proof, namely in a technical theorem relating pseudoeffectivity of a sufficiently stable reflexive sheaf with trivial determinant to the vanishing of its orbifold second Chern class. We also provide examples of Calabi–Yau varieties of small dimension with singularities in codimension 2.

In the second part, we present many results toward a classification of those quotients of an abelian variety by a finite group acting freely in codimension 2 that admit a Calabi–Yau resolution. This is equivalent to classifying Calabi-Yau manifolds with a partial vanishing of the second Chern class. While Oguiso constructed two examples in dimension 3, we show that there are none in dimension 4. We also show that, up to isogeny, there are only two abelian varieties admitting such finite quotients in each dimension: $\left(E_{\frac{-1+i\sqrt{3}}{2}}\right)^n$ and $\left(E_{\frac{-1+i\sqrt{7}}{2}}\right)^n$. As for the finite group acting, we show that it is generated by its elements admitting fixed points, and classify its subgroups $\mathrm{Pstab}(a)$ that have a common fixed point a: they are elementary abelian 3-groups or 7-groups. Finally, our results imply that no quotient of an abelian variety by a finite group acting freely in codimension 3 admits a simply-connected crepant resolution.

The goal of the third part is to establish the Cone Conjecture for so-called Schoen pairs, generalizing the work by Grassi and Morrison on the Calabi–Yau threefolds constructed by Schoen. In order to prove it, we completely describe the nef cone of Schoen varieties, using their description as fiber products over $\mathbb{P}^1$. This is joint work with Hsueh-Yung Lin and Long Wang.

In the fourth part, we prove that a smooth projective variety X of dimension $n \geq 4$, respectively $n \geq 5$, with strictly nef third, respectively fourth, exterior power of the tangent bundle is a Fano variety. We also classify X under the assumption that $\rho(X) \neq 1$. Finally, we prove that if the $(n-1)$-th exterior power of the tangent bundle is nef and X is rationally connected, then X is a Fano variety.

Keywords: Calabi-Yau varieties, hyperkähler varieties, abelian varieties, positivity of the (co)tangent sheaf, Chern classes, McKay correspondence, crepant resolution, nef cone, Kawamata-Morrison Cone Conjecture, rationally connected varieties, Fano varieties.

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CHAPTER 1

INTRODUCTION

1.1 Introduction. Ever since the Greek antiquity, correspondences between geometric shapes and equations have been at the root of geometry, algebra, and arithmetics at once. Diophantine arithmetics considers integral, or rational solutions to polynomial equations, while geometry and algebra accept real, or complex solutions. A complex projective variety X is the set of solutions to a system of irreducible complex polynomial equations in the complex projective space $\mathbb{P}^n$. Smoothness of the variety X is equivalent to an algebraic property of the corresponding system of polynomial equations, as says the Jacobi criterion.

Smooth complex projective varieties are complex manifolds: as such, they come naturally equipped with a tangent bundle. Positivity properties of this tangent bundle recover information about the initial variety. For instance, if the tangent bundle is ample, the initial variety is a projective space, whereas if the tangent bundle is numerically flat, the initial variety is an étale quotient of an abelian variety. However, the tangent bundle of a given variety is rarely fully understood. What we might know, given a variety, is some invariants attached to its tangent bundle, the Chern classes.

Chern classes are a useful tool in many ways. Alike Stiefel-Whitney classes, they can provide obstructions to embedding certain varieties into low dimensional projective spaces, as in [195], where it is proven that for $d \geq 3$, a d-dimensional abelian variety cannot be embedded in $\mathbb{P}^{2d}$. Chern classes (notably of the (co)tangent bundle) satisfy inequalities, as we will illustrate in Section 2.7. Some of the equality cases in these inequalities are characterized as peculiar geometric situations as well. These equality cases are noteworthy, as they allow to recover information on the geometry of X from purely numerical invariants.

This thesis, as its title suggests, deals with positivity of vector bundles, mirrored into positivity of its Chern classes. The goal is fourfold: in Part I, we prove that a slightly positive reflexive sheaf with vanishing first Chern class has vanishing second Chern class. Still in Part I, but also in Part II, we use information about a second Chern class (a positivity condition in Part I, and a vanishing in Part II) to derive information about a variety (about the positivity of its tangent bundle in Part I, about the variety itself in Part II). In Part III, we change from varieties of trivial first Chern class to varieties with non-negative first Chern class. A conjecture, the Kawamata-Morrison Cone Conjecture, describes the cone of non-negative divisors on such varieties. We prove it in a particular case. Finally, Part IV is about proving that some varieties with non-negative first Chern class, under a positivity assumption on some vector bundle canonically associated to them, actually have positive first Chern class.

Let us sketch our work in more detail.

1.2 Part I. Part I deals with the positivity of the (co)tangent sheaf of singular K-trivial varieties. In dimension 2, a dichotomy was observed in [148] between K3 surfaces, whose tangent and cotangent bundle are not pseudoeffective, and abelian varieties, whose tangent and cotangent bundle are trivial, hence pseudoeffective. We generalize this result to klt K-trivial varieties of arbitrary dimension. The theorem can be presented as follows: we know that, up to finite quasiétale cover, any K-trivial variety decomposes as a product of singular Calabi-Yau, singular irreducible holomorphic symplectic, and abelian varieties, in its so-called singular Beauville-Bogomolov decomposition [70, 55, 86, 8]. The point is that pseudoeffectivity of the tangent, or reflexivized cotangent sheaf of a variety detects an abelian factor in its singular Beauville-Bogomolov decomposition.

Theorem 1.1. *Let X be a normal projective variety with klt singularities and numerically trivial K_X. If its tangent or reflexivized cotangent sheaf is pseudoeffective, then there is a quasiétale finite cover $\tilde{X} \to X$ such that $q(\tilde{X}) \neq 0$. Equivalently, the singular Beauville-Bogomolov decomposition of X has an abelian factor of positive dimension. In particular, if X is a singular Calabi-Yau or IHS variety in the sense of Definition 5.2, then neither $\mathcal{T}_X$ nor its dual $\Omega_X^{[1]}$ is pseudoeffective.*

This result generalizes [86, Theorem 1.6], which makes a technical hypothesis of smoothness in codimension 2. The proof uses the singular Beauville-Bogomolov decomposition theorem and the following result.

Theorem 1.2. *Let X be a normal projective variety with klt singularities of dimension n, H a $\mathbb{Q}$-Cartier ample divisor on X. Consider $\mathcal{E}$ a reflexive sheaf on X such that $c_1(\mathcal{E}) \cdot H^{n-1} = 0$, the sheaves $\mathcal{E}$ and $S^{[l]}\mathcal{E}$, for some $l \geq 6$, are H-stable, and $\mathcal{E}$ is pseudoeffective. Then $c_1(\mathcal{E})^2 = c_2(\mathcal{E}) = 0$.*

Moreover, there is a quasiétale finite Galois covering $\nu : \tilde{X} \to X$, such that $\nu^{[*]}\mathcal{E}$ is locally-free, has a numerically trivial determinant, and is $\mathrm{Gal}(\tilde{X}/X)$-equivariantly flat on $\tilde{X}$, i.e., comes from a $\mathrm{Gal}(\tilde{X}/X)$-equivariant representation of $\pi_1(\tilde{X})$. In particular, $\nu^{[*]}\mathcal{E}$ is numerically flat, and

$$c_1(\nu^{[*]}\mathcal{E}) = 0, \quad c_2(\nu^{[*]}\mathcal{E}) = 0.$$

This theorem generalizes [86, Theorem 1.1], by again removing an assumption of smoothness in codimension 2. Additionally, it is motivated by a criterion of algebraic integrability for foliations that was recently developed [22, 29, 55], and that is a key ingredient in the proof of the singular Beauville-Bogomolov decomposition theorem too.

As for its proof, first note that positivity of a sheaf is not preserved by birational modifications, hence we could not simply terminalize X into $\tilde{X}$ and use the mentioned theorems on $\tilde{X}$. Hence, the strategy is to first reduce the generalization of Theorem 1.2, Theorem 3.3, to a statement on a klt surface S. By one of the standard construction for orbifold Chern classes, we then construct from the positive reflexive sheaf $\mathcal{E}$ on the surface S with quotient singularities a locally free sheaf $\hat{\mathcal{E}}$ on a finite Galois cover $\hat{S}$ of S: the Chern classes of $\hat{\mathcal{E}}$ correspond to the orbifold Chern classes of $\mathcal{E}$. We then play back-and-forth between different notions of positivity for $\mathcal{E}$ and $\hat{\mathcal{E}}$.

We finally provide examples of 2409 klt Calabi-Yau threefolds that are not smooth in codimension 2, to which our theorem applies. We extract them from the database of quasismooth Calabi-Yau hypersurfaces in wellformed weighted projective spaces by Kreuzer and Skarke [115]. This work is drawn from the author's [66].

1.3 Part II. In Part II, we focus on smooth Calabi-Yau varieties X admitting a nef and big divisor D with $c_2(X) \cdot D^{n-2} = 0$. Equivalently, by Theorem 2.57, these Calabi-Yau varieties are resolutions of quotients of abelian varieties by a finite group acting freely in codimension 2. In dimension 3, these are precisely the Calabi-Yau varieties of type III_0 in the classification of Calabi-Yau algebraic fiber spaces [152], that are determined by Oguiso in [155]: there are exactly two rigid instances of them. Describing such varieties plays an important role in the classification of contractions of Calabi-Yau manifolds. This classification, and in particular [155], is for instance used in [157] to prove that in a general Calabi-Yau hypersurface X in a smooth Fano fourfold, every nef divisor D satisfies $c_2(X) \cdot D > 0$.

We pursue the same purpose in higher dimension, and achieve it in dimension 4: there are no simply-connected crepant resolutions of quotients of abelian fourfolds by finite groups acting freely in codimension 2. We obtain partial results indicating that examples should be sparse in arbitrary dimension too, proving, e.g., the following theorem, where we denote by j the first primitive third root of unit and by u_7 the quadratic integer $\frac{-1+i\sqrt{7}}{2}$.

Theorem 1.3. *Let A be an abelian variety of dimension n and G be a finite group acting freely in codimension 2 on A. If A/G has a crepant resolution that is a Calabi-Yau manifold, then*

(1) *A is isogenous to ${E_j}^n$ or to ${E_{u_7}}^n$, and G is generated by its elements that admit fixed points.*

(2) *For every translated abelian subvariety $W \subset A$, there is $k \in \mathbb{N}$ such that the pointwise stabilizer*

$$\mathrm{PStab}(W) := \{g \in G \mid \forall w \in W,\ g(w) = w\}$$

is isomorphic to ${\mathbb{Z}_3}^k$ if A is isogenous to ${E_j}^n$, or to ${\mathbb{Z}_7}^k$ if A is isogenous to ${E_{u_7}}^n$.

(3) *For every translated abelian subvariety $W \subset A$, if $\mathrm{PStab}(W)$ is isomorphic to*

- *${\mathbb{Z}_3}^k$, then there are k generators of it such that their matrices are similar to*
$$\mathrm{diag}(\mathbf{1}_{n-3}, j, j, j),$$
and the j-eigenspaces of these matrices are in direct sum.
- *${\mathbb{Z}_7}^k$, then there are k generators of it such that their matrices are similar to*
$$\mathrm{diag}(\mathbf{1}_{n-3}, \zeta_7, {\zeta_7}^2, {\zeta_7}^4),$$
and all eigenspaces of these matrices with eigenvalues other than 1 are in direct sum.

Finally, we prove that there are no simply-connected crepant resolutions of quotients of abelian varieties by finite groups acting freely in codimension 3.

On the way, we also prove a result in the spirit of a conjecture by Ito and Reid in McKay correspondence [94], and find a counterexample to the conjecture itself.

Conjecture 1.4. *Let $G < \mathrm{GL}_n(\mathbb{C})$. If the quotient $\mathbb{C}^n/G$ admits a crepant resolution, then every maximal cyclic subgroup of G is generated by a junior element.*

As a counterexample, we propose a representation of the group $\mathrm{SL}_2(\mathbb{F}_3)$ in $\mathrm{GL}_6(\mathbb{C})$, with a maximal cyclic subgroup $\mathbb{Z}_4$ that is not generated by a junior element, but such that the quotient $\mathbb{C}^6/\mathrm{SL}_2(\mathbb{F}_3)$ admits a crepant resolution (this last part being checked with help of the software `Macaulay2` [69]). We correct the conjecture with a milder result.

Proposition 1.5. *Let $G < \mathrm{GL}_n(\mathbb{C})$. If the quotient $\mathbb{C}^n/G$ admits a crepant resolution, then G is generated by junior elements.*

This part relies on the numerous properties of automorphisms of small dimensional abelian varieties [17, Chapter 13], and on heavy computational group theory. We make good use of the `SmallGrp` package of the software `GAP` [67]. The results of this part are the subject of a preprint [64].

1.4 Part III. In Part III, which is joint work with Hsueh-Yung Lin and Long Wang, we describe the nef cone of some Calabi-Yau pairs whose underlying varieties are obtained by fiber products of certain rationally-connected manifolds fibred over $\mathbb{P}^1$. This generalizes a construction of Calabi-Yau threefolds from rational elliptic surfaces due to Schoen [179]. Our description of the nef cone enables us to prove the Kawamata-Morrison Cone Conjecture for our examples, i.e., that the action of a certain subgroup $\mathrm{Aut}(X, \Delta)$ of the automorphism group on the nef effective cone of X admits a rational polyhedral fundamental domain. The Kawamata-Morrison Cone Conjecture is a long-standing conjecture for klt pairs (and notably varieties) with trivial canonical class, and although it was proven in full generality in dimension 2 [191], it is still widely open in dimension 3. We refer the curious reader to [124] for a survey of its history and implications, as well as to Chapter 17.

The decomposition of the nef cone is proven by *ad hoc* methods, relying heavily on the fact that we consider fiber products over a curve (see Examples 19.5, 19.6). The result is the following.

Theorem 1.6. *For $i = 1, 2$, let $\phi_i : W_i \to C$ be a surjective morphism with connected fibers from a smooth projective variety to a smooth projective curve. Assume that*

- *the variety $W = W_1 \times_C W_2$ is smooth;*
- *it holds*
$$p_1^* N^1(W_1)_{\mathbb{R}} + p_2^* N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}},$$
where p_i denotes the projection from W onto W_i.

Then $p_1^*\mathrm{Nef}(W_1) + p_2^*\mathrm{Nef}(W_2) = \mathrm{Nef}(W)$.
As a consequence, $p_1^*\mathrm{Amp}(W_1) + p_2^*\mathrm{Amp}(W_2) = \mathrm{Amp}(W)$.

Grassi and Morrison [68, Proposition 3.1] had already proven such a nef cone decomposition for the fiber product of two rational elliptic surfaces with their natural fibrations to $\mathbb{P}^1$.

The existence of the rational polyhedral domain then follows, using a result by Totaro [190, Theorem 8.2] for rational elliptic surfaces, a result by Kollár [19] for ample hypersurfaces in Fano manifolds, a result by Looijenga [132] on the existence of a fundamental domain for a linear action on a cone, and by carefully choosing subgroups of the automorphism group of the factors of the fiber product, that are large enough to both act extensively on the nef cone, and small enough that their product embeds in $\mathrm{Aut}(X, \Delta)$. This strategy allows to bypass any decomposition result at the level of automorphism groups, while such a decomposition [151, Corollary 2.3] was crucial to Grassi and Morrison's proof [68]. These results should be available on the arXiv soon.

1.5 Part IV.

In Part IV, we investigate some varieties with nef first Chern class, namely smooth varieties X such that an exterior power of the tangent bundle $\bigwedge^r T_X$ is strictly nef.

Definition 1.7. A line bundle L on a normal projective variety X is *strictly nef* if, for every curve C in X, the intersection number $L \cdot C$ is positive. A vector bundle E is *strictly nef* if, on its associated projectivized bundle $\mathbb{P}(E)$, the tautological line bundle is strictly nef.

For $r = \dim(X)$, a conjecture by Campana and Peternell [28, Problem 11.4] predicts that smooth varieties with strictly nef anticanonical divisor are Fano varieties. It is quite a surprising conjecture, as in general, a strictly nef divisor has no reason to be ample [80, Chapter 1, Example 10.6], but if true, it would provide a simplified version of Nakai-Moishezon criterion for ampleness of $-K_X$. Until now, this conjecture is however only known in dimension up to 3, by the work of Maeda [134] and Serrano [181].

We investigate a similar question for different values of r.

Question 1.8. Let X be a smooth projective variety of dimension n. Let $1 \leq r \leq n$ be an integer, and assume that $\bigwedge^r T_X$ is strictly nef. Is X a Fano manifold?

In the case $r = 1$, Li, Ou and Yang [128] prove that X must be a projective space. In the case $r = 2$ and $\dim(X) \geq 3$, they prove that X must be a projective space or a quadric hypersurface, hence a Fano manifold. We also prove that X is Fano when $r = 3$ and $\dim(X) \geq 4$, when $r = 4$ and $\dim(X) \geq 5$, and when $r = \dim(X) - 1$. In the first two cases, we present a classification of those varieties X which additionally satisfy $\rho(X) \neq 1$, based on the theory of Mori contractions of large length.

Theorem 1.9. *Let X be a smooth projective rationally connected variety of dimension $n \geq 4$ such that for each rational curve C in X, we have $-K_X \cdot C \geq n - 1$. Then either $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$, or X is a Fano variety of Picard rank $\rho(X) = 1$.*

Corollary 1.10. *Let X be a smooth projective variety of dimension at least 4 such that the vector bundle $\bigwedge^3 T_X$ is strictly nef. Then either $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$, or X is a Fano variety of Picard rank $\rho(X) = 1$.*

Theorem 1.11. *Let X be a smooth projective rationally connected variety of dimension $n \geq 6$ such that for each rational curve C in X, we have $-K_X \cdot C \geq n - 2$. Then either X is isomorphic to $\mathbb{P}^3 \times \mathbb{P}^3$ or X is a Fano variety of Picard rank $\rho(X) = 1$.*

Corollary 1.12. *Let X be a smooth projective variety of dimension at least 5 such that the vector bundle $\bigwedge^4 T_X$ is strictly nef. Then either X is isomorphic to one of the following Fano varieties*

$$\mathbb{P}^2 \times Q^3;\ \mathbb{P}^2 \times \mathbb{P}^3;\ \mathbb{P}(T_{\mathbb{P}^3});\ \mathrm{Bl}_\ell(\mathbb{P}^5) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1));\ \mathbb{P}^3 \times \mathbb{P}^3$$

or X is a Fano variety of Picard rank $\rho(X) = 1$.

Theorem 1.13. *Let X be a smooth projective rationally connected variety of dimension n such that the vector bundle $\bigwedge^{n-1} T_X$ is nef. Then X is a Fano variety.*

The proof of this last theorem relies on Chern classes inequalities *à la* Demailly, Peternell, Schneider [47], and on the Hirzebruch-Riemann-Roch formula. Our result and its proof are reminiscent of the fact known by [47], that a smooth rationally connected variety X such that T_X is nef is a Fano variety. Note that, building on this theorem, [198, Proposition 1.4] very recently gave an affirmative answer to Question 1.8 in general. This work appears in the author's preprint [65].

1.6 Future directions. There still are many mysteries regarding positivity of Chern classes. For instance, it is still not known if the second Chern class of a variety with trivial first Chern class is always pseudoeffective, in the sense that it lies in the closure of the effective cone of codimension 2 cycles. In dimension 3, the dual of the cone of pseudoeffetive 1-cycles being the nef cone, the Inequality 2.51 asserts this pseudoeffectivity. However, in higher dimension, the closure of the cone generated by complete intersections of ample divisors is no longer dual to the pseudoeffective cone, and thus the pseudoeffectivity of $c_2(X)$ is not implied by Inequality 2.51. On pathologies of the nef cone and the pseudoeffective cone for higher codimension cycles, see [43] and [62]. By work of Ottem [162], it is known that on the variety of lines of a cubic fourfold (which is an irreducible holomorphic symplectic fourfold), the second Chern class is not big, i.e., not in the interior of the pseudoeffective cone. To our knowledge, more is not known.

Closer to our work, note that the classification of the finite quotients A/G that are smooth in codimension 2 and admit a Calabi-Yau resolution in Part II would need to be extended to higher dimensions. Under the additional assumption that the group G is abelian, Theorem 7.6 and the results of Section 12 prove that, if A is an abelian variety of dimension n and G is a finite abelian group acting freely in codimension 2 on A such that A/G admits a Calabi-Yau resolution X, then $n = 3$ and X is X_3 or X_7. Also note that G is abelian if and only if any two junior elements g, h of G commute, which by our results can be checked *via* their matrices acting on a vector space V of dimension 3, 4, 5, or 6. Standard finite group theory allows us to explicitly bound the order of $\langle g, h\rangle$ depending on this dimension and the isogeny type of A. If the dimension is 3 or 4, the bounds are reasonable enough to launch a computer-assisted search through all possible abstract groups $\langle g, h\rangle$. Among these, the only groups which, in a faithful 3 or 4-dimensional representation, are generated by two junior elements of the same type, are $\mathbb{Z}_3$, $\mathbb{Z}_7$, and the finite simple group $\mathrm{SL}_3(\mathbb{F}_2)$ of order 168. But a geometric argument on fixed loci excludes $\mathrm{SL}_3(\mathbb{F}_2)$, whence the wished contradiction. This reproves the classification of [155] in dimension 3, and settles Theorem 7.5. When V has dimension 5 or 6, we could also bound the order of $\langle g, h\rangle$ explicitly, but the bounds obtained in this way are too large for the `SmallGrp` library. In order to rpove that G is abelian, one needs to build a reasonably smaller finite list of possibilities for the abstract group $\langle g, h\rangle$, and to rule out those potential groups in the list other than $\mathbb{Z}_3$, $\mathbb{Z}_7$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_7 \times \mathbb{Z}_7$. This is our work in progress in dimension 5.

There are open questions about the known quotients of dimension 3 too. For instance, the discussion of rational curves on quotients of abelian varieties is a topic of long-lasting interest [109, 91], connected to the topic of the gonality of curves on abelian varieties [165, 196]. It would be interesting to prove that the number of rational curves on $\left(E_{\frac{-1+i\sqrt{7}}{2}}\right)^3/\mathbb{Z}_7$ is finite, or that the uniruled divisors in $(E_j)^3/\langle \mathrm{diag}(j,j,j)\rangle$ are exactly those induced by inclusions of the two-dimensional factor (as $(E_j)^2/\langle \mathrm{diag}(j,j)\rangle$ is a rational surface). Such considerations might lead to a proof of the Cone Conjecture for Oguiso's Calabi-Yau threefolds X_7 and X_3. In fact, these two Calabi-Yau threefolds present a dichotomy: X_7 has a finite automorphism group, and the quotient $\left(E_{\frac{-1+i\sqrt{7}}{2}}\right)^3/\mathbb{Z}_7$ has a rational polyhedral nef cone, while X_3 has an infinite automorphism group, and the quotient $(E_j)^3/\langle \mathrm{diag}(j,j,j)\rangle$ has a circular nef cone. However, it is not clear how to describe the nef cones of X_7 and X_3, as their relative Picard numbers above the quotients they resolve are quite large (21, respectively 27). A first attempt could be to find an alternative prove of the Cone Conjecture for the Kummer surface, which would bypass the use of lattice theory coming along with the realm of K3 surfaces, and the description of its Mori cone by [114].

Another question stemming from Part II is the following: are there sufficient criteria for affine quotients $\mathbb{C}^n/G$ by finite non-abelian groups G acting freely in codimension 1 to admit a crepant resolution? In the case that they admit a crepant resolution, how many do they have? In the abelian case, toric geometry provides a powerful tool to bring partial answers to these questions, see e.g. [176], a tool which lacks the non-abelian case. Partial results in the non-abelian case include [127, 12, 54, 13].

It would also be worthwhile to work on finite quotients of abelian varieties containing some pseudo-reflections. For instance, there is a structure theorem for smooth finite quotients of abelian varieties, due to [7, 6], and a more general structure theorem building finite quotients of abelian varieties from Q-abelian finite quotients of abelian varieties and $\mathbb{Q}$-Fano finite quotients of abelian varieties [183]. It seems to us that $\mathbb{Q}$-Fano finite quotients of abelian varieties are worth more investigation, e.g., along the lines of [183, Question 5.4]: are they toric?

Finally, Part IV leaves the classification of Fano varieties X with Picard number one and $\bigwedge^3 T_X$ or $\bigwedge^4 T_X$ strictly nef open. We conjecture that, if X is Fano variety of Picard number one, $\bigwedge^3 T_X$ is ample if and only if X is a del Pezzo manifold.

We hope to investigate some of these questions in the future.

CHAPTER 2

PRELIMINARIES

2.1 Notations and conventions. We work over the field of complex numbers $\mathbb{C}$. Varieties (and in particular curves) are always assumed irreducible and reduced. We use the expressions "smooth projective variety" and "projective manifold" interchangeably. We refer to [81] for scheme theory, [63] for intersection theory, [42] for birational geometry, in particular Mori theory, [121, 122] for positivity notions, [108] for rational curves and their deformations. As regards group theory, we refer to [188, 172], notably for p-group properties and Sylow theory, and to [87] for character theory.

When X is a projective variety, we denote by X_{reg} its smooth locus and by X_{sing} its singular locus. Note that, if X is normal, $X_{\mathrm{sing}} \subset X$ has codimension at least two.

If $\mathcal{E}$ is a coherent sheaf on a quasiprojective variety X, we denote by $\mathcal{E}^*$ its dual sheaf.

Divisors and line bundles. Let X be a normal projective variety. We identify the group $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ of line bundles up to isomorphism over X with the group of Cartier divisors up to linear equivalence on X. Recall that the exponential short exact sequence induces morphisms of abelian groups:

$$H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}).$$

We denote by $\mathrm{Pic}^0(X)$ the kernel of c_1, and by $\mathrm{NS}(X)$ the image of c_1, which we refer to as the *Néron-Severi group* of X. When k is field, we denote by $\mathrm{Pic}(X)_k$ the k-vector space $\mathrm{Pic}(X) \underset{\mathbb{Z}}{\otimes} k$, and by $\mathrm{NS}(X)_k$ the k-vector space $\mathrm{NS}(X) \underset{\mathbb{Z}}{\otimes} k$. We refer to $\mathrm{NS}(X)_{\mathbb{R}}$ as the *Néron-Severi space* of X. It is a theorem [81, Exercise V.1.8(b)] that the Néron-Severi space of a normal projective variety X is finite-dimensional. Its dimension is called the *Picard number* of X, and denoted by $\rho(X)$.

Cycles and intersection theory. Let X be a quasiprojective variety. We denote by $A^k(X)$, respectively $A_k(X)$, the $\mathbb{Z}$-module of cycles of codimension k, respectively of dimension k, up to rational equivalence. A k-cycle is called *effective* if it can be written as a linear combination of subvarieties of X of dimension k, with positive coefficients. Given two cycles C_1 and C_2, we write $C_1 = C_2$, or $C_1 \sim C_2$, if they are rationally equivalent.

In [63, Chapter 2], an intersection product is defined, that to a k-cycle C and a Cartier divisor D, associates a $(k-1)$-cycle, and that satisfies the crucial commutation property

$$D \cdot (D' \cdot C) = D' \cdot (D \cdot C).$$

From a k-cycle and k Cartier divisors, we can therefore define an intersection number through the natural group morphism $A_0(X) \to \mathbb{Z}$.

Given two 1-cycles C_1 and C_2, we say that they are *numerically equivalent* and write $C_1 \equiv C_2$ or $C_1 \underset{\text{num}}{\equiv} C_2$ if, for any Cartier divisor D on X, the intersection numbers $D \cdot C_1$ and $D \cdot C_2$ equate. Given two Cartier divisors D_1 and D_2, we say that they are *numerically equivalent* and write $D_1 \equiv D_2$ or $D_1 \underset{\text{num}}{\equiv} D_2$ if, for any 1-cycle C on X, the intersection numbers $D_1 \cdot C$ and $D_2 \cdot C$ equate. We denote by $N_1(X)_{\mathbb{R}}$ the real vector space generated by classes of 1-cycles, up to numerical equivalence. We denote by $N^1(X)_{\mathbb{R}}$ the real vector space generated by classes of Cartier divisors, up to numerical equivalence. It turns out that $N^1(X)_{\mathbb{R}} \cong \mathrm{NS}(X)_{\mathbb{R}}$.

Chern classes on a scheme. In [63, Chapter 3] is given a construction of Chern classes $c_i(E)$ for a locally free sheaf E on an arbitary scheme X. These Chern classes are defined as operators $c_i(E) : A_*(X) \to A_{*-i}(X)$. One can relate the axiomas given by Grothendieck [78] for Chern classes of locally free sheaves over smooth projective varieties with the properties constituting [63, Theorem 3.2].

Chern classes of a variety. When X is a smooth projective variety, we write $c_i(X) = c_i(T_X)$ for the Chern classes of the tangent bundle of X. When X is a normal projective variety of dimension n, we denote by $c_1(X) = -K_X$ the Weil divisor obtained by taking the Zariski closure of $c_1(X_{\text{reg}})$ in X, using the isomorphism $A_{n-1}(X) \xrightarrow{\sim} A_{n-1}(X_{\text{reg}})$ induced by the exact sequence [63, Proposition 1.8]. When X is smooth in codimension 2, e.g., when it is terminal [111, Corollary 5.18], the square of the first Chern class ${c_1}^2(X)$ and the second Chern class $c_2(X)$ are defined similarly, as the codimension 2 cycles obtained by taking the Zarsiki closure of $c_2(X_{\text{reg}})$ and $c_1(X_{\text{reg}})^2$ respectively. This process is actually more general, see [55, Definition 2.9]. Note that we will carefully distinguish between $c_1(X)^2$, when X is Gorenstein, and ${c_1}^2(X)$, when X is a normal projective variety that is smooth in codimension 2.

Finite morphisms. We will deal with various types of finite maps. Unless otherwise stated, all finite morphisms we speak about are surjective; we may well refer to them as finite coverings.
We say that a finite morphism is *quasiétale* if it is étale in codimension 1. By Zariski purity of the branch locus, a finite morphism $X \to Y$ between normal projective varieties is quasiétale if and only if its branch locus is a closed subscheme of Y_{sing}.
Following [72], we say that a finite morphism of normal varieties $X \to Y$ is *Galois* if it is the quotient map of X by a finite group action. The corresponding group is called the *Galois group* of the morphism.

2.2 Singularities of pairs, and Calabi-Yau pairs. Throughout this thesis, we will encounter various notions of singularities of the minimal model program (MMP). These singularities are defined for pairs (X, Δ).

Definition 2.1. A pair (X, Δ) is the data of a normal projective variety X and a $\mathbb{Q}$-Weil divisor (or for short, $\mathbb{Q}$-*divisor*) Δ that is effective on X such that the Weil divisor $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Notions that are defined with the canonical bundle for varieties, such as being Fano, or being K-trivial, extend to pairs, as the following definition suggests.

Definition 2.2. A pair (X, Δ) is called a *Fano pair*, respectively a *Calabi-Yau pair* if $-(K_X + \Delta)$ is ample, respectively numerically trivial.

To define our MMP-singularities, we first need a few notations.

Definition 2.3. Fix a pair (X, Δ). For any proper birational morphism $\varepsilon : Y \to X$, there is a formula

$$m(K_Y + \varepsilon_*^{-1}\Delta) = \varepsilon^* m(K_X + \Delta) + \sum a(E_i, X, \Delta) m E_i,$$

where the E_i are exceptional divisors in the exceptional locus of ε, and m is an integer such that $m(K_X + \Delta)$ is Cartier. The numbers $a(E_i, X, \Delta)$ are rational and are called the discrepancies of (X, Δ) with respect to E_i. They depend on the divisor E_i, but not on the choice of ε or m.

The minimum of the numbers $a(E_i, X, \Delta)$, running on all exceptional divisors E_i of all proper birational morphisms $Y \to X$, is denoted by $\mathrm{discrep}(X, \Delta)$.

Definition 2.4. If Δ is an $\mathbb{R}$-divisor, we can write $\Delta = \sum_i d_i D_i$ with $d_i \in \mathbb{R}$ and D_i Weil divisors. The round-down of Δ, which we denote by $\lfloor \Delta \rfloor$, is the divisor $\sum_i \lfloor d_i \rfloor D_i$.

Definition 2.5. A pair (X, Δ) is called

$$\begin{cases} \textit{terminal} \\ \textit{canonical} \\ \textit{Kawamata log terminal (klt)} \\ \textit{log canonical (lc)} \end{cases} \text{if } \mathrm{discrep}(X, \Delta) \begin{cases} > 0 \\ \geq 0 \\ > -1 & \text{and } \lfloor \Delta \rfloor = 0 \\ \geq -1 \end{cases}$$

A variety X is called *terminal, canonical, klt* or *lc* if the pair $(X, 0)$ is.

Example 2.6. A smooth variety is terminal.

2.3 Reflexive sheaves.

In this section, we mainly follow the following paper by Hartshorne [82].

Definition 2.7. A coherent sheaf $\mathcal{E}$ on a quasiprojective variety X is called *reflexive* if the canonical sheaf morphism $\mathcal{E} \to \mathcal{E}^{**}$ from $\mathcal{E}$ to its bidual is an isomorphism.

Lemma 2.8. [82, Corollary 1.2] *If $\mathcal{E}$ is a coherent sheaf, then $\mathcal{E}^*$ is reflexive.*

In particular, the tangent sheaf to a quasiprojective variety is a reflexive sheaf. Reflexive sheaves are not so far from locally free sheaves, as the following result specifies.

Lemma 2.9. [82, Corollary 1.4] *Let $\mathcal{E}$ be a reflexive sheaf on a smooth quasiprojective variety X. There is an open set $U \subset X$ whose complement has codimension at least three, such that $\mathcal{E}|_U$ is locally free.*

Definition 2.10. Let X be a quasiprojective variety. An open set $U \subset X$ is called a *big open set* if its complement has codimension at least two in X.

Definition 2.11. A coherent sheaf $\mathcal{E}$ on a quasiprojective variety X is called *normal* if for every open set U in X, for every big open subset V of U, the restriction map $\mathcal{E}(U) \to \mathcal{E}(V)$ is an isomorphism.

Lemma 2.12. [82, Prop.1.6] *Reflexive sheaves on normal quasiprojective varieties are normal.*

Corollary 2.13. *Let $f : \mathcal{E} \to \mathcal{F}$ be a sheaf morphism between two reflexive sheaves on a normal quasiprojective variety X. If there exists a big open set U of X such that $f|_U : \mathcal{E}|_U \to \mathcal{F}|_U$ is an isomorphism, then f is an isomorphism.*

Restriction to a closed subscheme does not in general preserve reflexivity: think of restricting the tangent sheaf to the cone over a quadric to a smooth line through the vertex. This tangent sheaf is reflexive by Lemma 2.8, but its restriction is not locally free, hence not reflexive by Lemma 2.9. However, the following result holds, and is to come handy in Chapter 4.

Lemma 2.14. *Let X be a normal quasiprojective variety, let $\mathcal{E}$ be a reflexive sheaf on X, and let H be a globally generated Cartier divisor on X. For a general member $Y \in |H|$, $\mathcal{E}|_Y$ is reflexive.*

Proof. Let $U := \{(x, Y)|x \in Y\} \subset X \times |H|$ be the universal family associated to our linear system, with $p : U \to X$ and $q : U \to |H|$. the natural projections. As H is globally generated, the fibers of p are hyperplanes in $|H|$. Hence, the Hilbert polynomial of a fiber is constant, so p is flat by [81, Theorem III.9.9]. So by [82, Proposition 1.8], $p^*\mathcal{E}$ is reflexive. Now by [50, Theorem 12.2.1], for $Y \in |H|$ general, $p^*\mathcal{E}|_{q^{-1}(Y)} = \mathcal{E}|_Y$ is reflexive. □

Let $\mathcal{E}$ be a reflexive sheaf on a variety X. Recall the reflexivization functor $\mathcal{F} \mapsto \mathcal{F}^{**}$ enables to perform general algebraic operations in the category of reflexive sheaves.

Notably, we will denote by

- $S^{[m]}\mathcal{E}$ the reflexivization of the m-th symmetric power of $\mathcal{E}$ (for $m \in \mathbb{N}$),
- $\nu^{[*]}\mathcal{E}$ the reflexivization of the pullback of $\mathcal{E}$ (for $\nu : Y \to X$ a surjective morphism).

The reflexive pullback behaves quite differently than the standard pullback, as we will see in Remark 2.25. However,

Lemma 2.15. *Let $p : X \to Y$ be a proper dominant morphism between normal projective varieties, with all fibers of the same dimension. The functor $p^{[*]}$ from the category of reflexive sheaves on Y to that of reflexive sheaves on X is left-exact.*

Proof. Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of reflexive sheaves on Y, and denote by $Y_{\text{sing}} \subset Z \subset Y$ a closed subscheme of codimension at least 2 such that our reflexive sheaves are locally-free on $Y \setminus Z$. Reflexive pullback a priori only gives morphisms

$$p^{[*]}\mathcal{E} \to p^{[*]}\mathcal{F} \to p^{[*]}\mathcal{G},$$

whose composition is zero. By [185, Lemma 31.12.7], the kernel K of the morphism $p^{[*]}\mathcal{F} \to p^{[*]}\mathcal{G}$ is reflexive. There is a natural morphism from $p^{[*]}\mathcal{E}$ to the kernel K, which restricts to an isomorphism over $X \setminus p^{-1}(Z)$ (as pullback is exact for locally free sheaves). As both sheaves are reflexive and $p^{-1}(Z)$ has codimension 2, they are isomorphic over all X. □

We could not produce any counterexample to the right-exactness of this functor. Push-forward is better behaved.

Lemma 2.16. *Let $p : X \to Y$ be a proper dominant morphism between normal projective varieties, with all fibers of the same dimension. Then if $\mathcal{E}$ is a reflexive sheaf on X, $p_*\mathcal{E}$ is still reflexive.*

Definition 2.17. Let X be a normal projective variety with a finite group G acting on it. Denote by ϕ the map $G \to \mathrm{Aut}(X)$. A coherent sheaf $\mathcal{E}$ on X is called a *G-sheaf* if for every open set $U \subset X$, there is a pushforward map $\mathcal{E}(U) \to \mathcal{E}(\phi(g)(U))$ that commutes with restrictions and glueings.

Example 2.18. Let $p : X \to Y$ be a finite Galois morphism between normal projective varieties, with Galois group G. Then for any coherent sheaf $\mathcal{E}$ on Y, $p^*\mathcal{E}$ and thus $p^{[*]}\mathcal{E}$ are G-sheaves.

Lemma 2.19. [71, Lemmas B3, B4] *Let $p : X \to Y$ be a finite Galois morphism between normal projective varieties, with Galois group G. Then the functor $\mathcal{E} \mapsto (p_*\mathcal{E})^G$ associating a sheaf on Y to a G-sheaf on X is exact, and sends reflexive G-sheaves to reflexive sheaves.*

Corollary 2.20. *Let $p : X \to Y$ be a finite Galois morphism between normal projective varieties, with Galois group G. Let $\mathcal{E}$ be a reflexive sheaf on Y. Then $(p_*p^{[*]}\mathcal{E})^G \cong \mathcal{E}$.*

Proof. As they both are reflexive sheaves, and as there is a canonical map $\mathcal{E} \to (p_*p^{[*]}\mathcal{E})^G$, it is enough to show that this map is an isomorphism on the locus where $\mathcal{E}$ is locally free, and that is what a local computation shows. $\square$

2.4 Positivity notions for reflexive sheaves.

We first recall the standard positivity notions for line bundles.

Positivity notions for line bundles. A line bundle L on a projective variety X is

- *globally generated* if the sheaf morphism $H^0(X, L) \otimes \mathcal{O}_X \to L$ is surjective;
- *very ample* if it is globally generated and if the induced morphism $X \to \mathbb{P}(H^0(X, \mathcal{O}_X(mL))^*)$ is an embedding;
- *ample* if it has a very ample positive multiple;
- *strictly nef* if for every curve C in X, the intersection number $L \cdot C$ is positive;
- *nef* if for every curve C in X, the intersection number $L \cdot C$ is non-negative;
- *effective* if it has a non-zero section;
- *pseudoeffective* if there is a sequence $L_n \in \mathbb{N}^1(X)_{\mathbb{R}}$ such that for each n, the $\mathbb{R}$-divisor L_n is a positive linear combination of effective line bundles, and the sequence $(L_n)_{n\in\mathbb{N}}$ converges to L in $\mathbb{N}^1(X)_{\mathbb{R}}$ endowed with the natural Euclidean topology.

These definitions motivate the introduction of the convex cones of ample, nef, pseudoeffective, and effective divisors in $\mathrm{NS}(X)_{\mathbb{R}}$,

$$\mathrm{Amp}(X) \subset \mathrm{Nef}(X) \subset \overline{\mathrm{Eff}}(X) \supset \mathrm{Eff}(X).$$

In Part III, we also study the following convex cones: the *nef effective cone*

$$\mathrm{Nef}^e(X) := \mathrm{Nef}(X) \cap \mathrm{Eff}(X),$$

and the *positive nef cone* $\mathrm{Nef}^+(X)$, which is the convex hull of $\mathrm{Nef}(X) \cap \mathrm{NS}(X)_{\mathbb{Q}}$ in $\mathrm{NS}(X)_{\mathbb{R}}$. These two cones play a central role in the Kawamata–Morrison Cone Conjecture.

Nefness for coherent sheaves. Let us recall that a coherent sheaf $\mathcal{E}$ on a normal variety X has a projectivization $\mathbb{P}(\mathcal{E})$ with a canonical, so-called tautological, line bundle ζ on it and a natural morphism $p : \mathbb{P}(\mathcal{E}) \to X$ with a natural sheaf quotient map: $p^*\mathcal{E} \twoheadrightarrow \zeta$. An account on this set-up is given in [49, Chapter 4]. We recall the universal property of this construction: for any scheme C with a morphism $q : C \to X$, to give a morphism $\nu : C \to \mathbb{P}(\mathcal{E})$ commuting with the projections to X is equivalent to giving a line bundle L over C together with a sheaf surjection $q^*\mathcal{E} \twoheadrightarrow L$.

Projectivizations are standardly used for generalizing positivity notions of line bundles to coherent sheaves, as follows.

Definition 2.21. Let $\mathcal{E}$ be a coherent sheaf on a normal variety X. It is called *ample, strictly nef, or nef* if the tautological line bundle ζ on $\mathbb{P}(\mathcal{E})$ is ample, strictly nef, or nef respectively.

Remark 2.22. This coincides with [120, Definition 6.1.1] when the sheaf $\mathcal{E}$ is locally free. Note that for a torsion-free coherent sheaf $\mathcal{E}$, the scheme $\mathbb{P}(\mathcal{E})$ may well have several irreducible components. Somehow, several of these components may be relevant for studying the positivity of $\mathcal{E}$: not only the mere one which is dominant onto X, but also components which may be contracted to a proper non-zero dimensional locus of X. Such components don't exist for a reflexive sheaf on a normal projective surface: so in this case, nefness is easier to study.

Proposition 2.23. *We have the following properties:*

- *if $Y \subset X$ is a normal subvariety, and $\mathcal{E}$ is a nef coherent sheaf on X, then $\mathcal{E}|_Y$ is nef [119, Proposition 7];*
- *conversely, nefness of a coherent sheaf $\mathcal{E}$ is enough to be checked on all curves of X [131, Proposition 3.2];*
- *if $f : Y \to X$ is a finite dominant morphism of normal varieties and $\mathcal{E}$ is a coherent sheaf on X, $\mathcal{E}$ is nef if and only if $f^*\mathcal{E}$ is;*
- *if $f : Y \to X$ is a proper birational morphism resolving the singularities of a normal variety X and $\mathcal{E}$ is a coherent sheaf on X such that $f^*\mathcal{E}$ is nef, then $\mathcal{E}$ is nef.*

Proof. Let $\mathcal{E}$ be a coherent sheaf on a normal variety X and let $f : Y \to X$ be a proper dominant morphism. By [49, 4.1.3.1], we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(f^*\mathcal{E}) & \xrightarrow{f'} & \mathbb{P}(\mathcal{E}) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

with a tautological sheaf compatibility $\zeta_{f^*\mathcal{E}} = f'^*\zeta_{\mathcal{E}}$. Let C be a curve in $\mathbb{P}(f^*\mathcal{E})$. Then

$$\zeta_{f^*\mathcal{E}} \cdot C = \zeta_{\mathcal{E}} \cdot f'_*C,$$

which is positive if $\mathcal{E}$ is nef. So if $\mathcal{E}$ is nef, $f^*\mathcal{E}$ is nef.

Assume moreover that f is finite, and let C be a curve in $\mathbb{P}(E)$. Then

$$\zeta_{\mathcal{E}} \cdot C = \frac{1}{\deg f}\zeta_{f^*\mathcal{E}} \cdot f'^{-1}(C),$$

which is positive if $f^*\mathcal{E}$ is nef. So if $f^*\mathcal{E}$ is nef, $\mathcal{E}$ is nef. □

Lemma 2.24. *Let X be a normal projective variety, and Z be a closed proper subscheme of X that is locally generated by a regular sequence. Then the ideal sheaf $\mathcal{I}_Z$ is not nef.*

Proof. Note that the blow-up $\mathrm{Bl}_Z(X)$ and the projectivization $\mathbb{P}(\mathcal{I}_Z)$ coincide by [85, 3.10]. Under this identification, the tautological sheaf on $\mathbb{P}(\mathcal{I}_Z)$ corresponds to the ideal sheaf of the exceptional divisor $\mathcal{O}_{\mathrm{Bl}_Z(X)}(-E)$, by [185, Lemma 31.32.4]. Intersecting with the strict transform of a curve that passes through a point of Z but is not contained in Z, we see that it is not nef. □

Remark 2.25. Interestingly enough, the *reflexive* pullback of a non-nef reflexive sheaf by a finite morphism may be nef, as shows the following example. Let X be the finite quotient of an abelian surface A by the involution $i = -\mathrm{id}_A$. Since $p : A \to X$ is a finite quasiétale cover, the reflexive sheaves $p^{[*]}\mathcal{T}_X$ and $\mathcal{T}_A$ are the same, in particular are nef.

We are going to prove that $\mathcal{T}_X$ itself is not nef. By Corollary 2.20, we have $\mathcal{T}_X = p_*(\mathcal{O}_A \oplus \mathcal{O}_A)^{\langle -\mathrm{id}_2\rangle}$. As the action is diagonal, $\mathcal{T}_X = (p_*\mathcal{O}_A)^{\langle -\mathrm{id}_2\rangle} \oplus (p_*\mathcal{O}_A)^{\langle -\mathrm{id}_2\rangle}$. Denote by $\mathcal{F}$ the sheaf $(p_*\mathcal{O}_A)^{\langle -\mathrm{id}_2\rangle}$. We compute locally: let $V \subset X, U = p^{-1}(V) \subset A$ be affine open sets with local coordinates $(x, y) \in \mathbb{C}^2 \simeq U$ so that $p|_U$ ramifies only at $(0, 0)$. The quotient map $p : U \to V$ rewrites:

$$\begin{aligned} \mathbb{C}[u, v, w]/(uv - w^2) \cong \mathcal{O}_X(V) &\to \mathbb{C}[x, y] \cong \mathcal{O}_A(U) \\ u, v, w &\mapsto x^2, y^2, xy, \end{aligned}$$

so its image $\mathbb{C}[x^2, y^2, xy]$ identifies with the local ring $\mathcal{O}_X(V)$. Hence,

$$\mathcal{F}(V) \simeq \{f \in \mathbb{C}[x, y] \mid \forall\, x, y,\ f(x, y) = -f(-x, -y)\} = x\,\mathbb{C}[x^2, y^2, xy] \oplus y\,\mathbb{C}[x^2, y^2, xy],$$

so that $\mathcal{F}^{\otimes 2}(V) \simeq u\,\mathcal{O}_X(V) \oplus v\,\mathcal{O}_X(V) \oplus w\,\mathcal{O}_X(V) = \mathcal{I}_{\mathrm{Sing}(X)}(V)$. This isomorphism is actually global:

$$\mathcal{F}^{\otimes 2} \cong \mathcal{I}_{\mathrm{Sing}(X)}.$$

Ideal sheaves are not nef by Lemma 2.24, so $\mathcal{F}^{\otimes 2}$ is not nef, so by [119, Proposition 2], $\mathcal{F}$ is not nef. □

Pseudoeffectivity for reflexive sheaves. For reflexive sheaves, pseudoeffectivity is standardly defined not through projectivization: the main reason for that is that the pushforwards of powers of the tautological sheaf on $\mathbb{P}(E)$ are the symmetric powers of $\mathcal{E}$, whereas we are interested in the potential sections of the reflexivized symmetric powers of $\mathcal{E}$.

Definition 2.26. Let $\mathcal{E}$ be a reflexive sheaf on a normal projective variety X. It is considered *pseudoeffective* if there is an ample Cartier divisor H on X such that for all $c > 0$, there are integers i, j such that $i > cj > 0$ and

$$h^0(X, \mathrm{Sym}^{[i]}\mathcal{E} \otimes \mathcal{O}_X(jH)) \neq 0.$$

Note that if $\mathcal{E}$ is pseudoeffective with respect to a certain ample Cartier divisor H, then it is actually pseudoeffective with respect to any ample Cartier divisor [86, Lemma 2.3]. Generalizing this definition to any coherent sheaf is not obvious [85].

Example 2.27. The sheaf $\mathcal{T}_X$ in Remark 2.25 is pseudoeffective, as $\mathcal{T}_X = \mathcal{F} \oplus \mathcal{F}$ with $S^{[2]}\mathcal{F} \cong \mathcal{O}_X$.

Definition 2.28. Let $\mathcal{E}$ be a reflexive sheaf on a normal projective variety X. Denote by $\mathbb{P}'(\mathcal{E})$ the normalization of the unique dominant component of $\mathbb{P}(\mathcal{E})$ onto X. Let P be a resolution of $\mathbb{P}'(\mathcal{E})$, such that the birational morphism $r : P \to \mathbb{P}'(\mathcal{E})$ over X is an isomorphism precisely over the open locus $X_0 \subset X_{\mathrm{reg}}$ where $\mathcal{E}$ is locally-free.

Denoting by π the morphism $P \to \mathbb{P}(\mathcal{E})$ and by $\mathcal{O}_P(1)$ the pullback of the tautological bundle of $\mathbb{P}(\mathcal{E})$ by π, [148, V.3.23] asserts that one can choose (often not uniquely) an effective divisor Λ supported in the exceptional locus of r such that

$$\zeta := \mathcal{O}_P(1) \otimes \mathcal{O}_P(\Lambda)$$

satisfies $\pi_*\zeta^{\otimes m} \simeq S^{[m]}\mathcal{E}$ for all $m \in \mathbb{N}$. Such ζ is called a *tautological class* of $\mathcal{E}$.

As said in [86, Lemma 2.3], with the same notations as previously, ζ is pseudoeffective on P if and only if $\mathcal{E}$ is pseudoeffective as a reflexive sheaf.

Proposition 2.29. *Let X be a normal projective variety, H an ample $\mathbb{Q}$-Cartier divisor, $\mathcal{E}$ a pseudoeffective reflexive sheaf on X. Then for m big and divisible enough, for a general element $D \in |mH|$, the sheaf $\mathcal{E}|_D$ is reflexive and pseudoeffective.*

Proof. Let $U \subset X_{\mathrm{reg}}$ be a big open set on which $\mathcal{E}$ is locally-free. For m big and divisible enough and for a general element D in $|mH|$, $U \cap D$ is a big open set of D. By Bertini theorem and by Lemma 2.14, we can assume D is a normal subvariety of X and $\mathcal{E}|_D$ is reflexive.

Let us fix a $c > 0$ and take i, j integers such that $i > cj > 0$ and $h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) > 0$. Up to taking a smaller j (possibly negative), we can assume that

$$h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X((j-m)H)) = 0.$$

By normality of reflexive sheaves,

$$\begin{aligned} h^0(D, S^{[i]}(\mathcal{E}|_D) \otimes \mathcal{O}_D(jH)) &= h^0(U \cap D, S^i(\mathcal{E}|_{U\cap D}) \otimes \mathcal{O}_{U\cap D}(jH)) \\ &\geq h^0(U, S^i(\mathcal{E}|_U) \otimes \mathcal{O}_U(jH)) \\ &\quad - h^0(U, S^i(\mathcal{E}|_U) \otimes \mathcal{O}_U((j-m)H)) \\ &= h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) \\ &\quad - h^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X((j-m)H)) \\ &> 0, \end{aligned}$$

where the second equality comes from tensoring by $S^i(\mathcal{E}|_U) \otimes \mathcal{O}_U(jH)$ and going to cohomology in the following exact sequence:

$$0 \to \mathcal{O}_U(-mH) \to \mathcal{O}_U \to \mathcal{O}_{U\cap D} \to 0.$$

$\square$

This result can be slightly generalized.

Proposition 2.30. *Let X be a normal projective variety, L a globally generated line bundle, $\mathcal{E}$ a pseudoeffective reflexive sheaf on X. Then for a very general element $D \in |L|$, the sheaf $\mathcal{E}|_D$ is reflexive and pseudoeffective.*

The idea of this proof stems from Lemma 2.14. By passing through the universal family U for the linear system $|L|$, it is enough to establish the following two results.

Lemma 2.31. *Let X and U be normal projective varieties, and let $p : U \to X$ be a flat morphism, that is surjective and has connected fibers. If $\mathcal{E}$ is a reflexive pseudoeffective sheaf on X, then $p^*\mathcal{E}$ is a reflexive pseudoeffective sheaf on U.*

Proof. By [82, Proposition 1.8], $p^*\mathcal{E}$ is reflexive, and so is $p^*S^{[i]}\mathcal{E}$ for any $i > 0$. In particular, by normality, $p^*S^{[i]}\mathcal{E} = S^{[i]}p^*\mathcal{E}$ for any $i > 0$. Fixing an ample Cartier divisor H on X, and an effective ample Cartier divisor A on U, we set $\tilde{H} := p^*H + A$, an ample Cartier divisor on U. We then fix $c > 0$, and note that for any $i > cj > 0$, it holds

$$\begin{aligned} h^0(U, S^{[i]}p^*\mathcal{E} \otimes \mathcal{O}_U(j\tilde{H})) &\geq h^0(U, p^*(S^{[i]}\mathcal{E} \otimes \mathcal{O}_U(jH))) \\ &= h^0(X, S^{[i]}\mathcal{E} \otimes \mathcal{O}_U(jH)) \end{aligned}$$

where we use the fact that p is flat, that it satisfies $p_*\mathcal{O}_U = \mathcal{O}_X$, and the projection formula. Hence, $p^*\mathcal{E}$ is pseudoeffective. □

Lemma 2.32. *Let U and Y be normal projective varieties, let $q : U \to Y$ be a proper morphism that is surjective and has connected fibers, and whose general fiber is irreducible. If $\mathcal{E}$ is a reflexive pseudoeffective sheaf on U, and y is a very general point in Y, then $\mathcal{E}|_{q^{-1}(y)}$ is a reflexive pseudoeffective sheaf on Y.*

Proof. By [50, Theorem 12.2.1], for $y \in Y$ general, $\mathcal{E}|_{q^{-1}(y)}$ is reflexive, and so is $\left(S^{[i]}\mathcal{E}\right)|_{q^{-1}(y)}$, which coincides with $S^{[i]}\left(\mathcal{E}|_{q^{-1}(y)}\right)$ by normality. Let us fix an ample divisor H on U. Note that for any positive integer n, there are $i > nj > 0$ such that $S^{[i]}\mathcal{E} \otimes \mathcal{O}_U(jH)$ has a non-zero section s_n. The set $\{y \in Y \mid s_n|_{q^{-1}(y)} = 0\}$ is Zariski-closed in Y, and so for a general y, the reflexive sheaf $\left(S^{[i]}\mathcal{E} \otimes \mathcal{O}_U(jH)\right)|_{q^{-1}(y)}$ admits a non-zero section. Hence, for a very general point $y \in Y$, $\mathcal{E}_{q^{-1}(y)}$ is pseudoeffective. □

We will use several times the following result [85, Lemma 3.15].

Proposition 2.33. *Let $\mathcal{E}$ be a reflexive sheaf on a normal projective variety X, and $f : Y \to X$ be a finite dominant morphism of normal projective varieties. Then $\mathcal{E}$ is pseudoeffective if and only if $f^{[*]}\mathcal{E}$ is.*

Definition 2.34. Let D be a $\mathbb{Q}$-Cartier divisor on a normal projective variety X. We define its *stable base locus*

$$\mathbb{B}(D) := \bigcap_{m \in M} \mathrm{Bs}(mD),$$

where $M \subset \mathbb{N}$ is the set of all m such that mD is Cartier and $\mathrm{Bs}(mD)$ is the base locus of the linear system $|mD|$.
We then define its *restricted base locus*

$$B_-(D) := \bigcup_{n \in \mathbb{N}^*} \mathbb{B}\left(D + \frac{1}{n}A\right),$$

where A is an arbitrary ample divisor (the locus does not depend on the choice of A by [57, Proposition 1.19]).

Of course, a nef $\mathbb{Q}$-divisor has an empty restricted base locus. To that extent, the restricted base locus measures the non-nefness of a pseudoeffective line bundle. However, not all curves of a restricted base locus $B_-(D)$ must be D-non-positive, even in the simpler case where D is a pseudoeffective line bundle on a smooth surface and $B_-(D)$ is the negative part of its Zariski decomposition.

2.5 Stability for a torsion-free sheaf.

Definition 2.35. Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal projective variety X of dimension n. For an ample $\mathbb{Q}$-Cartier divisor H on X, $\mathcal{E}$ is said to be *H-semistable*, if for some integer m such that mH is Cartier, for all torsion-free non-zero proper subsheaf $\mathcal{F} \subset \mathcal{E}$, it holds

$$\frac{c_1(\mathcal{F}) \cdot (mH)^{n-1}}{\mathrm{rk}(\mathcal{F})} \leq \frac{c_1(\mathcal{E}) \cdot (mH)^{n-1}}{\mathrm{rk}(\mathcal{E})}.$$

If the inequality is strict for all such $\mathcal{F}$, we say that $\mathcal{E}$ is *H-stable*.

Let $\mathcal{E}$ be a torsion-free coherent sheaf on a normal projective variety X. For any ample $\mathbb{Q}$-Cartier divisor H on X, and for some m big and divisible enough, $n-1$ general members of $|mH|$ cut out a smooth curve C on which $\mathcal{E}|_C$ is still torsion free by [50, Theorem 12.2.1], hence locally free. A generalization of a well-known Mehta–Ramanathan result says that stability behaves well under some well-chosen restrictions; we recall it as it is stated in [86, Lemma 2.11]. This result shows that, for a locally-free sheaf $\mathcal{E}$, testing stability on the locally-free proper subsheaves is the same as testing stability on the torsion free coherent proper subsheaves: Our notion of (semi)stability is no different from the notion usually defined for locally-free sheaves.

Lemma 2.36. *Let X be a normal projective variety of dimension n, and H an ample Cartier divisor on X. Let $\mathcal{E}$ be a torsion-free coherent sheaf on X, that is stable with respect to H. Then there is m_0 such that, for all $m \geq m_0$, and for $D_1, \dots, D_k$ general elements of $|mH|$ with $k \in [\![1, n-1]\!]$, if we denote by Y the complete intersection $D_1 \cap \ldots \cap D_k$, $\mathcal{E}|_Y$ is stable with respect to $H|_Y$.*

Remark 2.37. Note that the converse is clearly true.

Stability a priori weakens through finite Galois reflexive pullbacks:

Lemma 2.38. *Let $p : Y \to X$ be a finite Galois cover of normal projective varieties of dimension n, G its Galois group, H an ample $\mathbb{Q}$-Cartier on X, $\mathcal{E}$ be a reflexive sheaf on X. Let $\mathcal{F} := p^{[*]}\mathcal{E}$. Then, if $\mathcal{E}$ is H-stable, $\mathcal{F}$ is p^*H-semistable.*

Proof. Suppose that $\mathcal{E}$ is H-stable. By Lemma 2.36, on a smooth curve C cut out by $n-1$ very general elements of the linear system defined by a suitable multiple of H, the now locally-free sheaf $\mathcal{E}|_C$ is still $H|_C$-stable. In particular, [120, Lemma 6.4.12] applies; so the pullback sheaf $\mathcal{F}|_{p^{-1}(C)}$ is $p^*H|_C$-semistable. Hence, $\mathcal{F}$ is H-semistable. □

Note that positivity and stability of a zero-slope locally-free sheaf are related by Miyaoka's result [140], [120, Proposition 6.4.11]:

Proposition 2.39. *Let $\mathcal{E}$ be a vector bundle on a smooth curve C. If $\mathcal{E}$ is semistable and $c_1(\mathcal{E}) = 0$, then $\mathcal{E}$ is nef.*

More subtle than the mere stability of $\mathcal{E}$ is the stability of $\mathcal{E}$ and some of its symmetric powers. A first issue is that the torsion freeness of $\mathcal{E}$, an assumption that is crucial to make sense of stability in our definition, does not imply torsion freeness of the symmetric powers of $\mathcal{E}$. Nevertheless, we provide a few results.

Lemma 2.40. *Let $\mathcal{E}$ be a locally free sheaf on a normal projective variety X of dimension n. Suppose that for some positive integer r, the symmetric power $S^r\mathcal{E}$ is stable. Then $\mathcal{E}$ is stable as well.*

Proof. Let $\mathcal{F}$ be a locally-free non-zero subsheaf of $\mathcal{E}$. Then $S^r\mathcal{F}$ is a locally free non-zero subsheaf of $S^r\mathcal{E}$, and we can compute its slope with respect to an ample Cartier divisor H on X:

$$\frac{c_1(S^r\mathcal{F})\cdot H^{n-1}}{\mathrm{rk}(S^r\mathcal{F})} = r\frac{c_1(\mathcal{F})\cdot H^{n-1}}{\mathrm{rk}(\mathcal{F})}.$$

If $\mathcal{F}$ is proper in $\mathcal{E}$, then $S^r\mathcal{F}$ is proper in $S^r\mathcal{E}$; so the stability of $S^r\mathcal{E}$ implies that of $\mathcal{E}$. □

Corollary 2.41. *Let $\mathcal{E}$ be a reflexive sheaf on a normal projective variety X of dimension n. Suppose that for some positive integer r, the reflexivized symmetric power $S^{[r]}\mathcal{E}$ is stable. Then $\mathcal{E}$ is stable as well.*

Proof. It follows from Lemmas 2.36, 2.40, and from the fact that a reflexive sheaf on a smooth curve is locally-free. □

Remark 2.42. We recall an interesting fact stated in [9, Corollary 6, following remark]. If $\mathcal{E}$ is a locally-free stable sheaf on a smooth projective variety X, then the following are equivalent

- $S^r\mathcal{E}$ is stable for some $r \geq 6$;
- $S^r\mathcal{E}$ is stable for any $r \geq 6$.

Whether or not the stability of all $S^{[r]}\mathcal{E}$ for $l \in \mathbb{N}$ could boil down to the stability of some $S^{[r]}\mathcal{E}$ for a finite amount of r's remains an open question.

Nevertheless, this remark allows us to rewrite the result [86, Proposition 1.3] in the following way.

Lemma 2.43. *Let $\mathcal{E}$ be a locally-free sheaf on a smooth curve C. Assume that for some $r \geq 6$, $S^r\mathcal{E}$ is stable, and that $c_1(\mathcal{E}) = 0$. Denoting by ζ the tautological bundle on $\mathbb{P}(\mathcal{E})$, ζ is nef and satisfies:*

$$\zeta^{\dim Z}\cdot Z > 0,$$

for any closed proper subvariety $Z \subset \mathbb{P}(\mathcal{E})$.

We can not emphasize enough that the reflexive pullback $p^{[*]}\mathcal{E}$ of a H-stable reflexive sheaf $\mathcal{E}$ by a finite dominant morphism p is merely p^*H-semistable and a priori not stable, let alone his reflexive symmetric powers. However, the conclusive property of Lemma 2.43 is preserved by reflexive pullback.

Remark 2.44. Let $\mathcal{E}$ be a reflexive sheaf on a normal projective variety X, and $C \subset X$ a smooth curve such that $\mathcal{E}$ is locally-free in an analytical neighborhood of C, such that the tautological bundle ζ on $\mathbb{P}(\mathcal{E}|_C)$ is nef and

$$\zeta^{\dim Z} \cdot Z > 0$$

holds for any closed proper subvariety $Z \subset \mathbb{P}(\mathcal{E}|_C)$.

Let $p : \hat{X} \to X$ be a finite dominant morphism, where $\hat{X}$ is a normal projective variety. Denote $\hat{C} := p^{-1}(C)$, $\hat{\mathcal{E}} := p^{[*]}\mathcal{E}$ and $\hat{\zeta}$ the tautological bundle of $\mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}})$. If we have that $p^*(\mathcal{E}|_C) = \hat{\mathcal{E}}|_{\hat{C}}$, then the following diagram is Cartesian with tautological compatibility $\hat{\zeta} = q^*\zeta$.

$$\begin{tikzcd} \mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}}) \arrow[r, "q"] \arrow[d, "\hat{\pi}"] & \mathbb{P}(\mathcal{E}|_C) \arrow[d, "\pi"] \\ \hat{C} \arrow[r, "p"] & C \end{tikzcd}$$

Hence, $\hat{\zeta}$ is nef and satisfies, for any closed proper subvariety $Z \subset \mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}})$:

$$\hat{\zeta}^{\dim Z} \cdot Z > 0.$$

Remark 2.45. The case in which this remark will be relevant for us is when X is a normal projective surface with an ample $\mathbb{Q}$-Cartier divisor H, C is a smooth curve arising as a very general element of $|mH|$, for m big and divisible enough, and $p : \hat{X} \to X$ is the morphism constructed in Section 2.6, so that $\hat{\mathcal{E}}$ is locally-free. In this set-up, [75, Proposition 3.11.1] grants the additional assumption $p^*(\mathcal{E}|_C) = \hat{\mathcal{E}}|_{\hat{C}}$.

2.6 Constructions and properties of orbifold Chern classes. Here we recall a standard construction for orbifold first and second Chern classes of a reflexive sheaf $\mathcal{E}$ on a normal projective variety X, whose singularities in codimension 2 are all quotient singularities. Note that normal projective klt varieties fall into this framework by the classical result [168, Corollary 1.14], [71, Proposition 9.4]. References for this matter include [147, 133, 75, 76], [106, Chapter 10, 11].

We first make plain what we mean by quotient singularities.

Definition 2.46. Let X be a normal quasiprojective variety. We say that X has a *quasiétale Q-structure* if there is a finite collection of quasiprojective varieties $(V_\alpha)_{\alpha \in A}$ together with morphisms

$$V_\alpha \overset{p_\alpha}{\to} V_\alpha/G_\alpha \overset{p'_\alpha}{\to} X$$

such that V_α is smooth, p'_α is étale, G_α is finite and acts faithfully freely in codimension 1 (so that p_α is quasiétale), and the union $\bigcup_{\alpha \in A} p'_\alpha(p_\alpha(V_\alpha))$ covers X.

Lemma 2.47. [147, Section 2, p.277] *Let X be a normal quasiprojective variety equipped with a quasiétale Q-structure $(V_\alpha, G_\alpha, p'_\alpha)_{\alpha \in A}$. Then there is a normal finite Galois cover $\hat{X}$ of X with group G such that for each α, we have a commutative diagram*

$$\begin{tikzcd} \hat{V}_\alpha \arrow[r, "\hat{p}_\alpha"] \arrow[d, "i_\alpha"] & V_\alpha \arrow[r, "p_\alpha"] & V_\alpha/G_\alpha \arrow[d, "p'_\alpha"] \\ \hat{X} \arrow[rr, "p"] & & X \end{tikzcd}$$

where i_α is the inclusion of an open subvariety, $\hat{p}_\alpha$ is Galois of group $H_\alpha \triangleleft G$, and p_α is Galois of group $G/H_\alpha \simeq G_\alpha$.

Let X be a normal projective variety, whose singularities in codimension 2 are all quotient singularities. Then X contains a normal quasiprojective subvariety Y with $\operatorname{codim}_X(X \setminus Y) \geq 3$ that admits a quasiétale Q-structure, by [75, Proposition 3.10]. Let us call the whole data $(X, Y, \hat{Y}, p)$ an *unfolding* of X. Note that if $(X, Y, \hat{Y}, p)$ is an unfolding of a surface X, then $Y = X$.

Now let $\mathcal{E}$ be a reflexive sheaf on X, and let us construct a sheaf $\hat{\mathcal{E}}$ that is locally free on an unfolding $\hat{Y}$ of X. First note that $\hat{Y}$ is normal, hence Cohen-Macaulay in codimension 2. So, up to removing from Y a finite union of subvarieties of codimension at least 3 in X, we can assume that $\hat{Y}$ is Cohen-Macaulay. Let $\hat{\mathcal{E}}$ denote the sheaf obtained from gluing together the $(\hat{p}_\alpha^* {p_\alpha}^{[*]} {p'_\alpha}^* \mathcal{E})_{\alpha \in A}$. Since $\hat{Y}$ is Cohen-Macaulay, the morphisms $\hat{p}_\alpha$ are flat [75, Remark 3.7], hence $\hat{\mathcal{E}}$ is reflexive, hence it coincides with $p^{[*]}\mathcal{E}$. On the other hand, as V_α is smooth, ${p_\alpha}^{[*]} {p'_\alpha}^* \mathcal{E}$ is locally free in codimension 2, hence $\hat{\mathcal{E}}$ is locally free in codimension 2. Up to replacing Y by a smaller unfolding, we thus make sure that the reflexive sheaf $\hat{\mathcal{E}} = p^{[*]}\mathcal{E}$ is locally free on $\hat{Y}$.

We are going to use $\hat{\mathcal{E}}$ to define the Chern classes of $\mathcal{E}$. We define a first, a squared first and a second orbifold Chern class of $\mathcal{E}$ as multilinear forms on $\mathrm{NS}(X)^{n-1}$, respectively on $\mathrm{NS}(X)^{n-2}$ by:

$$
\begin{aligned}
\hat{c}_1(\mathcal{E}) \cdot H_1 \cdots H_{n-1} &= \frac{1}{m^{n-1} \cdot |G|} c_1(\hat{\mathcal{E}}) \cdot p^*(mH_1) \cdots p^*(mH_{n-1}), \\
\hat{c_1}^2(\mathcal{E}) \cdot H_1 \cdots H_{n-2} &= \frac{1}{m^{n-2} \cdot |G|} c_1(\hat{\mathcal{E}})^2 \cdot p^*(mH_1) \cdots p^*(mH_{n-2}), \\
\hat{c}_2(\mathcal{E}) \cdot H_1 \cdots H_{n-2} &= \frac{1}{m^{n-2} \cdot |G|} c_2(\hat{\mathcal{E}}) \cdot p^*(mH_1) \cdots p^*(mH_{n-2}),
\end{aligned}
$$

where $H_1, \ldots, H_{n-1}$ are ample $\mathbb{Q}$-classes, and m is big and divisible enough that general elements of $p^*(mH_1), \ldots, p^*(mH_{n-1})$ cut out a complete intersection smooth curve in $\hat{Y}$ and general elements of $p^*(mH_1), \ldots, p^*(mH_{n-2})$ a complete intersection normal surface in $\hat{Y}$.

As stated in [75, Theorem 3.13.2], these orbifold Chern classes are compatible with general restrictions, and so is the unfolding construction [75, Proposition 3.11].

If X is a normal klt variety, we denote by $\hat{c}_1(X), \hat{c_1}^2(X), \hat{c}_2(X)$ the corresponding Chern classes for the reflexive sheaf $\mathcal{T}_X$.

Example 2.48. Let X be a smooth projective variety of dimension n, G be a finite group acting on X freely in codimension 1. Then $(X/G, X/G, X, p)$ is an unfolding of X/G, and $\mathcal{T}_X = p^{[*]}\mathcal{T}_{X/G}$, so

$$
\hat{c}_2(X/G) \cdot H_1 \cdots H_{n-2} = c_2(X) \cdot p^* H_1 \cdots p^* H_{n-2}.
$$

for any ample divisors $H_1, \cdot H_{n-2}$ on X/G. In particular, if Z is a finite quasiétale quotient of a torus, then $\hat{c}_2(Z) \cdot H_1 \cdots H_{n-2} = 0$ for all ample divisors $H_1, \ldots, H_{n-2}$. More is to be said about this example in Section 2.7.

Definition 2.49. Let $x \in X$ be a normal singularity. We define the local fundamental group $\pi_1^{\mathrm{loc}}(X, x)$ as the group $\pi_1(B \setminus \{x\})$, where B is the intersection of X with a small Euclidean ball containing x.

Example 2.50. Let S be a normal klt surface. Following [106, Definition 10.7], we define its orbifold Euler number as

$$e_{\mathrm{orb}}(S) = e(S) - \sum_{p \in S_{\mathrm{sing}}} 1 - \frac{1}{|\pi_1^{\mathrm{loc}}(S,p)|}.$$

By [106, Theorem 10.8], we then have $\hat{c}_2(S) = e_{\mathrm{orb}}(S)$.

2.7 Some inequalities for (orbifold) Chern classes. The first two parts of this thesis focus on minimal varieties, i.e. varieties with nef canonical divisor. The following inequality relates the first and the second Chern class of a smooth minimal variety.

Theorem 2.51. [140, Theorem 1.1] *Let X be a smooth projective variety of dimension n, with K_X nef. Then for any ample line bundles $H_1, \dots, H_{n-2}$ on X,*

$$(3c_2(X) - c_1(X)^2) \cdot H_1 \cdot \dots \cdot H_{n-2} \geq 0.$$

A partial equality case in Theorem 2.51 is entirely characterized.

Theorem 2.52. [210, Theorem 1.1], [77, p.4-5] *Let X be a smooth projective variety of dimension n. Suppose that $c_1(X) = 0$ and that for some ample divisor H, $c_2(X) \cdot H^{n-2} = 0$. Then X is uniformized by $\mathbb{C}^n$.*

Let us discuss how this picture generalizes to the singular setting. Several results build up to generalize Theorem 2.51. The first one is in a very singular setting, in dimension 2. It was used, as we advertised, to answer a boundedness question in [141].

Theorem 2.53. [138, Theorem 0.1] *Let X be a normal projective surface with log canonical singularities such that $\kappa(X) \geq 0$. Then $3\hat{c}_2(X) - \hat{c}_1{}^2(X) \geq 0$.*

The second one is in a mildly singular setting, but in dimension three.

Theorem 2.54. [173, Theorem 1.2] *Let X be a normal projective threefold with isolated log canonical singularities such that K_X is movable. Then for any ample divisor H on X, $3c_2(X) \cdot H - c_1{}^2(X) \cdot H \geq 0$.*

The third one is in arbitrary dimension, and in the klt setting.

Theorem 2.55. [79, Theorem B] *Let X be a normal projective variety of dimension n with klt singularities such that K_X is nef. Let m be a positive integer such that mK_X is Cartier. Let ν be the largest integer such that $(mK_X)^\nu$ is not numerically trivial. Then, for $i = \min(\nu, n-2)$ and for $j = n-2-i$, it holds*

$$\left(\hat{c}_2(X) - \frac{n}{2(n+1)}\hat{c}_1{}^2(X)\right) \cdot (mK_X)^i \cdot H^j \geq 0.$$

Note that those generalizations are initially stated in the broader context of pairs with standard coefficients. As in Theorem 2.52, the partial equality case is classified.

Theorem 2.56. [133, Theorem 1.2] *Let X be a normal projective variety of dimension n. Then X is a quotient of an abelian variety by a finite group action that is free in codimension 1 if and only if X is klt, $c_1(X) = 0$, and for some ample divisors $H_1, \dots, H_{n-2}$ on X,*

$$\hat{c}_2(X) \cdot H_1 \cdots H_{n-2} = 0.$$

Let us close this section with two inequalities that have little to do with minimal varieties. As we discussed in Example 2.48, the second orbifold number of the singular Kummer surface vanishes, whereas its resolution, a K3 surface, has Euler number 24. This phenomenon generalizes into the following inequality, which will be important to prove Theorem 2.119, a motivation for Part II. A similar inequality is established in [74, Claim 7.1].

Theorem 2.57. *Let X be a normal klt variety of dimension n, and let $\varepsilon : \tilde{X} \to X$ be a resolution of X. For any ample divisor H on X, it holds*

$$c_2(\tilde{X}) \cdot (\varepsilon^* H)^{n-2} \geq \hat{c}_2(X) \cdot H^{n-2}$$

and equality occurs if and only if X is smooth in codimension 2.

Proof. This is claimed in dimension 3 by [182, Proposition 1.1], and [133, Remark 1.5] claims that it generalizes it to arbitrary dimension, but let us include a proof, as we could not entirely follow [182].

First, up to replacing H be a large multiple of itself, we can assume that it is very ample. Let S be a normal complete intersection surface cut out by $n-2$ general members of the linear system $|H|$. Let S' be its strict transform in $\tilde{X}$, which is in fact cut out by the strict transforms of the $n-2$ general members of $|H|$, i.e., by $n-2$ general members of $\varepsilon^*|H|$. We are left showing that

$$c_2(T_{\tilde{X}}|_{S'}) \geq \hat{c}_2(\mathcal{T}_X|_S).$$

Consider the local diagram of an unfolding of X.

$$\begin{tikzcd}
\hat{V}_\alpha \arrow[r, "\hat{p}_\alpha"] \arrow[d, "i_\alpha"] & V_\alpha \arrow[r, "p_\alpha"] & V_\alpha/G_\alpha \arrow[d, "p'_\alpha"] \\
\hat{Y} \arrow[rr, "p"] & & Y
\end{tikzcd}$$

Note that by [75, Proposition 3.11], there is a compatible unfolding of S, namely

$$\begin{tikzcd}
\hat{W}_\alpha \arrow[r, "\hat{p}_\alpha"] \arrow[d, "i_\alpha"] & W_\alpha \arrow[r, "p_\alpha"] & W_\alpha/G_\alpha \arrow[d, "p'_\alpha"] \\
\hat{S} \arrow[rr, "p"] & & S
\end{tikzcd}$$

where $W_\alpha \subset V_\alpha$ is cut out by $n-2$ general members in the linear system ${p_\alpha}^* {p'_\alpha}^* |H|$.

Note that V_α and W_α are smooth and

$${p_\alpha}^{[*]} {p'_\alpha}^* \mathcal{T}_S|_{p_\alpha(W_\alpha)\cap S} = T_{W_\alpha}$$

$${p_\alpha}^{[*]} {p'_\alpha}^* \mathcal{T}_X|_{p_\alpha(V_\alpha)\cap S} = T_{V_\alpha}|_{W_\alpha}$$

So we get an exact sequence of locally free sheaves on W_α

$$0 \to {p_\alpha}^{[*]} {p'_\alpha}^* \mathcal{T}_S|_{p_\alpha(W_\alpha)} \to {p_\alpha}^{[*]} {p'_\alpha}^* \mathcal{T}_X|_{p_\alpha(V_\alpha)\cap S} \to {p_\alpha}^* {p'_\alpha}^* \mathcal{O}_S(H)^{\oplus n-2} \to 0$$

As pullback is exact for locally free sheaves, we can pullback by $\hat{p}_\alpha$ and glue the locally free sheaves obtained onto $\hat{S}$, to get

$$0 \to p^{[*]}\mathcal{T}_S \to p^{[*]}\mathcal{T}_X|_S \to p^*\mathcal{O}_S(-mH)^{\oplus n+2} \to 0,$$

hence

$$\hat{c}_2(T_X|_S) = \hat{c}_2(S) + \binom{n-2}{2} m^2 (H|_S)^2 + (n-2)m\hat{c}_1(S) \cdot H|_S.$$

Since $\tilde{X}$ and S' are smooth, we also have

$$0 \to T_{S'} \to T_{\tilde{X}}|_{S'} \to \mathcal{O}_{S'}(\varepsilon^*(mH))^{\oplus n-2} \to 0$$

so

$$c_2(T_{\tilde{X}}|_{S'}) = c_2(S') + \binom{n-2}{2} m^2 (\varepsilon^* H|_{S'})^2 + (n-2)mc_1(S') \cdot \varepsilon^* H|_{S'}.$$

We are left proving that $c_2(S') = e(S') \geq e_{\mathrm{orb}}(S) = \hat{c}_2(S)$, but this follows from the exceptional locus in S' being a union of trees of $\mathbb{P}^1$'s

$$\begin{aligned} e(S') &= e(S_{\mathrm{reg}}) + \sum_{p \in S_{\mathrm{sing}}} e(\text{a tree of } \mathbb{P}^1\text{'s}) \\ &\geq e(S_{\mathrm{reg}}) + 2|S_{\mathrm{sing}}| \\ &\geq e(S) + |S_{\mathrm{sing}}| \\ &\geq e_{\mathrm{orb}}(S). \end{aligned}$$

and equality holds if and only if S_{sing} is empty, i.e., X is smooth in codimension 2. □

Finally, this last inequality states that the Chern numbers of a positive vector bundle are, in some sense, positive. It will come handy in Sections 4.1 and 24.

Theorem 2.58. [47, Corollary 2.6] *Let X be a smooth projective variety of dimension n, and E be a nef vector bundle on X. Then for any $1 \leq r \leq n$, any Chern monomial $c_{i_1}(E) \cdots c_{i_k}(E)$ with $i_1 + \ldots + i_k = r$ and any ample line bundle H on X satisfy*

$$c_1(E)^r \cdot H^{n-r} \geq c_{i_1}(E) \cdots c_{i_k}(E) \cdot H^{n-r} \geq 0.$$

2.8 An introduction to valuation theory for singularities. Recall that an *integral valuation* v on a field K is a function $\nu : K \to \mathbb{Z} \cup \{+\infty\}$ that satisfies, for all $a, b \in K$,

- $v(a) = +\infty$ if and only if $a = 0$;
- $v(a+b) \geq \min(v(a), v(b))$;
- $v(ab) = v(a) + v(b)$.

A *discrete valuation* is an integral valuation which is surjective onto $\mathbb{Z} \cup \{+\infty\}$.

Example 2.59. We say that E is a divisor *over* a normal complex analytic variety X if there is a *partial resolution* of X, i.e., a normal complex analytic variety $\tilde{X}$ with a proper birational morphism $\varphi : \tilde{X} \to X$, such that E is a φ-exceptional prime divisor. We say that the partial resolution φ *realizes* E. To such a divisor we associate a discrete valuation on the function field of X:

$$v_E : f \in k(X) \mapsto \mathrm{ord}_E(f \circ \varphi) \in \mathbb{Z} \cup \{+\infty\},$$

which does not depend on the partial resolution φ chosen. A valuation of this form is called a *divisorial valuation*.

To propose a second example, we first need to introduce the notion of a junior element. It will play an important role in Part II.

Definition 2.60. Let g be a matrix in $\mathrm{GL}_n(\mathbb{C})$. Assume that it has finite order d. Since $g^d = \mathrm{id}$, g is diagonalizable and has eigenvalues of the form $e^{2i\pi a_k/d}$, for integers $a_k \in [\![0, d-1]\!]$. The *age* of g is set to be the number $\frac{a_1+\ldots+a_n}{d}$. If it equals 1, we say that g is *junior*.

Example 2.61. [94] Let $g \in \mathrm{SL}_n(\mathbb{C})$ be a matrix of finite order d. We can take coordinates $x_1, \ldots, x_n$ on $\mathbb{C}^n$ that diagonalize g, so that for any $k \in [\![1, n]\!]$, $g^* x_k = e^{2i\pi a_k/d} x_k$, with $a_k \in [\![0, d-1]\!]$. We define the integral valuation:

$$v_g : x_k \in k(\mathbb{C}^n) \mapsto a_k \in \mathbb{Z} \cup \{+\infty\}.$$

If g is junior, then $a_1, \ldots, a_n$ have no common prime divisor, and thus v_g is then a discrete valuation.

The following theorem has a reinterpretation in terms of valuations.

Theorem 2.62. [94] *Let $\mathbb{C}^n/G$ be a finite Gorenstein quotient singularity, and let X be a terminalization of it. Then there is a natural one-to-one correspondence between conjugacy classes of junior elements in G and prime exceptional divisors in X.*

Remark 2.63. Note that the correspondence in Theorem 2.62 is just the identification of the set of divisorial valuations v_E, when E is a crepant divisor over $\mathbb{C}^n/G$, and the set of valuations v_g, when g is a junior element in G.

Definition 2.64. Let X, Y be normal complex analytic varieties, and $p : X \to Y$ be a finite Galois morphism of group G. Let v, w be discrete valuations on the function fields $k(X)$ and $k(Y)$. Note that $k(Y)$ identifies with the invariant subfield $k(X)^G$ of $k(X)$.
The *ramification index* $\mathrm{Ram}(v, k(Y))$ of v over $k(Y)$ is the unique non-negative integer such that:

$$v(k(Y)^*) = \mathrm{Ram}(v, k(Y))\mathbb{Z}.$$

We say that v is an *extension* of w to $k(X)$ if:

$$w = \frac{1}{\mathrm{Ram}(v, k(Y))} v \mid_{k(Y)} .$$

If v is an extension of w, then by [211, Ch.VI, Par.12], the set of all extensions of w is exactly $\{v \circ g \mid g \in G\}$. In particular, all extensions of w have the same ramification index.

Remark 2.65. When considering divisorial valuations, ramification indices and extension properties carry a geometrical meaning. Let X, Y be normal complex analytic varieties endowed with their sheaves of holomorphic functions $\mathcal{H}_X$ and $\mathcal{H}_Y$. Let $p : X \to Y$ be a finite Galois morphism of group G, and let E, F be prime divisors in X, Y. The local rings $\mathcal{H}_{Y,E}$ and $\mathcal{H}_{X,F}$ are discrete valuation rings for the valuations v_E and v_F.

If we assume that F dominates E, then $p : X \to Y$ induces an injective morphism of local rings $p^\sharp : \mathcal{H}_{Y,E} \to \mathcal{H}_{X,F}$ by [185, Lem.29.8.6]. The maximal ideals $m_E \subset \mathcal{H}_{Y,E}$ and $m_F \subset \mathcal{H}_{X,F}$ relate by $p^\sharp(m_E) = m_F^r$, where r is the ramification index of F over

E. Hence $v_F|_{\mathcal{H}_{Y,E}} = rv_E$, i.e., v_F is an extension of v_E to $k(X)$ with ramification index $\mathrm{Ram}(v_F, k(Y)) = r$.

Conversely, if we assume that v_F is an extension of v_E to $k(X)$, then the structure sheaf map $p^\sharp : \mathcal{H}_Y \to p_*\mathcal{H}_X$ sends the ideal sheaf $\mathcal{I}_E$ to a subsheaf of $p_*\mathcal{I}_F$, so F dominates E.

Another important concept when considering ramification of valuations over subfields is the following.

Definition 2.66. Let X, Y be normal complex analytic varieties, and $p : X \to Y$ be a finite Galois morphism of group G. Let v be a discrete valuation on $k(X)$. Let $R_v \subset k(X)$ be the valuation ring, and $m_v \subset R_v$ be the unique maximal ideal. We define the *inertia group*

$$G_T(v) := \{g \in G \mid \forall\, x \in R_v,\ gx - x \in m_v\}.$$

Proposition 2.67. [211, p.77, Cor.] *If the residue field R_v/m_v has characteristic zero, then the inertia group G_T is cyclic of order* $\mathrm{Ram}(v, k(Y))$.

Proposition 2.68. [94, Cor.2.7 and p.11, Par.1] *Suppose that U is an open simply-connected subset of $\mathbb{C}^n$, G is a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ stabilizing U, and $p : U \to U/G = Y$ is the quotient map. Let $h \in \mathrm{GL}_n(\mathbb{C})$ be a junior element. Then:*

$$G_T(v_h) = G \cap \langle h \rangle.$$

2.9 Automorphisms of abelian varieties. Let us first recall the basic definitions.

Definition 2.69. A *complex torus* is a quotient of the form $\mathbb{C}^n/\Lambda$, where Λ is a lattice in $\mathbb{C}^n$. A complex torus of dimension $n = 1$ is called an *elliptic curve*.

Definition 2.70. An *abelian variety* A is a complex torus that admits a holomorphic embedding into a projective space.

Definition 2.71. We denote by $\mathrm{Aut}(A)$ the set of biholomorphisms from A to A. We call them *automorphisms* of A.

Remark 2.72. Contrarily to [17], we do not require automorphisms of A to fix $0 \in A$.

Example 2.73. [17, Example 4.1.13] All elliptic curves are abelian varieties.

The starting point when studying automorphisms of abelian varieties is the following proposition.

Proposition 2.74. [17, Proposition 1.2.1] *Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Let $g : X \to X$ be a holomorphic map. Then there exists a unique matrix $M(g) \in \mathrm{GL}_n(\mathbb{C})$ and a unique point $T(g) \in X$ such that $M(g)(\Lambda) \subset \Lambda$ and, for all $x \in X$,*

$$g(x) = M(g)x + T(g).$$

Whereas every point defines a translation, not every matrix defines a holomorphic map on a torus. There is namely the following constraint.

Proposition 2.75. [17, Proposition 1.2.3] *Let $X = \mathbb{C}^n/\Lambda$ be a complex torus. Let $g : X \to X$ be a holomorphic map. Then the characteristic polynomial of $M(g) \oplus \overline{M(g)}$ has rational coefficients.*

We derive the following corollary, that will be used extensively in Part II.

Lemma 2.76. *Let A be an abelian variety of dimension n, and $g \in \mathrm{Aut}(A)$ of finite order. Denote by $P(g)$ the characteristic polynomial of $M(g)$. Then $P(g)\overline{P(g)}$ is a product of cyclotomic polynomials.*

Proof. By [17, Proposition 1.2.3], $P(g)\overline{P(g)}$ is a polynomial over $\mathbb{Q}$. Since g has finite order, the roots of this polynomial are roots of unity. Remembering that cyclotomic polynomials are the minimal polynomials of roots of unity over $\mathbb{Q}$, an easy induction shows that there is a product Π of cyclotomic polynomial that has the exact same roots as $P(g)\overline{P(g)}$. But since both cyclotomic polynomials and characteristic polynomials are unitary, it means that $P(g)\overline{P(g)} = \Pi$. $\square$

Even with this constraint, not every matrix can act on every abelian variety.

Example 2.77. [17, Corollary 13.3.4] Let E be an elliptic curve. If $M(g)$ is an automorphism of E of finite order $d \geq 3$ such that $M(g)(0) = 0$, then we are in one of the folowing three cases:

d	3	4	6
$M(g)$	j	i	$-j$
E	E_j	E_i	E_j

where $j = e^{2i\pi/3}$ and $E_z = \mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$.

We will discuss partial results in higher dimension in Part II, in the following direction: knowing the matrix of an automorphism of A, what can we say about A? This has to do with the theory of abelian varieties with complex multiplication (CM). We will recall some background on them and a useful proposition, following [17, 13.3] and [184].

Definition 2.78. A number field is *totally real* if for every embedding of it into the complex numbers, its image lies in the real line. It is *totally complex* if it cannot be embedded into the real numbers.

Definition 2.79. A *CM-field* is a totally complex quadratic extension of a totally real number field.

Example 2.80. Clearly, $\mathbb{Q}[j]$ and $\mathbb{Q}[i]$ are CM-fields. Every cyclotomic field $\mathbb{Q}[\zeta_n]$ is clearly totally complex, and is in fact a CM-field. Indeed, the number field $\mathbb{Q}[\zeta_n + \zeta_n^{-1}]$ is totally real, and ζ_n is a root of the quadratic equation

$$(2X - \zeta_n - {\zeta_n}^{-1})^2 - (\zeta_n + {\zeta_n}^{-1})^2 + 4 = 0.$$

More importantly for us, defining the following quadratic integers

$$u_7 = \frac{-1 + i\sqrt{7}}{2},\ u_8 = i\sqrt{2},\ u_{15} = \frac{1 + i\sqrt{15}}{2},\ u_{20} = i\sqrt{5},\ u_{24} = i\sqrt{6},$$

and the following algebraic integer, whose square is a quadratic integer

$$u_{16} = i\sqrt{4 + 2\sqrt{2}}.$$

then clearly, for each $k \in \{7, 8, 15, 16, 20, 24\}$, $\mathbb{Q}[u_k]$ is a CM-field.

Definition 2.81. A *CM-type* of a CM-field K of degree $2g$ over $\mathbb{Q}$ is the data of embeddings $\{\sigma_1, \dots, \sigma_g\}$ of K into $\mathbb{C}$ that are pairwise distinct and pairwise non-conjugated.

Example 2.82. There are exactly two CM-types for $\mathbb{Q}[j]$: either the natural embedding, or the conjugated embedding. An example of a CM-type for $\mathbb{Q}[\zeta_7]$ is given by

$$\begin{aligned}\sigma_1 :&\zeta_7 \mapsto \zeta_7\\ \sigma_2 :&\zeta_7 \mapsto {\zeta_7}^2\\ \sigma_3 :&\zeta_7 \mapsto {\zeta_7}^4\end{aligned}$$

It has in fact eight different CM-types.

Definition 2.83. Given a CM-field K and a CM-type $\Sigma = \{\sigma_1, \dots, \sigma_g\}$, an *abelian variety of CM-type* (K, Σ) is an abelian variety A of dimension g such that there is an embedding $\rho : K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A)$ such that, when taking matrices,

$$M \circ \rho : K \to \mathcal{M}_g(\mathbb{C})$$

is conjugated in $\mathrm{GL}_g(\mathbb{C})$ to $\sigma_1 \oplus \ldots \oplus \sigma_g$, where $\mathcal{M}_g(\mathbb{C})$ denotes the algebra of $g \times g$ square matrices with complex coefficients.

Proposition 2.84. [17, Proposition 13.3.1] *To any CM-field K and CM-type Σ, one can associate an abelian variety of type (K, Σ).*

Clearly, any two isogenous abelian varieties have the same CM-types. A converse is true.

Proposition 2.85. [184, p.45, Corollary of Theorem 2] *Any two abelian varieties of the same CM-type are isogenous.*

Definition 2.86. [169, p.108] Let A be a finite dimensional $\mathbb{Q}$-algebra. A $\mathbb{Z}$-*order*, or *order* in A is a subring O of A such that O is a finitely generated $\mathbb{Z}$-module and

$$\mathbb{Q} \cdot O := \left\{ \sum_{i=1}^{n} q_i o_i \mid q_i \in \mathbb{Q}, o_i \in O, n \in \mathbb{N} \right\} = A.$$

Example 2.87. By [169, Chapter 12], if K is a number field, viewing it as a $\mathbb{Q}$-algebra, its ring of integers is the unique maximal order in it.

Definition 2.88. An abelian variety A of CM-type (K, Σ) is called *principal* if $\mathrm{End}(A) \cap \rho(K)$ is the maximal order in $\rho(K) \simeq K$.

Remark 2.89. If A is an abelian variety of CM-type (K, Σ) such that the embedding $K \subset \mathrm{End}_{\mathbb{Q}}(A)$ sends the ring of integers of K to a subring of $\mathrm{End}(A)$, then A is principal.

Example 2.90. The elliptic curve E_j is of CM-type $(\mathbb{Q}[j], \sigma)$. It is principal because $\mathrm{End}(E_j) = \mathbb{Z}[j]$ is a maximal order in $\mathbb{Q}[j]$. The elliptic curve E_{2j} is of CM-type $(\mathbb{Q}[j], \sigma)$ too. It is not principal because $\mathrm{End}(E_{2j}) = \mathbb{Z}[2j]$ is not a maximal order in $\mathbb{Q}[j]$.

Proposition 2.91. [184, p.60, Proposition 17] *The number of principal non-isomorphic abelian varieties of a same CM-type (K, Σ) is the class number of K. In particular, there is exactly one if and only if the ring of integers of K is a principal ideal domain.*

Definition 2.92. An abelian variety is said to be *simple* if it contains no proper abelian subvariety of positive dimension. A CM-type is called *primitive* if every abelian variety of this CM-type (or equivalently, one abelian variety of this CM-type) is simple.

We can determine purely algebraically whether a given CM-type is primitive.

Definition 2.93. Let K_0 be a totally real number field. An element $x \in K_0$ is *totally positive* if for every embedding $K \hookrightarrow \mathbb{R}$, the image of x is a positive number.

Example 2.94. The unit 1 is totally positive, -1 is clearly not totally positive, and $1+\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ is positive in the natural embedding, but not totally positive as $1-\sqrt{2} < 0$ is its image by the conjugated embedding of $\mathbb{Q}[\sqrt{2}]$ in $\mathbb{R}$.

Proposition 2.95. [184, p.69, Proposition 27] *A CM-type (K, Σ) is primitive if and only if there exists a totally real number field K_0, a totally positive element $\eta \in K_0$, and an element $\zeta \in K$ such that*

- $-\zeta^2 = \eta$;
- $K_0[\zeta] = \mathbb{Q}[\zeta] = K$;
- *Σ is the set of embeddings $\sigma : K \hookrightarrow \mathbb{C}$ such that* $\mathrm{Im}(\sigma(\zeta)) > 0$;
- *for all $\zeta' \neq \zeta$ conjugated to ζ over $\mathbb{Q}$, the ratio $\zeta'\zeta^{-1}$ is not totally positive.*

Example 2.96. Consider $K = \mathbb{Q}[u_{16}]$ in the notation of Example 2.80. It has degree 2 over $K_0 = \mathbb{Q}[\sqrt{2}]$. Fix $\eta = 4+2\sqrt{2}$ and $\zeta = u_{16}$. Then the first two hypotheses hold. Let us consider the fourth hypothesis. The conjugates of u_{16} over $\mathbb{Q}$ are $-u_{16}$, $v_{16} := i\sqrt{4-2\sqrt{2}}$, and $-v_{16}$. Clearly, all ratios are negative, except for $\frac{v_{16}}{u_{16}} = \sqrt{2}-1$. But as we can define an embedding $K_0 \hookrightarrow \mathbb{R}$ that sends $\sqrt{2}$ to $-\sqrt{2}$, and as $-\sqrt{2}-1$ now is negative, the ratio $\frac{v_{16}}{u_{16}}$ is not totally positive.

So setting Σ as the set of embeddings $\sigma : K \hookrightarrow \mathbb{C}$ such that $\mathrm{Im}(\sigma(u_{16})) > 0$, we obtain a primitive CM-type (K, Σ). As K has degree 4 over $\mathbb{Q}$, the corresponding simple abelian varieties have dimension 2.

Finally, we recall without proof a result [184, p.46, Proof of Theorem 3] that we use in Part II.

Lemma 2.97. *Let $K = \mathbb{Q}(\alpha)$ be a totally imaginary quadratic extension of $\mathbb{Q}$ of degree $2m$. Let F be a finite Galois extension of K, of degree $2r$ over $\mathbb{Q}$. Let $\{\varphi_i\}_{1\leq i \leq r}$ be morphisms of $\mathbb{Q}$-algebras defined from F to $\mathbb{C}$ such that:*

$$\mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(F, \mathbb{C}) = \mathrm{Vect}_{\mathbb{Q}}\left(\varphi_1, \overline{\varphi_1}, \dots, \varphi_r, \overline{\varphi_r}\right).$$

Suppose also that no two of the restrictions $\varphi_i|_K$ are conjugate.

Then we can restrict m of these morphisms, defining $\psi_j = \varphi_{i_j}|_K$ for some distinct i_j with $j \in [\![1, m]\!]$, such that:

$$\mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(K, \mathbb{C}) = \mathrm{Vect}_{\mathbb{Q}}\left(\psi_1, \overline{\psi_1}, \dots, \psi_m, \overline{\psi_m}\right).$$

We obtain a $\mathbb{Z}$-algebra $\Delta := \mathbb{Z}[(\psi_1(\alpha), \dots, \psi_m(\alpha))]$ that is a lattice of rank $2m$ in $\mathbb{C}^m$. The complex torus $A := (\mathbb{C}^m/\Delta)^{n/m}$ is an abelian variety of CM-type $(F, \{\varphi_i\}_{1\leq i\leq f})$.

2.10 A reminder in Sylow theory. As we will use them thoroughly in Part II, we recall a few facts about p-groups and the three Sylow theorems.

Definition 2.98. [172, Corollary 4.3] Let p be a prime number. A *p-group* is a group in which every element has order a power of p. Equivalently, it is a group whose order is a power of p.

We use the following result to bound the p-Sylow subgroups of a group of matrices acting on an abelian variety, in which commutation is rare: see the proofs in Chapter 10 for more precise statements.

Theorem 2.99. [172, Theorem 4.4] *Let p be a prime number. Let S be a finite non-trivial p-group. Then the center $Z(S)$ of S contains an element of order p.*

A corollary is the following result.

Corollary 2.100. *Let p be a prime number. Let S be a finite p-group. If the maximal abelian subgroup of S is cyclic of order p, then S itself is cyclic of order p.*

Note that, although we do not need this stronger result in this thesis, the following nice generalization holds, bounding the order of a p-group in which commutation is rare.

Theorem 2.101. [188, Corollary 2, p.94] *Let p be a prime number. Let S be a finite p-group of order p^s. If a normal abelian subgroup N of S of maximal order has order p^n, then $s \leq \frac{n(n+1)}{2}$.*

Remark 2.102. Obviously, the same inequality holds *a fortiori* if N is a non-normal abelian subgroup of maximal order.

Let us now move on to Sylow theory.

Definition 2.103. Let p be a prime number. Let G be a finite group. A *p-Sylow subgroup* of G is a maximal p-subgroup of G.

The following three theorems often go, by order, under the name of the first, second and third Sylow theorems.

Theorem 2.104. [172, Theorem 4.14] *Let p be a prime number. Let G be a finite group of order $p^s m$, with p and m coprime. Then there exists a p-Sylow subgroup of G of order p^s.*

Theorem 2.105. [172, Theorem 4.12] *Let p be a prime number. Let G be a finite group. Then any two p-Sylow subgroups of G are conjugated.*

In particular, all p-Sylow subgroups have the same order, determined by the first Sylow theorem.

Theorem 2.106. [172, Theorem 4.12] *Let p be a prime number. Let G be a finite group. Then the number n_p of p-Sylow subgroups of G is congruent to one modulo p and divides $|G|$.*

A last important point is the following theorem about finding a normal complement to a Sylow subgroup.

Definition 2.107. Let G be a finite group, and S be a p-Sylow subgroup in G. We say that S admits a normal complement if there exists a normal subgroup $N \triangleleft G$ such that $G \cong N \rtimes S$.

It is important to note that if G has to be generated by elements of order prime to p, then a p-Sylow subgroup of G cannot have a normal complement. Indeed, if $G \cong N \rtimes S$, all elements of order prime to p belong to N, therefore generate a subgroup of N, which is proper in G.

The following result yields the existence of a normal complement in some cases.

Theorem 2.108. [172, Theorem 7.50] *Let G be a finite group and let p be a prime number dividing the order of G. Let S be a p-Sylow subgroup of G. If S is abelian and $N_G(S) = C_G(S)$, then S admits a normal complement.*

2.11 A primer on Mori theory. Mori theory is a vast branch of birational geometry. For our purpose though, only results tracing back to the eighties will be needed, namely the base point free theorem, a few theorems around contractions, and the Cone Theorem. We follow [111] in the treatment of these results, but the reader might alternatively refer to [42]. Let us first present the base point free theorem, and the contraction theorems.

Theorem 2.109. Base point free Theorem [111, Theorem 3.3] *Let (X, B) be a projective klt pair, and let D be a nef Cartier divisor on X such that for some positive rational number a, the divisor $aD - (K_X + B)$ is nef and big. Then there exists m_0 such that, for all $m \geq m_0$, the linear system $|mD|$ is base point free.*

Definition 2.110. A *contraction* is a proper morphism of normal varieties $h : X \to Y$ such that $h_*\mathcal{O}_X = \mathcal{O}_Y$. Equivalently, it is a proper surjective morphism with connected fibers.

Definition 2.111. The Mori cone $\overline{NE}(X)$ of a normal projective variety X is the closure of the cone generated by classes of effective 1-cycles in $N_1(X)$.

Example 2.112. [111, Example 1.23(4)] The Mori cone of $\mathbb{P}^2$ blown-up in the 9 base points of a general pencil of cubics, whose members are all ireducible, is a ten-dimensional cone generated by infinitely many classes of (-1)-curves, which correspond to sections of the anticanonical elliptic fibration onto $\mathbb{P}^1$.

In general, describing the Mori cone of a given variety is strenuous, and equivalent to describing its dual cone, i.e., the nef cone defined in Section 2.4. However, describing the contractions of a variety X allows to say some things about part of its Mori cone, as the Cone Theorem will soon state formally. Before that, let us describe some special contractions in more detail.

Definition 2.113. An *extremal ray* R of a closed convex cone C is a halfline $R \subset C$ such that for all $u, v \in C$ with $u + v \in R$, u, v belong to R. It is called *K_X-negative* if there exists $Z \in R$ such that $K_X \cdot Z < 0$.

Definition 2.114. [111, Definition 1.25] Let X be a normal projective variety, and R be an extremal ray of $\overline{NE}(X)$. A *contraction associated to R* is a contraction $h : X \to Y$ such that for every curve C in X, $h(C) = \{\mathrm{pt}\}$ if and only if $[C] \in R$.

Definition 2.115. A contraction of a K_X-negative extremal ray is called a *Mori contraction.*

Example 2.116. Consider the surface S that is $\mathbb{P}^2$ blown-up in two distinct, non infinitesimally close points, and let $\ell \subset S$ be the strict transform of a line through these two points. Then ℓ is a K_S-negative (-1)-curve on S, hence it spans a K_S-negative extremal ray [42, Lemma 6.2(b)]. There is a Mori contraction associated to ℓ, which is a map $S \to \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2.117. Contraction Theorem [111, Theorem 3.7(3) and (4)], [42, Theorem 7.39] *Let (X, B) be a normal projective variety with klt singularities, and let R be a $K_X + B$-negative extremal ray of $\overline{NE}(X)$. Then there is a contraction $h : X \to Y$ associated to R, and moreover*

- $-(K_X + B)$ *is h-ample;*
- *if D is a Cartier divisor and for all curve C contracted by h, it holds $D \cdot C = 0$, then there exists E a Cartier divisor on Y such that $D \sim h^*E$;*
- $\rho(X) = \rho(Y) + 1.$

A little more can be said about X and Y when $h : X \to Y$ is a contraction that is *divisorial*, i.e., it is a birational contraction and its exceptional locus has a component of codimension 1 in X. A birational contraction that is not divisorial is called a *small contraction.*

Proposition 2.118. [111, Proposition 2.5] *Let (X, B) be a normal projective $\mathbb{Q}$-factorial pair, and let $h : X \to Y$ be a divisorial contraction of a $(K_X + B)$-negative rational extremal ray. Then the exceptional locus of h consists in one irreducible divisor E.*

An important corollary of these theorems is the following fact presented in the introduction of Part II. Let us present a detailed proof of it.

Theorem 2.119. *Let X be a Calabi-Yau manifold of dimension n. The following are equivalent:*

(i) *There is a nef and big divisor D on X such that $c_2(X) \cdot D^{n-2} = 0$.*

(ii) *There is an abelian variety A and a finite group G acting freely in codimension 2 on A such that X is a crepant resolution of A/G.*

If it satisfies these conditions, X is called a Calabi-Yau manifold of type n_0.

Proof. First assume that there is a nef and big divisor D on X such that $c_2(X) \cdot D^{n-2} = 0$. Then by the base point free theorem, as K_X is trivial, there is an integer m such that $|mD|$ is base point free. This linear system induces a morphism $\phi : X \to \mathbb{P}(H^0(X, mD)^*)$, whose image we denote by Y. Let H be an ample divisor on X, let H' be an ample Cartier divisor on Y such that $\phi^*H' = D$. As D is big, the variety Y has dimension n. Thus, ϕ is a birational contraction.

Let us follow [193] and say more about ϕ. Note that for j large enough, $jD - H$ is big (as the big cone is open). Hence for k large enough, both $jkD - kH$ and $(k-1)H$ are effective. So $jkD - H$ is effective. Moreover, it is ϕ-antiample. So, for $\varepsilon > 0$ small enough, setting $B := \varepsilon(jkD - H)$, the pair (X, B) is klt by [111, Corollary 2.35(2)] and ϕ is a $(K_X + B)$-negative contraction.

Note that as K_X is trivial, $\phi_*\omega_X = \phi_*\mathcal{O}_X = \mathcal{O}_Y$. Moreover, $(\phi_*\omega_X)^{**}$ and ω_Y are reflexive sheaves which coincide in codimension 1, so by normality of reflexive sheaves, they coincide. Hence $\omega_Y \simeq \mathcal{O}_Y$. In particular, Y has canonical singularities. Moreover, by Theorem 2.55 and Theorem 2.57,

$$0 = c_2(X) \cdot D^{n-2} \geq \hat{c}_2(Y) \cdot H'^{n-2} \geq 0.$$

As equality holds, Y is smooth in codimension 2. Moreover, by Theorem 2.56, Y is a finite quotient of an abelian variety A by a group G acting freely in codimension 1. As G contains no pseudoreflection, A/G_{sing} is exactly the union of the fixed loci of the element of G, so smoothness in codimension 2 implies that G acts freely in codimension 2.

Conversely, assume that X is a crepant resolution of A/G, with G acting freely in codimension 2. Denote by H' an ample divisor on A/G, by D its pullback to X. Clearly, D is nef and big. By the equality case in Theorem 2.57, $c_2(X) \cdot D^{n-2} = c_2(A/G) \cdot H'^{n-2} = 0$. This concludes the proof. □

We can now conclude this exposition with the Cone Theorem.

Theorem 2.120. [111, Theorem 3.7(1) and (2)] *Let (X, B) be a normal projective klt pair. Then there are countably many rational curves $C_i \subset X$ such that $0 < -(K_X + B) \cdot C_i \leq 2\dim(X)$ and*

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B\geq 0} + \sum_{i\in I} \mathbb{R}_+[C_i].$$

Moreover, for any ample divisor H and for any $\varepsilon > 0$, there exists $I_0 \subset I$ finite such that

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+B+\varepsilon H\geq 0} + \sum_{i\in I_0} \mathbb{R}_+[C_i].$$

2.12 Families of deformations of rational curves. In this section, we work with the scheme $\mathrm{RatCurves}^n(X)$, which is the normalization of the scheme parametrizing rational curves in X. As a rational curve is a 1-cycle, we also work with the scheme $\mathrm{Chow}(X)$ parametrizing 1-cycles in X, and the natural map $\mathrm{RatCurves}^n(X) \to \mathrm{Chow}(X)$.

Definition 2.121. Let X be a normal projective variety. Recall by [108, I.3.21, II.2.11] that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Univ}^{rc}(X) \arrow[r, "f'"] \arrow[d, "\phi"'] \arrow[rr, bend left, "\mathrm{ev}"] & \mathrm{Univ}^{\mathrm{chow}}(X) \arrow[r] \arrow[d, "\pi"] & X \\ \mathrm{RatCurves}^n(X) \arrow[r, "f"] & \mathrm{Chow}(X) & \end{tikzcd}$$

where ϕ is a $\mathbb{P}^1$-bundle, and for every rational curve C in X, there exists (finitely many and at least) a point $p \in \mathrm{RatCurves}^n(X)$ such that $\mathrm{ev}(\phi^{-1}(p)) = C$.

If C is a rational curve in X, we may denote by $\mathcal{V}$ a family of deformations of C, that is an irreducible component of $\mathrm{RatCurves}^n(X)$ containing a point corresponding to C. We then define

$$\mathrm{Locus}(\mathcal{V}) := \mathrm{ev}(\phi^{-1}(\mathcal{V})) \subset X.$$

We say that the family $\mathcal{V}$ is *covering* if $\mathrm{Locus}(\mathcal{V}) = X$. We say that the family $\mathcal{V}$ is *unsplit* if $\mathcal{V}$ is proper over $\mathrm{Spec}(\mathbb{C})$, or equivalently if $f(\mathcal{V})$ is closed in $\mathrm{Chow}(X)$.

For $x \in \mathrm{Locus}(\mathcal{V})$, we define $\mathcal{V}_x := \phi(\mathrm{ev}^{-1}(x))$ the family of deformations of C through x. We finally define $\mathrm{Locus}(\mathcal{V}_x) := \mathrm{ev}(\phi^{-1}(\mathcal{V}_x)) \subset X$.

We will use the following lemma repeatedly.

Lemma 2.122. [108, Corollary IV.2.6] *Let X be a smooth projective variety. Let $\mathcal{V}$ be a family of deformations of a rational curve C in X. If $\mathcal{V}$ is unplit, then*

$$\dim \mathrm{Locus}(\mathcal{V}) + \dim \mathrm{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + \dim(X) - 1.$$

The following lemma produces unsplit families of rational curves.

Lemma 2.123. *Let X be a smooth projective variety. Suppose that $-K_X \cdot C > 0$ for every rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$, and let C be a rational curve such that $\pi(C) \neq \{\mathrm{pt}\}$ and such that*

$$-K_X \cdot C = \min\{-K_X \cdot B \mid B \text{ rational curve in } X, \pi(B) \neq \{\mathrm{pt}\}\}.$$

Then the family of deformations of C is unsplit.

Proof of Lemma 2.123. Let $\mathcal{V}$ be the family of deformations of C. Suppose that it is splitting, i.e.,

$$C \underset{\text{num}}{\equiv} \sum_i a_i C_i,$$

with rational curves C_i and coefficients $a_i \geq 1$ such that $\sum_i a_i \geq 2$. Since $-K_X$ is positive on rational curves, we have $-K_X \cdot C_i < -K_X \cdot C$ for all i. So, by minimality of $-K_X \cdot C$, the fibration π contracts all curves C_i. Let H be an ample divisor on Y. We obtain $\sum_i a_i C_i \cdot \pi^* H = 0$, contradiction. □

The interaction of fibred Mori contractions and unsplit families of rational curves is especially nice, as the following lemma says.

Lemma 2.124. *Let X be a smooth projective variety. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$, and let C be a rational curve such that $\pi(C) \neq \{\mathrm{pt}\}$ and such that its family of deformations $\mathcal{V}$ is unsplit. Then, for any $x \in \mathrm{Locus}(\mathcal{V})$,*

$$\dim \mathrm{Locus}(\mathcal{V}_x) \leq \dim Y.$$

Proof of Lemma 2.124. We claim that $\pi|_{\mathrm{Locus}(\mathcal{V}_x)}$ is finite onto its image. If it is not, it contracts a curve B to a point: for some ample divisor H on Y, we have $B \cdot \pi^* H = 0$. By [3, Lemma 4.1], the numerical class of $B \subset \mathrm{Locus}(\mathcal{V}_x)$ is a multiple of $C \in N_1(X)_{\mathbb{Q}}$, whence $C \cdot \pi^* H = 0$, which is a contradiction. So $\pi|_{\mathrm{Locus}(\mathcal{V}_x)}$ is finite onto its image: this implies $\dim \mathrm{Locus}(\mathcal{V}_x) \leq \dim Y$. □

PART I

POSITIVITY OF THE (CO)TANGENT SHEAF ON SINGULAR CALABI-YAU VARIETIES

CHAPTER 3
INTRODUCTION

Complex algebraic varieties with trivial canonical class are of great importance in birational geometry. Indeed, they appear naturally as possible minimal models in the Minimal Model Program (MMP), and come in quite diverse geometrical families. Since higher-dimensional MMP generally gives rise to singular minimal models, understanding singular projective varieties with trivial canonical class is particularly relevant. Recently, three papers [86, Theorem 1.5], [70], [55] achieved a singular decomposition result for these varieties :

Theorem 3.1. *Let X be a normal projective variety with klt singularities, with K_X numerically trivial. Then there exists a normal projective variety $\tilde{X}$ with at most canonical singularities, which comes with a quasiétale finite cover $f : \tilde{X} \to X$ and decomposes as a product:*

$$\tilde{X} \cong A \times \prod_i Y_i \times \prod_j Z_j,$$

where A is a smooth abelian variety, the Y_i are singular Calabi-Yau varieties and the Z_j are singular irreducible holomorphic symplectic (IHS) varieties, as defined in Section 5.2.

May it seem an expected generalization of the smooth Beauville-Bogomolov decomposition result [11], [10], this theorem however relies on serious results from each paper: [70] introduces algebraic holonomy and studies infinitesimal decompositions of the tangent sheaf $\mathcal{T}_X$; [55] deals with the abelian part in the infinitesimal decomposition through a positive characteristics argument, and proves an integrability criterion for the remaining subsheaves of $\mathcal{T}_X$; [86] establishes positivity results which add up to Druel's criterion to finish the proof. This proof was notably simplified by [26], shortcutting the positive characterictics argument. Furthermore, the two recent papers [36], [8] extend this decomposition result to the singular Kähler case by subtle algebraic approximation considerations.

Interestingly enough, the singular decomposition for a klt variety X may not be the same as the singular decomposition of its terminalisation. The typical example is a singular Kummer surface, which resolves by 16 blow-ups into a smooth K3 surface, but has the Beauville-Bogomolov type of an abelian surface. Other such intriguing examples are given in [70, Sect.14]. Compatibility of the singular Beauville-Bogomolov decomposition with terminalisation nevertheless holds for *some* klt varieties with trivial canonical class [55, Lemma 4.6]. This license to terminalise is essential in the current proof of [86, Theorem 1.5], as it involves positivity results [86, Theorem 1.1] for klt varieties which are smooth in codimension 2: any klt variety is not, but its terminalisation surely is.

Since these positivity results have a wider scope than the mere proof of the singular decomposition theorem, it is worth extending them to normal projective klt varieties. Our main theorem is:

Theorem 3.2. *Let X be a normal projective variety with klt singularities and numerically trivial K_X. If its tangent or reflexivized cotangent sheaf is pseudoeffective, then there is a quasiétale finite cover $\tilde{X} \to X$ such that $q(\tilde{X}) \neq 0$. Equivalently, the singular Beauville-Bogomolov decomposition of X has an abelian factor of positive dimension.*

In particular, if X is a singular Calabi-Yau or IHS variety in the sense of Def.5.2, then neither $\mathcal{T}_X$ nor its dual $\Omega_X^{[1]}$ is pseudoeffective.

Importantly enough, this theorem does not boil down to [86, Theorem 1.6] on a terminalisation of X; we inevitably have to deal with codimension 2 quotient singularities on X. In this perspective, we resort to the theory of orbifold Chern classes. It has been developped in the late eighties in connection to the abundance problem for threefolds [106], and we will extensively use some of its most recent developments, *inter alia* [133], [75], [76].

Let us present a brief outline of the proof, say for a variety X with pseudoeffective tangent sheaf.

The fact that $\mathcal{T}_X$ is pseudoeffective pullbacks and restricts to one factor in the Beauville-Bogomolov decomposition of X. Supposing by contradiction that X has no abelian part, we can reduce to a Calabi-Yau or IHS factor Z such that $\mathcal{T}_Z$ is pseudoeffective. The work of [70] also establishes that $\mathcal{T}_Z$ and all its symmetric powers are stable of slope zero with respect to any polarisation H. Finally, since Z is not abelian, its orbifold second Chern class satisfies $\hat{c}_2(\mathcal{T}_Z) \cdot H^{\dim X - 2} \neq 0$. This contradicts the following generalization of [86, Theorem 1.1]:

Theorem 3.3. *Let X be a normal projective variety with klt singularities of dimension n, H a $\mathbb{Q}$-Cartier ample divisor on X. Consider $\mathcal{E}$ a reflexive sheaf on X such that:*

- $\hat{c_1}(\mathcal{E}) \cdot H^{n-1} = 0$;
- *for some $l \geq 6$, $S^{[l]}\mathcal{E}$ is H-stable;*
- *$\mathcal{E}$ is pseudoeffective.*

Then $\hat{c_1}(\mathcal{E})^2 \cdot H^{n-2} = \hat{c}_2(\mathcal{E}) \cdot H^{n-2} = 0$.

Moreover, there is a finite Galois covering $\nu : \tilde{X} \to X$, étale in codimension 1, such that $\nu^{[*]}\mathcal{E}$ is locally-free, has a numerically trivial determinant, and is $\mathrm{Gal}(\tilde{X}/X)$-equivariantly flat on $\tilde{X}$, ie comes from a $\mathrm{Gal}(\tilde{X}/X)$-equivariant representation of $\pi_1(\tilde{X})$. In particular, $\nu^{[*]}\mathcal{E}$ is numerically flat, and, as symmetric multilinear forms on $\mathrm{NS}(X)$:

$$c_1(\nu^{[*]}\mathcal{E}) \equiv 0, \quad c_2(\nu^{[*]}\mathcal{E}) \equiv 0.$$

The hard part here is the first assertion on the vanishing of orbifold Chern classes, the rest follows from [133].

In Sections 2.3, 2.4, 2.5, we recalled and proved basics to reduce the proof of Theorem 3.3 to working on a normal projective klt surface S. A crucial ingredient is that orbifold Chern classes behave well under certain restrictions [75, Proposition

3.11]. In Section 2.6, we introduced an *unfolding* $p : \hat{S} \to S$, obtained by gluing together local finite Galois quasiétale resolutions of the singularities of S. The surface $\hat{S}$ may be as singular as S; importantly enough though, any reflexive sheaf $\mathcal{E}$ on S reflexively pulls back to a locally-free sheaf $\hat{\mathcal{E}}$ on $\hat{S}$. We investigate the relationship of $\mathcal{E}$ and $\hat{\mathcal{E}}$. The key of the proof of Theorem 3.3, presented in Chpater 4, is then to establish the nefness of $\hat{\mathcal{E}}$, which yields the Chern classes vanishing for $\hat{\mathcal{E}}$, hence for $\mathcal{E}$. Note that $\mathcal{E}$ may very well not be nef itself: see Remark 2.25.

As a conclusive remark, note that investigating pseudoeffectivity of the tangent and reflexivized cotangent sheaves of a variety with trivial canonical class requires knowledge of its singular Beauville-Bogomolov decomposition. To that extent, Theorem 3.2 cannot be used on an explicit variety before knowing a bare minimum about its geometry. In Section 6, we exhibit 2409 Calabi-Yau threefolds with singularities in codimension 2, among the 7555 wellformed quasismooth hypersurfaces of trivial canonical sheaf in weighted projective 4-dimensional spaces classified by [116, 115]. These examples stay out of the range of the earlier pseudoeffectivity result of [86, Theorem 1.6], but are covered by our Theorem 3.2.

CHAPTER 4

RESTRICTING TO A GENERAL SURFACE

We prove the following proposition in Section 4.2:

Proposition 4.1. *Let S be a normal projective klt surface, and H an ample $\mathbb{Q}$-Cartier divisor on S. Let $\mathcal{E}$ be a reflexive sheaf on S such that:*

- $\hat{c}_1(\mathcal{E}) \cdot H = 0$;
- *for some $l \geq 6$, $S^{[l]}\mathcal{E}$ is stable with respect to H;*
- *$\mathcal{E}$ is pseudoeffective.*

Then there is an unfolding $p : \hat{S} \to S$ as in Section 2.6 on which the locally-free sheaf $\hat{\mathcal{E}} = p^{[*]}\mathcal{E}$ is nef.

In Section 4.1, we explain how this result implies the first part of Theorem 3.3, namely the vanishing of the squared first and second orbifold Chern classes.

4.1 Consequences of Proposition 4.1.

We are going to combine Proposition 4.1 with the following corollary of Theorem 2.58.

Lemma 4.2. *Let S be a normal projective surface, H an ample $\mathbb{Q}$-Cartier divisor on S and $\mathcal{E}$ a locally-free sheaf on S. Assume that $\mathcal{E}$ is nef and $c_1(\mathcal{E}) \cdot H = 0$. Then:*

$$c_1(\mathcal{E})^2 = c_2(\mathcal{E}) = 0.$$

Proof. Let $\tilde{S} \xrightarrow{\varepsilon} S$ be the minimal resolution of S, $\tilde{H} = \varepsilon^* H$. Writing $\tilde{\mathcal{E}} := \varepsilon^*\mathcal{E}$, we get a nef locally-free sheaf on a smooth surface. The functoriality of Chern classes of locally-free sheaves by continuous pullbacks [139, XI-Lemma 1] guarantees $c_i(\tilde{\mathcal{E}}) = \varepsilon^* c_i(\mathcal{E})$ for $i = 1, 2$. In particular, $c_1(\tilde{\mathcal{E}}) \cdot \tilde{H} = 0$. By nefness, $c_1(\tilde{\mathcal{E}})^2 \geq 0$. Hence, by Hodge Index Theorem, $c_1(\tilde{\mathcal{E}})^2 = 0$ which yields, by [47, Proposition 2.1, Theorem 2.5], $c_2(\tilde{\mathcal{E}}) = 0$. So we obtain:

$$c_1(\mathcal{E})^2 = c_2(\mathcal{E}) = 0.$$

□

Proof of the first assertion in Theorem 3.3. Let a variety X, an ample $\mathbb{Q}$-Cartier divisor H, and a reflexive sheaf $\mathcal{E}$ be as in the asssumptions of Theorem 3.3. By Proposition 2.29, Lemma 2.36 and [75, Prop.3.11], we can consider an integer m big and divisible enough that $n - 2$ general members of $|mH|$ cut out a complete intersection normal projective klt surface S in X on which:

- $\mathcal{E}|_S$ and $(S^{[l]}\mathcal{E})|_S$ are still reflexive;
- as a consequence, $S^{[l]}(\mathcal{E}|_S) = (S^{[l]}\mathcal{E})|_S$;
- $S^{[l]}(\mathcal{E}|_S)$ remains $H|_S$-stable of zero slope;
- $\mathcal{E}|_S$ is pseudoeffective.

Then, by Proposition 4.1, there is a finite Galois cover $p : \hat{S} \to S$ such that the reflexive pullback $\hat{\mathcal{E}} := p^{[*]}\mathcal{E}|_S$ is a nef locally-free sheaf of zero slope. Lemma 4.2 yields:

$$c_1(\hat{\mathcal{E}})^2 = c_2(\hat{\mathcal{E}}) = 0,$$

so that, by construction, $\hat{c}_1{}^2(\mathcal{E}|_S) = \hat{c}_2(\mathcal{E}|_S) = 0$ and hence:

$$\hat{c}_1{}^2(\mathcal{E}) \cdot H^{n-2} = \hat{c}_2(\mathcal{E}) \cdot H^{n-2} = 0.$$

The first assertion in Theorem 3.3 is established. □

4.2 Proof of Proposition 4.1. Let S be a normal projective klt surface, and H an ample $\mathbb{Q}$-Cartier divisor on S. Let $\mathcal{E}$ be a reflexive sheaf on S such that:

- $\hat{c}_1(\mathcal{E}) \cdot H = 0$;
- for some $l \geq 6$, $S^{[l]}\mathcal{E}$ is stable with respect to H;
- $\mathcal{E}$ is pseudoeffective.

As in Section 2.6, we denote by $p : \hat{S} \to S$ a finite Galois cover on which the sheaf $\hat{\mathcal{E}} = p^{[*]}\mathcal{E}$ is locally-free. Let $\hat{H} := p^*H$ be an ample $\mathbb{Q}$-Cartier divisor on $\hat{S}$, $\hat{\pi} : \mathbb{P}(\hat{\mathcal{E}}) \to \hat{S}$ be the natural map and $\hat{\zeta}$ be the tautological bundle on $\mathbb{P}(\hat{\mathcal{E}})$.

Abiding by [86, Sect.3.2], we prove two lemmas. The first lemma uses the stability of $S^{[l]}\mathcal{E}$ to prove the ampleness of $\hat{\zeta}$ on certain subvarieties of $\mathbb{P}(\hat{\mathcal{E}})$.

Lemma 4.3. *Keep the notations. For any closed proper subvariety $Z \subset \mathbb{P}(\hat{\mathcal{E}})$, for m big and divisible enough and for a general curve $\hat{C} \in p^*|mH|$, the restricted tautological $\hat{\zeta}|_{Z\cap\hat{\pi}^{-1}(\hat{C})}$ is ample.*

Proof. If Z is contained in a fiber of $\hat{\pi}$, then $Z \cap \hat{\pi}^{-1}(\hat{C})$ is empty and there is nothing to prove. Assume now that $\hat{\pi}(Z)$ has dimension 1 or 2 in $\hat{S}$. Since p is finite, $p(\hat{\pi}(Z))$ has dimension 1 or 2 in S. Hence, for m big and divisible enough, a very general curve $C \in |mH|$ satisfies:

- C is a smooth curve inside the locus $S_0 \subset S_{\mathrm{reg}}$ where $\mathcal{E}$ is locally-free;
- $\hat{C} := p^{-1}(C)$ is general in the basepoint-free linear system $p^*|mH|$ and hence smooth too;
- consequently, we have locally-free sheaf isomorphisms $\hat{\mathcal{E}}|_{\hat{C}} = p^*\mathcal{E}|_C$ and $S^l(\mathcal{E}|_C) = (S^{[l]}\mathcal{E})|_C$;
- since mH is ample and $\hat{\pi}(Z)$ is not a point, $Z \cap \hat{\pi}^{-1}(\hat{C})$ is non empty;
- since Z is proper in $\mathbb{P}(\mathcal{E})$, $Z \cap \hat{\pi}^{-1}(\hat{C})$ is proper in $\hat{\pi}^{-1}(\hat{C})$;

- $S^l(\mathcal{E}|_C)$ remains $H|_C$-stable of zero slope, by Lemma 2.36; by Lemma 2.40, $\mathcal{E}|_C$ is $H|_C$-stable of zero slope as well.

Apply now Lemma 2.43 and Remark 2.44: they establish that $\hat{\zeta}|_{\hat{\pi}^{-1}(\hat{C})}$ is nef and that, for any closed proper variety $W \subset \hat{\pi}^{-1}(\hat{C}) = \mathbb{P}(\hat{\mathcal{E}}|_{\hat{C}})$,

$$\left(\hat{\zeta}|_{\hat{\pi}^{-1}(\hat{C})}\right)^{\dim W} \cdot W > 0.$$

Using this formula for any closed subvariety W of $Z\cap\hat{\pi}^{-1}(\hat{C})$, the Nakai-Moishezon criterion shows that $\hat{\zeta}|_{Z\cap\hat{\pi}^{-1}(\hat{C})}$ is ample. □

The second lemma is set at the higher level of $(\hat{S}, \hat{\mathcal{E}})$ directly. It uses the pseudo-effectivity and $\hat{H}$-semistability of the locally-free sheaf $\hat{\mathcal{E}}$, infered by Proposition 2.33, Corollary 2.41, and Lemma 2.38, but no other property of $\mathcal{E}$.

Lemma 4.4. *Keep the notations. If $\hat{\zeta}$ is not nef, then there is a closed proper subvariety W of $\mathbb{P}(\hat{\mathcal{E}})$ that is not contained in a fiber of $\hat{\pi}$ such that, for a very general curve $\hat{C} \in p^*|mH|$:*

$$\hat{\zeta}|_{W\cap\hat{\pi}^{-1}(\hat{C})} \textit{ is nef and not big.}$$

This result essentially relies on [86, Lemma 3.4].

Proof. Denote by $\mu : \tilde{S} \to \hat{S}$ the minimal resolution of $\hat{S}$, by $\tilde{\mathcal{E}} := \mu^*\hat{\mathcal{E}}$, by $\tilde{\zeta}$ the tautological bundle of $\mathbb{P}(\tilde{\mathcal{E}})$. We have a Cartesian diagram with compatibility of tautological bundles:

$$\begin{tikzcd}
\mathbb{P}(\tilde{\mathcal{E}}) \arrow[r, "\mu'"] \arrow[d, "\tilde{\pi}"] & \mathbb{P}(\hat{\mathcal{E}}) \arrow[d, "\hat{\pi}"] \\
\tilde{S} \arrow[r, "\mu"] & \hat{S}
\end{tikzcd}$$

Note that $\mathbb{P}(\tilde{\mathcal{E}})$ with its tautological $\tilde{\zeta}$ is a smooth modification of $\mathbb{P}(\hat{\mathcal{E}})$ just as in Definition 2.28. Hence, $\tilde{\zeta}$ is pseudoeffective.

We suppose that $\hat{\zeta}$ is not nef. In particular, $\tilde{\zeta}$ is pseudoeffective but not nef. Let $Z \subset B_-(\tilde{\zeta})$ be an irreducible component of maximal dimension. Since $\tilde{\zeta}$ is pseudoeffective, Z is proper in $\mathbb{P}(\hat{E})$. Note that Z contains a $\tilde{\zeta}$-negative curve N: its image $\mu'(N)$ must be a $\hat{\zeta}$-negative curve, hence it is not in a fiber of $\hat{\pi}$. So $\hat{\pi}(\mu'(Z))$ is not a point in $\hat{S}$.

Now, for m big and divisible enough, for a general curve $\hat{C} \in p^*|mH|$,

- $\hat{C}$ is a smooth curve in $\hat{S}_{\text{reg}}$; in particular, μ is an isomorphism over $\hat{C}$;
- $Z \cap \mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))$ is non-empty, and proper in $\mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))$;
- $\hat{\mathcal{E}}|_{\hat{C}}$ is nef by Lemma 2.36 and Proposition 2.39 and it has $\hat{H}|_{\hat{C}}$-slope zero;
- hence, $\tilde{\zeta}|_{\mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))}$ is nef too, and moreover its top power is zero;
- hence, by [86, Lemma 3.4] (which applies with a void condition for $k = \operatorname{codim}_{\mathbb{P}(\tilde{\mathcal{E}})} Z$, since Z was chosen with maximal dimension, i.e., with minimal codimension):

$$0 = \left(\tilde{\zeta}|_{\mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))}\right)^{\dim \mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))} \geq \left(\tilde{\zeta}|_{Z\cap\mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))}\right)^{\dim Z\cap\mu'^{-1}(\hat{\pi}^{-1}(\hat{C}))} \geq 0.$$

As μ' is an isomorphism over $\hat{C}$, $W := \mu'(Z)$ works well as the closed proper subvariety of $\mathbb{P}(\hat{\mathcal{E}})$ we want to construct. $\square$

We now combine these lemmas to establish Proposition 4.1.

Proof of Proposition 4.1. Suppose by contradiction that $\hat{\mathcal{E}}$ is not nef. Then Lemma 4.4 yields a closed proper subvariety W of $\mathbb{P}(\hat{\mathcal{E}})$ which satisfies, for m big and divisible enough and for a very general curve $\hat{C} \in p^*|mH|$:

$$\varnothing \neq W \cap \hat{\pi}^{-1}(\hat{C}) \subsetneq \hat{\pi}^{-1}(\hat{C}) \text{ and } \hat{\zeta}|_{W \cap \hat{\pi}^{-1}(\hat{C})} \text{ is nef and not big.}$$

The first condition shows that $\hat{\pi}(W)$ is not a point. So Lemma 4.3 applies, hence $\hat{\zeta}|_{W \cap \hat{\pi}^{-1}(\hat{C})}$ is ample, contradiction! $\square$

CHAPTER 5

PROOF OF THEOREM 3.3 AND OF THEOREM 3.2

5.1 Proof of Theorem 3.3. As it follows from the discussion in Section 4.1, Theorem 3.3 is halfway. Here is what remains to prove:

Theorem 5.1. *Let X be a normal projective klt variety of dimension n with an ample $\mathbb{Q}$-Cartier divisor H. Let $\mathcal{E}$ be a reflexive sheaf on X, such that:*

- *$\mathcal{E}$ is H-semistable;*
- *the following equalities hold:*

$$\hat{c_1}(\mathcal{E}) \cdot H^{n-1} = \hat{c_1}^2(\mathcal{E}) \cdot H^{n-2} = \hat{c_2}(\mathcal{E}) \cdot H^{n-2} = 0.$$

Then there is a finite Galois morphism $\nu : \tilde{X} \to X$, étale in codimension 1, such that $\nu^{[*]}\mathcal{E}$ is a locally-free sheaf with numerically trivial determinant, and is $\mathrm{Gal}(\tilde{X}/X)$*-equivariantly flat. Consequentially, $\nu^{[*]}\mathcal{E}$ is numerically flat and its first and second Chern classes are numerically trivial.*

Proof. We apply [133, Theorem 1.4] to obtain a finite Galois morphism $\nu : \tilde{X} \to X$, étale over X_{reg}, such that $\nu^{[*]}\mathcal{E}$ is locally-free with a numerically trivial determinant and $\mathrm{Gal}(\tilde{X}/X)$-equivariantly flat.

Let then $\varepsilon : \tilde{X}' \to \tilde{X}$ be a resolution of $\tilde{X}$ and $\mathcal{E}' := \varepsilon^*\nu^{[*]}\mathcal{E}$, which is a flat locally-free sheaf with a numerically trivial determinant on $\tilde{X}'$. As shown in [86, Rmk.2.6], $\mathcal{E}'$ is then numerically flat and its Chern classes vanish (as cohomological classes on $\tilde{X}'$). By Prop.2.23, $\nu^{[*]}\mathcal{E}$ is nef, hence numerically flat. Moreover, for any $\mathbb{Q}$-Cartier divisors $D_1, \dots, D_{n-2}$,

$$c_2(\nu^{[*]}\mathcal{E}) \cdot D_1 \cdots D_{n-2} = c_2(\mathcal{E}') \cdot \varepsilon^* D_1 \cdots \varepsilon^* D_{n-2} = 0,$$

so the Chern classes of $\nu^{[*]}\mathcal{E}$ are trivial, which completes the proof of the theorem. □

5.2 Proof of Theorem 3.2. We give a few definitions along the lines of Theorem 3.1:

Definition 5.2. Let X be a normal projective canonical variety of dimension $n \geq 2$. It is called:

- a *Calabi-Yau* variety if $h^0(Y, \Omega_Y^{[q]}) = 0$ for all integers $1 \leq q \leq n-1$ and all quasiétale finite covers $Y \to X$;

- an *irreducible holomorphic symplectic (IHS)* variety if there is a reflexive form $\sigma \in H^0(X, \Omega_X^{[2]})$ such that, for any quasiétale finite cover $f : Y \to X$, the reflexive form $f^{[*]}\sigma$ generates $H^0(Y, \Omega_Y^{[\cdot]})$ as an algebra for the wedge product.

We use the terms singular Calabi-Yau (resp. IHS) variety and Calabi-Yau (resp. IHS) variety interchangeably, unless explicitly said otherwise. They may both accidentally denote smooth varieties.

Definition 5.3. For the sake of a consistent terminology, let us call a *singular K3 surface*, or for short a *K3 surface*, a normal projective klt surface which has no finite quasiétale cover by an abelian variety. Equivalently, it is a Calabi-Yau variety or an IHS variety of dimension 2.

Definition 5.4. For the sake of a convenient vocabulary, let us define the *augmented irregularity* $\tilde{q}(X)$ of a normal projective klt variety X with trivial canonical class as the maximum of all irregularities $q(Y)$ of finite quasiétale covers Y of X. Note that it is precisely the dimension of the abelian part in the singular Beauville-Bogomolov decomposition of X.

Let us now proceed to prove Theorem 3.2.

Proof of Theorem 3.2. Let X be a normal projective klt variety of dimension at least 2 with trivial canonical class. Suppose that $\Omega_X^{[1]}$ is pseudoeffective (the same whole argument works just alike for the tangent sheaf $\mathcal{T}_X$) and assume by contradiction that $\tilde{q}(X) = 0$.

The singular Beauville-Bogomolov decomposition then reads:

$$f : \tilde{X} \to X \text{ and } \tilde{X} \cong \prod_i Y_i \times \prod_j Z_j,$$

with the same notations as in Theorem 3.1.

Remember that $f^{[*]}\Omega_X^{[1]} = \Omega_{\tilde{X}}^{[1]}$, since reflexive sheaves are normal and there is a big open set over which f is just a finite étale cover. By Proposition 2.33, $\Omega_{\tilde{X}}^{[1]}$ is pseudoeffective; it splits according to the product defining $\tilde{X}$. So there is a factor Y (Calabi-Yau or IHS) of $\tilde{X}$ such that $\Omega_Y^{[1]}$ is pseudoeffective [86, inductive argument in Proof of Theorem 1.6]. Now, $\Omega_Y^{[1]}$ satisfies all hypotheses of Theorem 3.3, the stability assumptions coming from [73, Prop.8.20] and [70, Rmk.8.3].

As a consequence, for some ample polarization H on Y, $\hat{c}_2(\Omega_Y^{[1]}) \cdot H^{\dim Y} = 0$, so that Y has a finite quasiétale cover by an abelian variety by [133, Theorem 1.4], contradiction! □

Remark 5.5. This pseudoeffectiveness result can be considered as an interesting improvement of the effectiveness result [70, Theorem 11.1], which says that $\tilde{q}(X) = 0$ if and only if, for all $m \in \mathbb{N}$, $h^0(X, S^{[m]}\Omega_X^{[1]}) = 0$.

Examples for Theorem 3.2 are to search among normal projective klt varieties with trivial canonical class singularities in codimension 2, which are plethoric. But singular varieties whose decomposition is known are not so numerous; and, for sure, one shall understand the Beauville-Bogomolov type of a given variety before telling anything about the positivity of its reflexivized cotangent sheaf.

Example 5.6. A first example to which Theorem 3.2 applies is the following [70, Par.14.2.2]: let F be a Fano manifold on which a finite group G acts freely in codimension 1. Suppose there is a smooth G-invariant element Y in the linear system $|-K_F|$. Then, Y is a smooth Calabi-Yau variety with a G-action. If the volume form on Y is preserved by this action, then $X := Y/G$ is a normal projective klt variety with trivial canonical class, and the morphism $Y \to X$ has no ramification divisor, hence it is quasiétale. The fact that the decomposition of X consists of a smooth Calabi-Yau manifold Y guarantees that X is a singular Calabi-Yau variety, as presented in Definition 5.2.

Although X may well have singularities in codimension 2, they merely stem from its global quasiétale quotient structure. In particular, [86, Theorem 1.6] actually proves the non-pseudoeffectiveness of $\mathcal{T}_X$ and $\Omega_X^{[1]}$, namely because it applies to Y and converts onto X through Proposition 2.33. Hence, the example is quite shallow: it has no real need for the machinery dealing with singularities in codimension 2 that Theorem 3.2 is about.

In the next chapter, we present better examples for Theorem 3.2, namely Calabi-Yau threefolds with singularities in codimension 2 that are not constructed as global quasiétale quotients of varieties which are smooth in codimension 2.

CHAPTER 6

THREEFOLDS IN THEOREMS 3.1 AND 3.2

In Section 5.2, we defined singular Calabi-Yau and IHS varieties. It follows from basic linear algebra that IHS varieties must have even dimension. In particular, the singular Beauville-Bogomolov decomposition for a normal projective klt variety X of dimension 3 is quite simple: $\tilde{X}$ has to be one of the following:

- a smooth abelian variety;
- a product $S \times E$, where S is a K3 surface as in Definition 5.3 and E is a smooth elliptic curve;
- a Calabi-Yau variety.

The aforementioned [133, Theorem 1.4] provides a criterion for identifying the purely abelian case by computing $\hat{c}_2(X)$.

One is then left with two cases: the singular threefold X may arise from a product $S \times E$, in which case $\mathcal{T}_X$ and $\Omega_X^{[1]}$ are pseudoeffective because of the abelian factor E; alternatively, X can be a genuine singular Calabi-Yau threefold. This second possibility is hard to identify, but, when it happens, it may give new examples for Theorem 3.2.

The next subsection is devoted to providing a necessary condition for a normal projective klt threefold to be finitely quasiétaly covered by a product $S \times E$.

6.1 Products of a K3 surface and an elliptic curve.

We are going to prove the following result:

Proposition 6.1. *Let X be a normal projective klt threefold with trivial canonical class. Suppose its Beauville-Bogomolov decomposition is of the form*

$$\tilde{X} = S \times E,$$

where S is a K3 surface and E a smooth elliptic curve. Then X has fibrations:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S/G_S \\ E/G_E & \end{tikzcd}$$

where G_E and G_S are finite subgroups of $\mathrm{Aut}(E)$ *and* $\mathrm{Aut}(S)$*. In particular,* $\rho(X) \geq 2$.

Let us first state a weak uniqueness result, guaranteeing that the statement of Proposition 6.1 makes sense. It is straightforward from the proof of the Beauville-Bogomolov decomposition theorem.

Proposition 6.2. *Let X be a normal projective klt variety with trivial canonical class. Then the number, types and dimensions of the factors of a finite quasiétale covering $\tilde{X} \to X$ as in Theorem 3.1 do not depend on the choice of that covering.*

A finite quasiétale morphism is not necessarily a quotient map by a finite group action free in codimension 1. In the smooth case however, [10, Lemma p.9] allows us to assume that the finite étale decomposition morphism $p : \tilde{X} \to X$ is Galois. Let us state a partial singular analog:

Proposition 6.3. *Let X be a normal projective klt variety with trivial canonical class. Take a finite quasiétale covering $f : \tilde{X} \to X$ as in Theorem 3.1. Suppose that all Calabi-Yau factors of $\tilde{X}$ have even dimension. Then there is a finite quasiétale Galois morphism $f' : \tilde{Z} \to X$, so that $\tilde{Z}$ splits into factors in the same number, types, and dimensions as $\tilde{X}$.*

Proof. By [72, Theorem 1.5], we can take a finite quasiétale Galois covering $g : Y \to X$ such that any finite morphism $Z \to Y$ étale over Y_{reg} is étale over Y. By purity of the branch locus, any quasiétale morphism $Z \to Y$ is then étale.

Note that Y is still a normal projective klt variety with trivial canonical class, hence has a singular Beauville-Bogomolov decomposition $h : Z \to Y$. By Proposition 6.2, the factors of Z have the same type as those of $\tilde{X}$. It writes:

$$Z = A \times \prod_i Y_i \times \prod_j Z_j,$$

where A is an abelian variety, Y_i Calabi-Yau varieties and Z_j IHS varieties. Since all Y_i and, of course, all Z_j have even dimension, by [70, Cor.13.3], they are simply connected.

Hence, finite étale fundamental groups equal: $\widehat{\pi_1}(Z) \simeq \widehat{\pi_1}(A)$. That is to say, any finite étale cover of Z actually stems from a finite étale cover of A.

We now use [72, Theorem 3.16]: there is a finite Galois morphism $\gamma : \tilde{Z} \to Z$ such that $\Gamma = g \circ h \circ \gamma : \tilde{Z} \to X$ is finite Galois and ramifies where $g \circ h$ does. So Γ is still quasiétale, in particular $h \circ \gamma : \tilde{Z} \to Y$ is quasiétale too. By construction of Y, $h \circ \gamma$ is then étale, so that γ is étale. By construction of Z, one has:

$$\tilde{Z} = A' \times \prod_i Y_i \times \prod_j Z_j,$$

where A' is a finite étale cover of the abelian variety A. Finally, $\Gamma : \tilde{Z} \to X$ is finite Galois quasiétale, and $\tilde{Z}$ splits as mandated. □

Remark 6.4. The main obstacle for generalizing this proposition is the fact that fundamental groups of odd-dimensional Calabi-Yau varieties are poorly understood [70, Sect.13.2]; most notably, they may not be finite.

Here is the last ingredient for the proof of Proposition 6.1:

Lemma 6.5. *Let S be a K3 surface as in Definition 5.3, E a smooth elliptic curve. Then:*

$$\mathrm{Aut}(S \times E) \cong \mathrm{Aut}(S) \times \mathrm{Aut}(E).$$

Proof. Let $\tilde{S}$ be the minimal resolution of S. It is a smooth K3 surface, so $\mathrm{Aut}(\tilde{S})$ is discrete. Moreover, the uniqueness of minimal resolution implies that any automorphism of S lifts to an automorphism of $\tilde{S}$, and this is obviously an injection. Hence, $\mathrm{Aut}(S)$ is discrete.

Let us now copy the argument by [10, Lemma p.8]. Let $u \in \mathrm{Aut}(S \times E)$. Since the projection $p_E : S \times E \to E$ is the Albanese map of $S \times E$, we can factor $p_E \circ u$ by it: there is $v \in \mathrm{Aut}(E)$ such that $p_E \circ u = v \circ p_E$. Hence, there is a map $w : E \to \mathrm{Aut}(S)$ which decomposes:

$$u : (s, e) \in S \times E \mapsto (w_e(s), v(e)).$$

Since $\mathrm{Aut}(S)$ is discrete, the map w is constant, so $u = (w_0, v)$. □

Proof of Proposition 6.1. Let X be a normal projective variety of dimension 3 with trivial canonical class. Suppose that there is a finite quasiétale cover $f : S \times E \to X$, where S is a singular K3 surface and E a smooth elliptic curve. By Proposition 6.3, we can assume that there is a finite group G acting on $S \times E$ such that f is the induced quotient map. By Lemma 6.5, G can be considered a subgroup of $\mathrm{Aut}(S) \times \mathrm{Aut}(E)$. As it acts diagonally, we have the following diagram:

$$\begin{tikzcd}
S \times E \arrow[r, "p_S"] \arrow[d, "p_E"] \arrow[dr, "f"] & S \arrow[dr] & \\
E \arrow[dr] & X \arrow[r] \arrow[d] & S/G_S \\
& E/G_E &
\end{tikzcd}$$

so that $\rho(X)$ is at least 2. □

6.2 Calabi-Yau hypersurfaces in weighted projective spaces. The aim of this last part is to provide examples of Calabi-Yau threefolds that are singular along curves, by establishing the following result.

Proposition 6.6. *Let $\mathbb{P} = \mathbb{P}(w_0, \dots, w_4)$ be a weighted projective space and $d = w_0 + \dots + w_4$ such that there is a general wellformed quasismooth hypersurface X of degree d in $\mathbb{P}$. Suppose that X contains no edge of $\mathbb{P}$. Then X is a singular Calabi-Yau in the sense of Definition 5.2.*

A general exposition to complete intersections in weighted projective spaces can be found in [89]. We stick to its terminology.

Let $\mathbb{P} = \mathbb{P}(w_0, \dots, w_4)$ be a wellformed 4-dimensional weighted projective space. There is a ramified quotient map: $p : \mathbb{P}^n \to \mathbb{P}$, by the finite diagonal group action of $\bigoplus_i \mathbb{Z}_{w_i}$ on $\mathbb{P}^n$. With homogeneous coordinates on either side, we can write:

$$p : [x_0 : \ldots : x_n] \in \mathbb{P}^n \mapsto [y_0 = x_0^{w_0} : \ldots : y_n = x_n^{w_n}] \in \mathbb{P}.$$

We denote by $\mathcal{O}_{\mathbb{P}}(1)$ the ample $\mathbb{Q}$-Cartier divisor on $\mathbb{P}$ whose pullback by p is $\mathcal{O}_{\mathbb{P}^n}(1)$. If the linear system $|\mathcal{O}_{\mathbb{P}}(d)|$ contains a wellformed quasismooth hypersurface, it actually contains a Zariski-open set of such hypersurfaces and we write X_d for a general one.

Singularities of general quasismooth hypersurfaces of dimension 3. Let X be a general quasismooth hypersurface of degree d and of dimension 3 in the weighted projective space $\mathbb{P}$. Then X is a full suborbifold of $\mathbb{P}$ (see [24, Def.5] for a definition, [53, Theorem 3.1.6] for a proof). In particular, $X_{\text{sing}} = X \cap \mathbb{P}_{\text{sing}}$, and at any point $x \in X \cap \mathbb{P}_{\text{sing}}$, writing that $\mathbb{P}$ is locally isomorphic to a quotient $\mathbb{C}^4/G_x$, X is locally isomorphic to $\mathbb{C}^3/G_x$ in a compatible way with inclusions. Hence, X has only quotient singularities, so it is klt. The locus X_{sing} is a finite union of curves and points, which may be of various types:

- a *vertex* in $\mathbb{P}$ is a point with $y_i = 1$ for a single $i \in [\![0, 4]\!]$ and $y_j = 0$ for all $j \neq i$. If $w_i \neq 1$, this vertex is a singular point in $\mathbb{P}$. It gives rise to a singular point in X if and only if it lies in it, ie w_i does not divide d.

- an *edge* in $\mathbb{P}$ is a line with equation $y_j = 0$ for all $j \in J$, for a certain $J \subset [\![0, 4]\!]$ of cardinal 3. If $\gcd(w_j)_{j \notin J} \neq 1$, the edge is in $\mathbb{P}_{\text{sing}}$. Recall that X is taken general in its linear system. Hence, an edge in $\mathbb{P}$ lies entirely in X if and only if $(w_j)_{j \notin J}$ do not partition d, in X_{sing} if and only if $(w_j)_{j \notin J}$ do not partition d and have a non-trivial common divisor. If an edge in $\mathbb{P}_{\text{sing}}$ does not lie entirely in X, it gives a finite amount of points in X_{sing}.

- a *2-face* in $\mathbb{P}$ is a 2-plane with equation $y_j = 0$ for all $j \in J$, for a certain $J \subset [\![0, 4]\!]$ of cardinal 2. If $\gcd(w_j)_{j \notin J} \neq 1$, the 2-face is in $\mathbb{P}_{\text{sing}}$. By quasismoothness, no 2-face lies entirely in X. Hence, any 2-face intersects X along an effective 1-cycle. In this way, 2-faces in $\mathbb{P}_{\text{sing}}$ may produce curves in X_{sing}.

Under the additional hypothesis that X contains no edge of $\mathbb{P}$, we can say more about singular loci.

Indeed, it is worth noticing that the restricted quotient map $p^{-1}(X) \to X$ is an unfolding of X, as defined in Section 2.6; we may write $\hat{X}$ for $p^{-1}(X)$. For establishing Prop.6.6, we will prove the following:

Lemma 6.7. *Let X be a general wellformed quasismooth hypersurface of dimension 3 in a weighted projective space $\mathbb{P}$ not isomorphic to $\mathbb{P}^4$. Assume that X has trivial canonical class and that it contains no edge of $\mathbb{P}$. Then $\hat{c}_2(X) \cdot \mathcal{O}_X(1) > 0$.*

In the course of the proof of this lemma, we will use the fact that X containing no edge of $\mathbb{P}$, $\hat{X}$ is smooth in codimension 2.

Remark 6.8. Note that the restricted finite map $\hat{X} = p^{-1}(X) \to X$ is certainly ramified along divisors, so that X, in the lucky case where it happens to be a singular Calabi-Yau threefold, is not at all constructed as a finite quasiétale global quotient, contrarily to the unsatisfying Example 5.6.

The proof that $\hat{X}$ is smooth in codimension 2 relies on the following lemma and remark:

Lemma 6.9. *Let X be a general quasismooth hypersurface of degree d in the weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_4)$. Suppose that it contains no edge of $\mathbb{P}$. Then the base locus $\mathrm{Bs}(\mathcal{O}_{\mathbb{P}}(d))$ has dimension 0.*

Proof. Let Z be an irreducible component of the base locus of $\mathcal{O}_{\mathbb{P}}(d)$, let us prove by induction on $\dim \mathbb{P}$ that it is a point. Suppose we are at the induction step where the ambient space $\mathbb{P}'$ has local coordinates $y_0, y_1, y_2, \ldots$ and dimension 4, 3 or 2.

Denote by H_i the hyperplane $\{y_i = 0\}$ in $\mathbb{P}'$, by $\mathbb{P}'_i$ the isomorphic weighted projective space $\mathbb{P}'(\ldots, \hat{w}_i, \ldots)$. By [14, Prop.4.A.3], we have an isomorphism between the restriction $\mathcal{O}_{\mathbb{P}'}(d) \otimes \mathcal{O}_{H_i}$ and the $\mathbb{Q}$-Cartier divisor $\mathcal{O}_{\mathbb{P}'_i}(d)$. This translates to global sections as a surjection:

$$H^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(d)) \twoheadrightarrow H^0(\mathbb{P}'_i, \mathcal{O}_{\mathbb{P}'_i}(d)), \tag{6.1}$$

which is given by setting $y_i = 0$ when considering the global sections as certain polonomials in the local coordinates of $\mathbb{P}'$.

The quasismoothness of X in $\mathbb{P}$ and the way the composite surjection

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \twoheadrightarrow H^0(\mathbb{P}', \mathcal{O}_{\mathbb{P}'}(d)),$$

writes in local coordinates yield a global section of $\mathcal{O}_{\mathbb{P}'}(d)$ of the form $y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2}$. In particular, there is an $i = 0, 1$ or 2 such that $Z \subset H_i \simeq \mathbb{P}'_i$. Moreover, by Eq.6.1, Z sits in the base locus of $\mathcal{O}_{\mathbb{P}'_i}(d)$.

Induction propagates from $\mathbb{P}' = \mathbb{P}$ down to when we obtain that Z is contained in an edge H_{ijk} of $\mathbb{P}$ and in the base locus $\mathrm{Bs}(\mathcal{O}_{\mathbb{P}_{ijk}}(d)) \subset \mathrm{Bs}(\mathcal{O}_{\mathbb{P}}(d)) \subset X$. Since X contains no edge of $\mathbb{P}$, Z is in $X \cap H_{ijk}$ of dimension 0, so it is a point. □

Remark 6.10. With the same notations and hypotheses, the intersection of X with any 2-face of $\mathbb{P}_{\mathrm{sing}}$ is a reduced curve.

Proof. As in the proof of Lemma 6.9, the intersection is scheme-theoretically defined by a general section of $\mathcal{O}_{\mathbb{P}_{ij}}(d)$. We are to show that such general section of $\mathcal{O}_{\mathbb{P}_{ij}}(d)$ is quasismooth in the weighted projective space $\mathbb{P}_{ij}$, hence it is a variety by [89, 3.1.6].

We use the arithmetical criterion for quasismoothness: since X contains no edge of $\mathbb{P}$, each pair w_a, w_b partitions d. We are left to check the criterion for $k = 1$: fix any $a \neq i, j$, we want to find $b \neq i, j$ such that w_a divides $d - w_b$. It is clear that there is a $b \in [\![0, 4]\!]$ satisfying that. As H_{ij} is a 2-face in $\mathbb{P}_{\mathrm{sing}}$, the greatest common divisor of all weights except w_i, w_j is non-trivial, divides d but neither w_i nor w_j (by wellformedness). In particular, since this greatest common divisor divides $w_b = d - \alpha w_a$, $b \neq i, j$, as wished. □

We can now deduce:

Proposition 6.11. *Let X be a general quasismooth hypersurface of degree d in a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_4)$, p the natural quotient $\mathbb{P}^4 \to \mathbb{P}$, $\hat{X} = p^{-1}(X)$. Suppose that X contains no edge of $\mathbb{P}$. Then $\hat{X}$ is smooth in codimension 2.*

Proof. The threefold $\hat{X}$ is general in the linear system $p^*|\mathcal{O}_{\mathbb{P}}(d)|$, whose base locus has dimension 0 by Lemma 6.9. By Bertini's theorem, $\hat{X}$ is smooth in codimension 2. □

Remark 6.12. The converse of Proposition 6.11 does not hold: for instance, the general quasismooth X_7 in $\mathbb{P}(1, 1, 1, 2, 2)$ contains the edge of equation $y_0 = y_1 = y_2 = 0$, but its unfolding is nevertheless smooth in codimension 2.

Example 6.13. The hypothesis of Proposition 6.11 is not that X contains no edge of $\mathbb{P}_{\text{sing}}$, but that it contains no edge of $\mathbb{P}$ at all: for instance, consider the general $X = X_{56}$ in $\mathbb{P}(2, 4, 9, 13, 28)$. It contains a single edge of $\mathbb{P}$, namely e of equation $y_0 = y_1 = y_4 = 0$. This edge does not actually lie in $\mathbb{P}_{\text{sing}}$, as 9 and 13 are coprime, but one can check that $\hat{X}$ has the curve $p^{-1}(e)$ in its singular locus (by computing the derivatives of the equation defining $\hat{X}$ in $\mathbb{P}^4$ along the curve $p^{-1}(e)$).

Example 6.14. The general wellformed quasismooth hypersurface $X = X_{1734}$ in $\mathbb{P}(91, 96, 102, 578, 867)$ contains no edge of $\mathbb{P}$. In particular, $\hat{X}$ is smooth in codimension 2 by Proposition 6.11.

Moreover, the curves of X_{sing} are precisely the intersections of X with all 2-faces of $\mathbb{P}_{\text{sing}}$, which we can list:

- $y_0 = y_1 = 0$ of type $\frac{1}{17}(6, 11)$,
- $y_0 = y_3 = 0$ of type $\frac{1}{3}(1, 2)$,
- $y_0 = y_4 = 0$ of type $\frac{1}{2}(1, 1)$.

It is possible to check the type of singularities of a general hypersurface of a given degree in a given weighted projective space by a simple computer program.

Proof of Proposition 6.6. As we said before, the main ingredient in the proof is Lemma 6.7.

Proof of Lemma 6.7. Let $p : \mathbb{P}^4 \to \mathbb{P}$ be the natural quotient map. Writing $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_4)$ with $(w_0, \ldots, w_4)$ not colinear to $(1, \ldots, 1)$, the morphism p has degree $w_0 \cdots w_4$, which we denote by N, and X has degree $w_0 + \ldots + w_4$, which we denote by d. We may also write s for the symmmetric elementary polynomial of degree 2 in the weights and q for the sum of their squares: $d^2 = q + 2s$.

Since X is a full suborbifold of $\mathbb{P}$, $\hat{X} := p^{-1}(X) \to X$ is an unfolding of X as defined in Section 2.6. Applying the left-exact functor of reflexive pullback (see Lemma 2.15) to the exact sequence:

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}}|_X \to -K_{\mathbb{P}},$$

we get another exact sequence:

$$0 \to p^{[*]}\mathcal{T}_X \to p^{[*]}\mathcal{T}_{\mathbb{P}}|_X \to p^{[*]}(-K_{\mathbb{P}}) \to \mathcal{Z} \to 0,$$

where the coherent sheaf $\mathcal{Z}$ is supported on the locus $p^{-1}(\operatorname{Sing} X) \subset \hat{X}$ of codimension at least 2.

Because of the last surjection, $\dim_{k(p)} \mathcal{Z} \otimes \mathcal{O}_p \leq 1$ for any closed point $p \in \tilde{X}$.

By Proposition 6.11, the unfolding $\hat{X}$ is smooth in codimension 2, so the usual second Chern class $c_2(\mathcal{Z})$ makes sense. Since usual Chern classes are additive, and $c_1(\mathcal{T}_X) = 0, c_1(\mathcal{Z}) = 0$:

$$\hat{c}_2(\mathcal{T}_X) \cdot \mathcal{O}_X(1) = \hat{c}_2(\mathcal{T}_{\mathbb{P}}|_X) \cdot \mathcal{O}_X(1) + \frac{1}{N} c_2(\mathcal{Z}) \cdot \mathcal{O}_{\hat{X}}(1).$$

By the Miyaoka-Yau inequality [76, Theorem 1.5], we have a positive contribution:

$$\hat{c}_2(\mathcal{T}_{\mathbb{P}}|_X) \cdot \mathcal{O}_X(1) = \hat{c}_2(\mathcal{T}_{\mathbb{P}}) \cdot (-K_{\mathbb{P}}) \cdot \mathcal{O}_{\mathbb{P}}(1) \geq \frac{4}{10}(-K_{\mathbb{P}})^3 \cdot \mathcal{O}_{\mathbb{P}}(1) = \frac{4d^3}{10N}.$$

Let us estimate the other summand. Take m big and divisible enough that $\mathcal{O}_{\hat{X}}(m)$ is very ample and S a general element in $|\mathcal{O}_{\hat{X}}(m)|$. By [106, Lemma 10.9],

$$c_2(\mathcal{Z}) \cdot \mathcal{O}_{\hat{X}}(1) = \frac{1}{m} c_2(\mathcal{Z}|_S) = -\frac{1}{m}\deg(\mathcal{Z}|_S)$$

Denote by $C_1, \ldots C_k$ the curves in X_{sing}. By Lemma 6.15, we can bound:

$$\begin{aligned}
\deg(\mathcal{Z}|_S) \leq \operatorname{Card}\left(S \cap \bigcup_{i=1}^{k} p^{-1}(C_i)\right) &= \sum_{i=1}^{k} N\mathcal{O}_X(m) \cdot C_i \\
&\leq Nm\mathcal{O}_X(1)^3 \sum_{0 \leq i < j \leq 4} w_i w_j \\
&= mNs(-K_{\mathbb{P}}) \cdot \mathcal{O}_{\mathbb{P}}(1)^3 \\
&= msd.
\end{aligned}$$

Finally putting the positive and negative part together,

$$\begin{aligned}
\hat{c}_2(X) \cdot \mathcal{O}_X(1) &> \frac{4d^3 - 10sd}{10N} \\
&= \frac{d(4q - 2s)}{10N} \\
&= \frac{d}{10N} \sum_{0 \leq i < j \leq 4} (w_i - w_j)^2 > 0.
\end{aligned}$$

□

Lemma 6.15. *Let X be a general wellformed quasismooth hypersurface of dimension 3 in a weighted projective space $\mathbb{P}$. Assume that X has trivial canonical class and contains no edge of $\mathbb{P}$. Then there are at most 10 curves in X_{sing}, with different cohomological classes in the list of the*

$$[\mathcal{O}_X(w_i) \cdot \mathcal{O}_X(w_j)] \in H^4(X; \mathbb{Q}), \ \text{for } 0 \leq i < j \leq 4.$$

Proof of Lemma 6.15. By Remark 6.10, each curve in X_{sing} is scheme-theoretically the complete intersection of X with a 2-face H_{ij} of $\mathbb{P}_{\text{sing}}$. This association being bijective, there are as many curves in X_{sing} as 2-faces in $\mathbb{P}_{\text{sing}}$, so at most 10. The curve that corresponds to the 2-face H_{ij} has cohomological class $[\mathcal{O}_X(w_i) \cdot \mathcal{O}_X(w_j)]$. □

Now we can finally establish Proposition 6.6:

Proof. Consider X a general wellformed quasismooth hypersurface of degree $d = w_0 + \ldots + w_4$ in a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_4)$. Suppose that X contains no edge of $\mathbb{P}$. If $\mathbb{P}$ is $\mathbb{P}^4$, X is smooth and there is nothing to prove. Let us assume $\mathbb{P} \ncong \mathbb{P}^4$. By Lemma 6.7, $\hat{c}_2(X) \cdot \mathcal{O}_X(1) \neq 0$, hence by [133, Theorem 1.4], X is not a finite quotient of an abelian threefold. Moreover, one has $\operatorname{Pic}(X) \simeq \mathbb{Z}$ [53, Theorem 3.2.4(i)], so Proposition 6.1 applies to X: it is not covered by a product of a K3 surface and an elliptic curve, hence its Beauville-Bogomolov decomposition consists of a single Calabi-Yau factor. By Lemmma 6.16, so X itself is a Calabi-Yau variety, in the sense of Definition 5.2. In particular, X has canonical (and not merely klt) singularities. □

Lemma 6.16. *Let X be a general quasismooth hypersurface in a weighted projective space $\mathbb{P}$. Then any finite quasiétale cover X' of X is trivial.*

Proof. Let X' be a finite quasiétale cover of X of degree d; note that by Zariski purity of branch locus, it is étale over X_{reg}. Let $C_X^* \subset \mathbb{C}^{n+1} \setminus \{0\}$ be the smooth cone over X, with the projection $q : C_X^* \to X$. The morphism $C' = X' \underset{X}{\times} C_X^* \to C_X^*$ is finite of degree d and étale over the big open set $q^{-1}(X_{\mathrm{reg}}) \subset C_X^*$. Normalizing, the map $\tilde{C}' \to C_X^*$ has degree d and is étale over a big open set as well. As C_X^* is smooth, this map is actually finite étale; by [53, Lemma 3.2.2(ii)], $\pi_1^{\text{ét}}(C_X^*) = \{1\}$ so $d = 1$. □

Examples for Proposition 6.6. General wellformed quasismooth hypersurfaces with trivial canonical class in 4-dimensional weighted projective spaces are classified in [116]. There is an explicit exhaustive list of the 7555 of them. In this list, 7238 elements are not smooth in codimension 2, and 2409 elements that are not smooth in codimension 2 also contain no edge of their ambient weighted projective space. These elements fulfill the hypotheses for Proposition 6.6, just as Example 6.14 did: so they are singular Calabi-Yau threefolds to which Theorem 3.2 applies.

The exhaustive enumerations of elements of the [116] classification satisfying additional properties were done by running a simple computer program on the database [115]. The program is available on my webpage https://math.unice.fr/~gachet/research.html.

Remark 6.17. For the sake of transparent terminology, let us explain why the varieties studied in [116] are the same as general quasismooth wellformed hypersurfaces of trivial canonical class in a 4-dimensional weighted projective space.

First, any variety that [116] calls a nondegenerate Calabi-Yau hypersurface is sitting in an open set of nondegenerate Calabi-Yau hypersurfaces of a given linear system. This is precisely what we refered to as a general quasismooth hypersurface of trivial canonical class in a weighted projective space.

The paper [116] classifies " tuples "of positive integers $d, \{\{w_0, \ldots, w_N\}\}$ such that:

- there are no " trivial variables ", ie $\frac{d}{2} \notin \{\{w_0, \ldots, w_N\}\}$,
- $N = 3$ or 4,
- there is a nondegenerate Calabi-Yau hypersurface of degree d in $\mathbb{P}(w_0, \ldots, w_N)$ with condition " $c = 9$ ".

Here, we use $\{\{\cdot\}\}$ to denote tuples where order does not matter, or equivalently sets where elements may appear with a certain multiplicity.

We claim that the map f:

$$d, \{\{w_0, \ldots, w_N\}\} \mapsto \begin{cases} d, \{\{w_0, \ldots, w_3, \frac{d}{2}\}\} & \text{if } N = 3 \\ d, \{\{w_0, \ldots, w_4\}\} & \text{else,} \end{cases}$$

is a one-to-one correspondence between the data of [116] and all tuples $d, \{\{w_0, \ldots, w_4\}\}$ such that $X_d \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ is a general quasismooth wellformed hypersurface with trivial canonical class. To prove this claim, let us compute the image of this injective map f: a tuple $d, \{\{w_0, \ldots, w_4\}\}$ is in the image of f if and only if at most one of the w_i equals $\frac{d}{2}$ and the general hypersurface of degree d in the projective space of weights $\{\{w_i \mid 2w_i \neq d\}\}$ is quasismooth, has trivial canonical class and satisfies the rewritten condition $c = 9$:

$$\sum_{i=0}^{4} 1 - \frac{2w_i}{d} = 3, \text{ ie } \sum_{i=0}^{4} w_i = d.$$

So, it is clear that the image by f of the [116] tuples with $N = 4$ is made of all $d, \{\!\{w_0, \ldots, w_4\}\!\}$ such that $X_d \subset \mathbb{P}(w_0, \ldots, w_4)$ is a general quasismooth wellformed hypersurface of trivial canonical class, and for all i, $2w_i \neq d$.

The image by f of the tuples with $N = 3$ is easily checked to stand for quasismooth hypersurfaces in weighted projective spaces of dimension 4. We check that the quasismooth hypersurfaces arising in that way are wellformed by a careful application of the criterion [89, 6.13], [51, Prop.2], together with elementary arithmetic. Notably, the fact that

$$\sum_{i=0}^{3} w_i = \frac{d}{2}$$

together with the quasismoothness conditions implies that either for all i,

$$\gcd(w_0, \ldots, \hat{w}_i, \ldots, w_3) = 1,$$

or $d \equiv 2 \bmod 4$ and for all i, $\gcd(w_0, \ldots, \hat{w}_i, \ldots, w_3) = 1$ or 2 helps to apply the criterion. As adjunction formula holds, these general quasismooth wellformed hypersurfaces in weighted projective spaces of dimension 4 have trivial canonical class. Conversely, precisely those general quasismooth wellformed hypersurfaces X_d of trivial canonical class in a $\mathbb{P}(w_0, \ldots, w_3, \frac{d}{2})$ are in the image by f of the $N = 3$ data.

Example 6.18. The wellformed quasismooth Calabi-Yau hypersurface X_{1734} in

$$\mathbb{P}(91, 96, 102, 578, 867)$$

comes from the $N = 3$ data (originally denoted $n = 4$) in [116], since $1734 = 2 \times 867$. The wellformed quasismooth Calabi-Yau hypersurface X_{120} in $\mathbb{P}(3, 7, 20, 40, 50)$ comes from the $N = 4$ data (originally denoted $n = 5$).

PART II

FINITE QUOTIENTS OF ABELIAN VARIETIES WITH A CALABI-YAU RESOLUTION

CHAPTER 7
INTRODUCTION

Since singularities are a byproduct of the Minimal Model Program, studying singular varieties with trivial canonical class, or singular K-trivial varieties, is an important question in the birational classification of complex algebraic varieties. From this point of view, the recent generalization of the Beauville-Bogomolov decomposition theorem for smooth K-trivial varieties ([11]) to klt K-trivial varieties ([70, 55, 86, 8]) is highly relevant. It indeed establishes that, after a finite quasiétale cover, any klt K-trivial variety is a product of a smooth abelian variety, some irreducible holomorphic symplectic varieties with canonical singularities, also called hyperkähler varieties, and some Calabi-Yau varieties with canonical singularities. These three main families of K-trivial varieties are the subject of large, mostly disjoint realms of the literature, ranging from the well-known theory of abelian varieties (exposed notably in the reference books [17, 184]), through the thriving study of hyperkähler varieties (see [41, 1, 96] for surveys), to the unruly "Zoo of Calabi-Yau varieties", populated by a huge amount of examples ([117, 118] for K3 surfaces and Calabi-Yau threefolds embedded as hypersurfaces in toric varieties only), and whose boundedness is yet not established (see [203, 204, 33, 48, 16] for recent breakthroughs).

A new feature appearing in the context of singular K-trivial varieties is that birational morphisms may change the type of the Beauville-Bogomolov decomposition. For example, Kummer surfaces are $K3$ surfaces, but arise as minimal resolutions of finite quasiétale quotients of abelian surfaces. Similar examples of dimension 3 are numerous, as in [155, 154, 153], and even less well understood in higher dimensions, *cf.* [39, 40, 163, 5, 25]. In arbitrary dimension, it is known that a crepant resolution or terminalization only changes the type of a klt K-trivial variety if its decomposition entails an abelian factor ([55, Prop.4.10]).[1]

This paper aims at describing changes of the type of a K-trivial variety through a birational morphism in the simplest case of higher dimension, i.e., when a singular variety with Beauville-Bogomolov decomposition of purely abelian type is resolved by a Calabi-Yau manifold. We work in the following set-up: By a Calabi-Yau manifold, we mean a smooth simply-connected complex projective variety of dimension n with trivial canonical bundle, without any global holomorphic differential form of degree $i \in [\![1, n-1]\!]$. Extending the terminology of [152], we define n-dimensional Calabi-Yau

[1] It reflects the more general fact that the Beauville-Bogomolov decomposition type of a klt K-trivial variety X with non-trivial fundamental group $\pi_1(X_{\mathrm{reg}})$ is not captured by its algebra of global holomorphic differential forms $H^0\left(X, \Omega_X{}^{[\cdot]}\right)$. Many examples supporting this fact are exposed in [70, Sec.14], and most notably, smooth K-trivial threefolds with Beauville-Bogomolov decomposition of pure abelian type and algebra of global differential forms generated by the volume form (as for a Calabi-Yau threefold) are classified in [159].

manifolds of type n_0 as follows.

Theorem 7.1. Theorem 2.119 *Let X be a Calabi-Yau manifold of dimension n. The following are equivalent:*

(i) *There is a nef and big divisor D on X such that $c_2(X) \cdot D^{n-2} = 0$.*

(ii) *There is an abelian variety A and a finite group G acting freely in codimension 2 on A such that X is a crepant resolution of A/G.*

If it satisfies these conditions, X is called a Calabi-Yau manifold of type n_0.

Calabi-Yau threefolds of type III_0 appear naturally when classifying extremal contractions of Calabi-Yau threefolds [152], and fit in a more general circle of ideas on how the cubic intersection form and the second Chern class determine the birational geometry of a Calabi-Yau threefold (see, e.g., the work of Wilson [205], Oguiso and Peternell [157]). Calabi-Yau threefolds of type III_0 were classified by Oguiso, as we now recall.

Theorem 7.2. [155] *There are exactly two Calabi-Yau threefolds X_3, X_7 of type III_0. They are the unique crepant resolution of ${E_j}^3$ quotiented by the group generated by $j\mathrm{id}_3$, and of ${E_{u_7}}^3$ quotiented by the group generated by:*

$$\begin{pmatrix} 0 & -8 & 7-10u_7 \\ 1 & -6-2u_7 & 11-u_7 \\ 0 & -1-2u_7 & 6+3u_7 \end{pmatrix}.$$

where $j = e^{2i\pi/3}$, $\zeta_7 = e^{2i\pi/7}$, $u_7 = \zeta_7 + {\zeta_7}^2 + {\zeta_7}^4 = \frac{-1+i\sqrt{7}}{2}$, and for any complex number $z \in \mathbb{C} \setminus \mathbb{R}$, we denote by E_z the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$.

Our first theorem restricts the isogeny type of A in arbitrary dimension.

Theorem 7.3. *Let A be an abelian variety of dimension n and G be a finite group acting freely in codimension 2 on A. If A/G has a crepant resolution that is a Calabi-Yau manifold, then A is isogenous to ${E_j}^n$ or to ${E_{u_7}}^n$ and G is generated by its elements that admit fixed points in A.*

Moreover, the local geometry of A/G is generally quite similar to the 3-dimensional model (see Theorem 7.6 below). Two important consequences of this are the following results.

Theorem 7.4. *Let A be an abelian variety and G be a finite group acting freely in codimension 3 on A. Then A/G has no simply-connected crepant resolution.*

Theorem 7.5. *Let A be an abelian variety and G be a finite group acting freely in codimension 2 on A. If A/G has a simply-connected crepant resolution, then $\dim(A) \neq 4$.*

Although local arguments are crucial to the proofs of these two results, they are not sufficient to conclude on their own, and we have to resort to global arguments involving the action on the abelian variety in the proofs.

On the one hand, the Calabi-Yau assumption is crucial in Theorem 7.3, as it rules out products of the 3-dimensional examples of Oguiso, e.g., $X_3 \times X_7$, which is a resolution of a finite quotient of ${E_j}^3 \times {E_{u_7}}^3$. On the other hand, Theorem 7.5 merely requires the simply-connectedness of a crepant resolution. Let us explain why. Note that, if A is an abelian variety and G is a finite group acting freely in codimension 2 on A, then A/G cannot have a holomorphic symplectic resolution X. Indeed, a holomorphic symplectic resolution provides $(A/G)_{\mathrm{reg}}$ with a holomorphic symplectic form. By [149, Thm, Cor.1] then, since A/G is smooth in codimension 2, it is terminal. As it is $\mathbb{Q}$-factorial as well, it thus admits no crepant resolution. By the Beauville-Bogomolov decomposition theorem, a smooth simply-connected K-trivial fourfold which is not holomorphic symplectic is a Calabi-Yau fourfold, whence the weaker assumption of Theorem 7.6.

The structure of the paper is as follows. Sections 8 to 15.2 build up to the proof of the main technical result.

Theorem 7.6. *Let A be an abelian variety of dimension n and G be a finite group acting freely in codimension 2 on A. If A/G has a crepant resolution that is a Calabi-Yau manifold, then*

(1) *A is isogenous to ${E_j}^n$ or to ${E_{u_7}}^n$, and G is generated by its elements that admit fixed points in A.*

(2) *For every translated abelian subvariety $W \subset A$, there is $k \in \mathbb{N}$ such that the pointwise stabilizer*

$$\mathrm{PStab}(W) := \{g \in G \mid \forall w \in W,\, g(w) = w\}$$

is isomorphic to ${\mathbb{Z}_3}^k$ if A is isogenous to ${E_j}^n$, or to ${\mathbb{Z}_7}^k$ if A is isogenous to ${E_{u_7}}^n$.

(3) *For every translated abelian subvariety $W \subset A$, if $\mathrm{PStab}(W)$ is isomorphic to*

- *${\mathbb{Z}_3}^k$, then there are k generators of it such that their matrices are similar to $\mathrm{diag}(\mathbf{1}_{n-3}, j, j, j)$, and the j-eigenspaces of these matrices are in direct sum.*
- *${\mathbb{Z}_7}^k$, then there are k generators of it such that their matrices are similar to $\mathrm{diag}(\mathbf{1}_{n-3}, \zeta_7, {\zeta_7}^2, {\zeta_7}^4)$, and all eigenspaces of these matrices with eigenvalues other than 1 are in direct sum.*

Our starting point in Section 8 is a necessary condition for a local quotient singularity to have a crepant resolution. The result is the following (Proposition 8.4): If $H \subset \mathrm{GL}_n(\mathbb{C})$ is a finite group, and $0 \in U \subset \mathbb{C}^n$ is an H-stable analytic open set such that U/H admits a crepant resolution, then H is generated by its so-called *junior* elements, i.e., elements M with eigenvalues $(e^{2i\pi a_k/d})_{1 \leq k \leq n}$ satisfying $0 \leq a_k \leq d-1$ and $\sum a_k = d$.

Matrices inducing actions on abelian varieties satisfy a rationality requirement [17, 1.2.3], which translates into arithmetic constraints on their characteristic polynomial. These constraints allow us to classify matrices of junior elements g acting on n-dimensional abelian varieties up to similarity: In Section 9, we prove that if a junior element g acts on an abelian variety in a way that the generated group $\langle g \rangle$ acts freely in codimension 2, then the matrix of g is of one of twelve possible types (see Proposition 9.2). In particular, the order of g and the number of non-trivial eigenvalues of g are bounded independently of the dimension n.

The next step is to show that ten out of the twelve types of junior elements can not belong to G, for a mix of local and global reasons. The proof spreads throughout

Chapters 10, 11, 13 and 14. Let us sketch the idea of the argument in the simplest case, namely if g is a junior element of composite order other than 6, with at most four non-trivial eigenvalues. If such a junior element g belongs to G, then some non-trivial power g^α is not junior, and has a larger fixed locus in A. Fix an irreducible component W of that larger fixed locus that is not in the fixed locus of g: the pointwise stabilizer $\mathrm{PStab}(W) \subset G$ does not contain g, but the power g^α. Now, as W has codimension less than 4, Chapter 10 shows that $\mathrm{PStab}(W)$ is cyclic generated by one junior element h, and thus, up to possibly replacing h by another junior generator of $\mathrm{Fix}(W)$, one has $g^\alpha = h^\alpha$. For well-chosen α, this is enough to yield $g = h$, and a contradiction.

This idea excludes seven out of the twelve types of junior elements (see Subsection 11.1). The three types of junior elements of order 6 are excluded by technical variations in the next sections. Ruling them out works along with classifying pointwise stabilizers in higher codimension: In codimension 4, Chapter 10 establishes cyclicity of the pointwise stabilizers and Section 11 deduces that junior elements with four non-trivial eigenvalues do not exist; in codimension 5 (Section 13), we first prove that junior elements with five non-trivial eigenvalues do not exist (Subsection 13.1), then deduce cyclicity of the pointwise stabilizers (Subsection 13.2). In codimension 6 (Section 14), we first classify pointwise stabilizers which do not contain junior elements of type $\mathrm{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$: they are isomorphic to $\mathbb{Z}_3, \mathbb{Z}_7, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_7 \times \mathbb{Z}_7$, or $\mathrm{SL}_2(\mathbb{F}_3)$ (Subsection 14.1). We use this partial classification to rule out junior elements with six non-trivial eigenvalues (Subsection 14.2), and we then finally refine the study of pointwise stabilizers in codimension 6 by ruling out $\mathrm{SL}_2(\mathbb{F}_3)$ (Subsection 14.3).

There finally remain two types of possible junior elements, which are those already appearing in dimension 3 in [155]: $\mathrm{diag}(\mathbf{1}_{n-3}, j, j, j)$ and $\mathrm{diag}(\mathbf{1}_{n-3}, \zeta_7, {\zeta_7}^2, {\zeta_7}^4)$.

This description of pointwise stabilizers in codimension up to 6 implies that any two junior elements admitting a common fixed point commute. Together with a simple argument about the isogeny type of A (see Section 12), it concludes the proof of Theorem 7.6. In fact, the idea that the existence of certain automorphisms on an abelian variety determines the isomorphism type of some special abelian subvarieties is general ([184]), and it applies crucially throughout this paper, starting in Chapter 10. From there, it is not so surprising that we are able to determine the isogeny type of A, interpreting the fact that A/G admits a Calabi-Yau resolution as an irreducibility property of the G-equivariant Poincaré decomposition of A.

Under the additional assumption that the group G is abelian, Theorem 7.6 and the results of Section 12 suffice to generalize Theorem 7.5 to higher dimensions, i.e., to the statement that, if A is an abelian variety of dimension n and G is a finite group acting freely in codimension 2 on A such that A/G admits a Calabi-Yau resolution X, then $n = 3$ and X is X_3 or X_7.

Also note that G is abelian if and only if any two junior elements g, h of G commute, which by our results can be checked *via* their matrices acting on a vector space V of dimension 3, 4, 5, or 6. Standard finite group theory allows us to explicitly bound the order of $\langle g, h \rangle$ depending on this dimension and the isogeny type of A. If the dimension is 3 or 4, the bounds are reasonable enough to launch a computer-assisted search through all possible abstract groups $\langle g, h \rangle$. Among these, the only groups which, in a faithful 3 or 4-dimensional representation, are generated by two junior elements of the same type, are $\mathbb{Z}_3$, $\mathbb{Z}_7$, and the finite simple group $\mathrm{SL}_3(\mathbb{F}_2)$ of order 168. But a geometric argument on fixed loci excludes $\mathrm{SL}_3(\mathbb{F}_2)$, whence the wished contradiction. This reproves the classification of [155] in dimension 3, and settles Theorem 7.5.

When V has dimension 5 or 6, we could also bound the order of $\langle g, h \rangle$ explicitly. For example, we could consider the image of the faithful representation $M \oplus \overline{M}$ in $\mathrm{SL}_{2\dim(V)}(\mathbb{Q})$, and use the classification of irreducible maximal finite integral matrix groups in dimension less than 12 by V. Felsch, G. Nebe, W. Plesken, and B. Souvignier to obtain a bound on the order of $\langle g, h \rangle$. But the bounds obtained in this way are too large for the `SmallGroup` library. One needs to better understand the arising matrix groups of larger order, and build a reasonably smaller finite list of possibilities for the abstract group $\langle g, h \rangle$. It will then remain to figure out geometric ways for ruling out those potential groups in the list other than $\mathbb{Z}_3$, $\mathbb{Z}_7$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_7 \times \mathbb{Z}_7$.

Some of our proofs resort to computer-searches among all finite groups of certain fixed orders (relying on the `SmallGroup` library of `GAP`). The computer-assisted results used in Subsection 10.3 were actually originally proven by hand using elementary representation theory and Sylow theory. Such arguments being standard in finite group theory, we chose to keep their exposition concise for the sake of readability, and preferred invoking computer-checked facts as black boxes when needed. This approach also has the advantage of better separating abstract group-theoretic arguments on G from properties of the particular representation $G \hookrightarrow \mathrm{GL}(H^0(T_A))$. All programs used are available in the Appendix.

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CHAPTER 8

SOME RESULTS IN MCKAY CORRESPONDENCE

Let us generalize the notion of a junior element, from matrices to automorphisms of abelian varieties.

Definition 8.1. If A is an abelian variety of dimension n and $g \in \mathrm{Aut}(A)$ has finite order, then g can be written as:

$$g : [z] \in A \mapsto [M(g)z + T(g)] \in A,$$

where $M(g)$ is a matrix of finite order in $GL_n(\mathbb{C})$, $T(g)$ a vector in $\mathbb{C}^n$. If g fixes any point a of A, it can be represented locally in a neighborhood of a by its matrix $M(g)$. Hence, it makes sense to say that the automorphism g is *junior* if g fixes at least one point in A and the matrix $M(g)$ is junior.

Remark 8.2. Note that if $g \in \mathrm{Aut}(A)$ admits a fixed point, then $\langle g \rangle$ contains no translation, so g and its matrix $M(g)$ have the same order.

Junior elements play a key role in the study of finite quotient singularities, as the previously mentioned Theorem 2.62 emphasizes. Quotient singularities are $\mathbb{Q}$-factorial, so they can not be resolved by small birational morphisms. This yields a simple corollary of Theorem 2.62.

Corollary 8.3. [94] *Let $\mathbb{C}^n/G$ be a finite Gorenstein quotient singularity, with G acting freely in codimension 1. If the singularity $\mathbb{C}^n/G$ admits a crepant resolution, then there is a junior element $g \in G$.*

In fact, [94, Par.4.5] conjectures that under the same hypotheses, if the singularity $\mathbb{C}^n/G$ admits a crepant resolution, then any maximal cyclic subgroup of G contains a junior element. A counterexample to this conjecture is however presented in Remark 14.15. In this section, we prove a weak version of that conjecture. We phrase it in an analytic set-up, as our later applications call for that, but the proof works in the affine set-up just as well.

Proposition 8.4. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group acting freely in codimension 1 on $\mathbb{C}^n$, and let $U \subset \mathbb{C}^n$ be a G-stable simply-connected analytic neighborhood of $0 \in \mathbb{C}^n$. If the singularity U/G admits a crepant resolution X_G, then the group G is generated by junior elements.*

Note that a singularity admitting a crepant resolution is Gorenstein. By [103][199], the existence of a crepant resolution X_G thus implies that $G \subset \mathrm{SL}_n(\mathbb{C})$.

In order to prove the proposition, we need some background in valuation theory. We refer for that to Section 2.8.

8.1 Proof of Proposition 8.4.

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ acting freely in codimension 1 on $\mathbb{C}^n$, and U be a G-stable simply-connected neighborhood of $0 \in \mathbb{C}^n$. Suppose that U/G has a crepant resolution X_G.

Set G_0 to be the subgroup of G generated by all junior elements. We have the following commutative diagram, constructed from the lower row up:

$$\begin{tikzcd} X \arrow[r, "\tilde{p}"] \arrow[d, "\varepsilon"'] & X_0 \arrow[r, "\tilde{q}"] \arrow[d, "\varepsilon_0"'] & X_G \arrow[d, "\varepsilon_G"] \\ U \arrow[r, "p"'] & U/G_0 \arrow[r, "q"'] & U/G \end{tikzcd}$$

The commutative squares containing the normal complex analytic varieties X_0, X are obtained by taking normalized fibred products. Since quotient singularities are locally $\mathbb{Q}$-factorial, all birational morphisms considered here are divisorial. The morphisms $p, q, \tilde{p}, \tilde{q}$ are finite, and $\varepsilon, \varepsilon_0, \varepsilon_G$ are proper birational.

The key fact is the following.

Lemma 8.5. *The prime exceptional divisors of ε_0 are crepant.*

Proof. Let E_0 be a prime exceptional divisor of ε_0, and denote by E its image in X_G. Since E is an exceptional divisor of ε_G, it is crepant. Let F be a prime exceptional divisor of ε dominating E_0. By Theorem 2.62 and Remark 2.63, there is a junior element $f \in G$ such that $v_F = v_f$. Since G_0 is generated by the junior elements of G, we have $f \in G_0$. We can compute the following ramification index.

$$|\mathrm{Ram}(E_0/E)| = \frac{|\mathrm{Ram}(F/E)|}{|\mathrm{Ram}(F/E_0)|} = \frac{|\mathrm{Ram}(v_f, k(U/G)|}{|\mathrm{Ram}(v_f; k(U/G_0))|} = \frac{|\langle f\rangle \cap G|}{|\langle f\rangle \cap G_0|} = 1$$

so E_0 is generically étale over E, hence crepant [111, Prop.5.20]. □

By this lemma, the finite proper morphism $\tilde{q} : X_0 \to X_G$ has no ramification divisor. By Zariski purity of the branch locus, since X_G is smooth, the morphism $\tilde{q}$ is unramified, hence étale by [81, Ex.III.10.3, Ex.III.10.9].

On the other hand, X_G is locally simply-connected by [107, Thm.7.5.2]: There is a contractible neighborhood V of $0 \in U/G$, such that $\varepsilon_G^{-1}(V)$ is simply-connected. Hence the following commutative diagram.

$$\begin{tikzcd} \varepsilon_0^{-1}(q^{-1}(V)) \arrow[r, "\tilde{q}"'] \arrow[d, "\varepsilon_0"] & \varepsilon_G^{-1}(V) \arrow[d, "\varepsilon_G"] \\ q^{-1}(V) \arrow[r, "q"'] & V \end{tikzcd}$$

As $\tilde{q}$ is étale, the pre-image $\varepsilon_0^{-1}(q^{-1}(V))$ is a disjoint union of $\deg(\tilde{q})$ copies of $\varepsilon_G^{-1}(V)$. Nevertheless, the morphism ε_0 has connected fibers and the base $q^{-1}(V)$ is itself connected, hence $\varepsilon_0^{-1}(q^{-1}(V))$ is connected, and

$$\deg(\tilde{q}) = \frac{|G|}{|G_0|} = 1,$$

so $G_0 = G$ and the proof of Proposition 8.4 is settled.

8.2 Global result along the same lines. We close this section with a global result along the same lines as Proposition 8.4.

Lemma 8.6. *Let G be a finite group acting freely in codimension 1 on an abelian variety A. Suppose that A/G has a resolution X_G that is simply-connected. Then G is generated by its elements admitting fixed points in A.*

Proof. Let $G_0 \triangleleft G$ be the normal subgroup of G generated by elements admitting fixed points. We want to prove that $G_0 = G$. We have a commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, "\tilde{q}"] \arrow[d, "\varepsilon_0"] & X_G \arrow[d, "\varepsilon_G"] \\ A/G_0 \arrow[r, "q"] & A/G \end{tikzcd}$$

By definition of G_0, for every $a \in A$, the stabilizers of a in G and G_0 coincide. Hence, q is étale, and $\tilde{q}$ is étale too by base change. But X_G is simply-connected and X_0 is connected, so $\deg(\tilde{q}) = 1$ and $G_0 = G$. □

Remark 8.7. If G is a finite group acting freely in codimension 1 on an abelian variety A so that A/G has a simply-connected crepant resolution, then G may still contain elements that admit no fixed point. Without loss of generality, we can assume that G contains no translation, up to replacing A by an isogenous abelian variety, but that is the best we can do.

CHAPTER 9

THE TWELVE TYPES OF JUNIOR ELEMENTS ON AN ABELIAN VARIETY

Section 8 just shows that, if we want a finite singular quotient of an abelian variety A/G to have a crepant resolution, the group G must contain some junior elements. The fact that in our set-up, G must also act freely in codimension 2 on A is restrictive enough that there are only twelve possibilities for the ranked vector of eigenvalues of a junior element $g \in G$.

Definition 9.1. Let g be a matrix in $\mathrm{GL}_n(\mathbb{C})$. Assume that it has finite order d. Since $g^d = \mathrm{id}$, g is diagonalizable and has eigenvalues of the form $e^{2i\pi a_k/d}$, for integers $a_k \in [\![0, d-1]\!]$ satisfying $a_1 \wedge \ldots \wedge a_n \wedge d = 1$. Ordering the integers a_k increasingly, we define the *ranked vector of eigenvalues* of g as the tuple $(e^{2i\pi a_k/d})_{1 \leq k \leq n}$.

Proposition 9.2. *Let A be an abelian variety of dimension n, and $g \in \mathrm{Aut}(A)$ be a junior element such that $\langle g \rangle$ acts freely in codimension 2. Then the order d of g and the ranked vector of eigenvalues of g are in one of the twelve columns of Table 9.1.*

d	3	4	6
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-3}, j, j, j)$	$(\mathbf{1}_{n-4}, i, i, i, i)$	$(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$
d	6	6	7
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$	$(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$	$\left(\mathbf{1}_{n-3}, \zeta_7, {\zeta_7}^2, {\zeta_7}^4\right)$
d	8	12	15
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-4}, \zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3)$	$(\mathbf{1}_{n-4}, \zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5)$	$(\mathbf{1}_{n-4}, \zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$
d	16	20	24
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-4}, \zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7)$	$(\mathbf{1}_{n-4}, \zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9)$	$(\mathbf{1}_{n-4}, \zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11})$

Table 9.1: Possible ranked vectors of eigenvalues for junior elements in G

For $d \in \mathbb{N}$, we denote $\zeta_d = e^{2i\pi/d}$, and in particular $j = e^{2i\pi/3}$ and $\omega = e^{2i\pi/6}$. For $k \in \mathbb{N}$, $\mathbf{1}_k$ refers to a sequence of k times the symbol 1 in a row.

The proof goes by elementary arithmetic and meticulous case disjunctions. The following terminology should simplify the exposition.

Definition 9.3. A *multiset* $\mathbf{A}$ is the data of a set A and a function $m : A \to \mathbb{Z}_{>0}$, called the *multiplicity function*. Intuitively, a multiset is like a set where elements are allowed to appear more than once.

If a multiset $\mathbf{A} = (A, m)$ is *finite*, i.e., its underlying set $A = \{a_1, \ldots, a_k\}$ is finite, we may write A in the following form:

$$\{\{\underbrace{a_1, \ldots, a_1}_{m(a_1)\text{ times}}, \ldots, \underbrace{a_k, \ldots, a_k}_{m(a_k)\text{ times}}\}\}.$$

Double-braces are used to avoid confusion between the multiset and the underlying set.

Let $\mathbf{A} = (A, m)$ be a finite multiset.
If $\alpha \in \mathbb{Z}_{>0}$ and , we denote by $\mathbf{A}^{*\alpha}$ the multiset $(A, \alpha m)$.
If A a subset of $\mathbb{Q}$, and p, q are rational numbers, with $q \neq 0$, we denote by $p + q\mathbf{A}$ the multiset $(p + qA, m)$.
The *cardinal* of $\mathbf{A}$ is:

$$|\mathbf{A}| := \sum_{a \in \mathbf{A}} m(a).$$

More generally, if $f : A \to \mathbb{Q}$ is a function, we define:

$$\sum_{a \in \mathbf{A}} f(a) := \sum_{a \in A} m(a) f(a).$$

If $\mathbf{A} = (A, m)$ and $\mathbf{B} = (B, n)$ are two multisets, we define their *union*:

$$\mathbf{A} \cup \mathbf{B} := (A \cup B, \mathbf{1}_A m + \mathbf{1}_B n),$$

where $\mathbf{1}_A$, $\mathbf{1}_B$ are the indicator functions of A and B.

Notation 9.4. For $d \in \mathbb{N}$, we denote by Φ_d the d-th cyclotomic polynomial, and by $\phi(d)$ the degree of Φ_d. In other terms, ϕ is the Euler indicator function.
For integers a, b, the greatest common divisor of a and b is denoted $a \wedge b$.

We establish a sequence of three useful lemmas.

Lemma 9.5. *Let u be a positive integer strictly greater than 2. Then we have:*

$$\frac{\phi(u)^2}{u} \leq 8 \textit{ or } \left(2 \mid u \textit{ and } \frac{\phi(u)^2}{u} \leq 4\right)$$
$$\Leftrightarrow \ u \in [\![3, 10]\!] \cup \{12, 14, 15, 16, 18, 20, 21, 24, 30, 36, 42\}.$$

Proof. Write $u = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < \ldots < p_k$ are prime numbers, and $\alpha_1, \ldots, \alpha_k$ positive integers, so that:

$$\frac{\phi(u)^2}{u} = \prod_{i=1}^{k} (p_i - 1)^2 p_i^{\alpha_i - 2}.$$

Each of the k factors of this product is greater or equal to 1, unless $p_1^{\alpha_1} = 2$ in which case the first factor is $\frac{1}{2}$.

Hence, if u satisfies:

$$\frac{\phi(u)^2}{u} \leq 8 \text{ or } \left(2 \mid u \text{ and } \frac{\phi(u)^2}{u} \leq 4\right),$$

then each factor satisfies:

$$(p_i - 1)^2 p_i^{\alpha_i - 2} \leq 8, \tag{9.1}$$

which yields $p_i \in \{2, 3, 5, 7\}$. Writing $u = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $\alpha, \beta, \gamma, \delta \geq 0$ and using Inequality (9.1) again bounds $\alpha \leq 4$, $\beta \leq 2$, $\gamma \leq 1$, $\delta \leq 1$. Among the finitely many possibilities left, it is easy to check that the solutions exactly are $u \in [\![3, 10]\!] \cup \{12, 14, 15, 16, 18, 20, 21, 24, 30, 36, 42\}$. □

Lemma 9.6. *Let $u \geq 2$ and $d \geq 3$ be integers, such that u divides d. Suppose that there are a positive integer α and a multiset $\mathbf{A}$ such that:*

$$\mathbf{A} \cup (d - \mathbf{A}) = \left\{ \left\{ a \in [\![1, d-1]\!] \mid u = \frac{d}{d \wedge a} \right\} \right\}^{*\alpha},$$

and such that the quantity:

$$S_{\mathbf{A},d}(u) := \sum_{a \in \mathbf{A}} \frac{a}{u(a \wedge d)}$$

satisfies $S_{\mathbf{A},d}(u) \leq 1$. Then u, $\frac{1}{d}\mathbf{A}$, α, $S_{\mathbf{A},d}(u)$ are classified in Table 9.2.

Proof. We consider the following function.

$$f : a \in A \cup (d - A) \mapsto \frac{a}{a \wedge d} = \frac{ua}{d}.$$

Clearly, f is an increasing function, and takes values in $\{\ell \in [\![1, u-1]\!] \mid \ell \wedge u = 1\}$. It is in fact a bijection, with converse

$$g : \ell \in \{\ell \in [\![1, u-1]\!] \mid \ell \wedge u = 1\} \mapsto \frac{d\ell}{u}.$$

So $|A| \geq \frac{\phi(u)}{2}$. The restriction $f|_A$ is injective, hence takes at least $\frac{\phi(u)}{2}$ distinct values in its image set, so that:

$$1 \geq S_{\mathbf{A},d}(u) = \frac{1}{u} \sum_{a \in \mathbf{A}} f(a) \geq \frac{\alpha}{u} \left(\sum_{\substack{1 \leq \ell \leq u/2 \\ \ell \wedge u = 1}} \ell \right). \tag{9.2}$$

Let us denote by $\Sigma(u)$ the sum $\sum_{\substack{1 \leq \ell \leq u/2 \\ \ell \wedge u = 1}} \ell$. We have the following coarse estimates:

$$u \geq \Sigma(u) \geq \sum_{\ell=1}^{\phi(u)/2} \ell \geq \frac{\phi(u)^2}{8}, \text{ and, if } u \text{ is even, } u \geq \Sigma(u) \geq \sum_{\ell=1}^{\phi(u)/2} 2\ell - 1 \geq \frac{\phi(u)^2}{4}.$$

Applying Lemma 9.5, these coarse estimates yield finitely many possibilities for u. Computing explicitly $\frac{1}{u}\Sigma(u)$ for the possible values and applying Inequality 9.2 again, we exclude a few of them, finally obtaining that:

$$u \in [\![2, 10]\!] \cup \{12, 14, 15, 16, 18, 20, 24\}.$$

For each u, we then list by hand the finitely many possibilities for the multiplicity α and the multiset $\frac{1}{d}\mathbf{A}$, and this is how we construct Table 9.2. □

u	α	$\frac{1}{d}A$	$S_{A,d}(u) \leq 1$
2	1	$\left\{\frac{1}{2}\right\}$	$\frac{1}{2}$
	2	$\left\{\frac{1}{2},\frac{1}{2}\right\}$	1
3	1	$\left\{\frac{1}{3}\right\},\left\{\frac{2}{3}\right\}$	$\frac{1}{3},\frac{2}{3}$
	2	$\left\{\frac{1}{3},\frac{1}{3}\right\},\left\{\frac{1}{3},\frac{2}{3}\right\}$	$\frac{2}{3},1$
	3	$\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\}$	1
4	1	$\left\{\frac{1}{4}\right\},\left\{\frac{1}{4}\right\}$	$\frac{1}{4},\frac{3}{4}$
	2	$\left\{\frac{1}{4},\frac{1}{4}\right\},\left\{\frac{1}{4},\frac{3}{4}\right\}$	$\frac{1}{2},1$
	3	$\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\}$	$\frac{3}{4}$
	4	$\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\}$	1
5	1	$\left\{\frac{1}{5},\frac{2}{5}\right\},\left\{\frac{1}{5},\frac{3}{5}\right\}$	$\frac{3}{5},\frac{4}{5}$
6	1	$\left\{\frac{1}{6}\right\},\left\{\frac{5}{6}\right\}$	$\frac{1}{6},\frac{5}{6}$
	2	$\left\{\frac{1}{6},\frac{1}{6}\right\},\left\{\frac{1}{6},\frac{5}{6}\right\}$	$\frac{1}{3},1$
	3	$\left\{\frac{1}{6},\frac{1}{6},\frac{1}{6}\right\}$	$\frac{1}{2}$
	4	$\left\{\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right\}$	$\frac{2}{3}$
	5	$\left\{\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right\}$	$\frac{5}{6}$
	6	$\left\{\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right\}$	1
7	1	$\left\{\frac{1}{7},\frac{2}{7},\frac{3}{7}\right\},\left\{\frac{1}{7},\frac{2}{7},\frac{4}{7}\right\}$	$\frac{6}{7},1$
8	1	$\left\{\frac{1}{8},\frac{3}{8}\right\},\left\{\frac{1}{8},\frac{5}{8}\right\}$	$\frac{1}{2},\frac{3}{4}$
	2	$\left\{\frac{1}{8},\frac{1}{8},\frac{3}{8},\frac{3}{8}\right\}$	1
9	1	$\left\{\frac{1}{9},\frac{2}{9},\frac{4}{9}\right\},\left\{\frac{1}{9},\frac{2}{9},\frac{5}{9}\right\}$	$\frac{7}{9},\frac{8}{9}$
10	1	$\left\{\frac{1}{10},\frac{3}{10}\right\},\left\{\frac{1}{10},\frac{7}{10}\right\}$	$\frac{2}{5},\frac{4}{5}$
	2	$\left\{\frac{1}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right\}$	$\frac{4}{5}$
12	1	$\left\{\frac{1}{12},\frac{5}{12}\right\},\left\{\frac{1}{12},\frac{7}{12}\right\}$	$\frac{1}{2},\frac{2}{3}$
	2	$\left\{\frac{1}{12},\frac{1}{12},\frac{5}{12},\frac{5}{12}\right\}$	1
14	1	$\left\{\frac{1}{14},\frac{3}{14},\frac{5}{14}\right\},\left\{\frac{1}{14},\frac{3}{14},\frac{9}{14}\right\}$	$\frac{9}{14},\frac{13}{14}$
15	1	$\left\{\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15}\right\},\left\{\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{8}{15}\right\}$	$\frac{14}{15},1$
16	1	$\left\{\frac{1}{16},\frac{3}{16},\frac{5}{16},\frac{7}{16}\right\}$	1
18	1	$\left\{\frac{1}{18},\frac{5}{18},\frac{7}{18}\right\},\left\{\frac{1}{18},\frac{5}{18},\frac{11}{18}\right\}$	$\frac{13}{18},\frac{17}{18}$
20	1	$\left\{\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20}\right\}$	1
24	1	$\left\{\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24}\right\}$	1

Table 9.2: Possibilities for u, $\frac{1}{d}A$, $\alpha, S_{A,d}(u)$ such that $S_{A,d}(u) \leq 1$

Lemma 9.7. *Let $k \in \mathbb{N}$. For each $m \in [\![1, k]\!]$, let $u_m \geq 2$ and $d_m \geq 3$ be integers, such that u_m divides d_m, and suppose that there are a positive integer α_m and a multiset $\mathbf{A_m}$ such that:*

$$\mathbf{A_m} \cup (d_m - \mathbf{A_m}) = \left\{\left\{a \in [\![1, d_m - 1]\!] \mid u_m = \frac{d_m}{d_m \wedge a}\right\}\right\}^{*\alpha_m}.$$

Suppose additionally that:

$$\sum_{m=1}^{k} S_{\mathbf{A_m}, d_m}(u_m) = 1.$$

Then the data of k and of all u_m, α_m, $\frac{1}{d_m}\mathbf{A_m}$ is classified in Table 9.3.

Proof. It is easily derived by hand from Table 9.2. □

$u_1, \ldots, u_k$	$\alpha_1, \ldots, \alpha_k$	$\frac{1}{d_1}A_1, \ldots, \frac{1}{d_k}A_k$	freeness in codimension 2
2	2	$\left\{\frac{1}{2}, \frac{1}{2}\right\}$	✗
$2, 3, 6$	$1, 1, 1$	$\left\{\frac{1}{2}\right\}, \left\{\frac{1}{3}\right\}, \left\{\frac{1}{6}\right\}$	✗
$2, 4$	$1, 2$	$\left\{\frac{1}{2}\right\}, \left\{\frac{1}{4}, \frac{1}{4}\right\}$	✗
$2, 6$	$1, 3$	$\left\{\frac{1}{2}\right\}, \left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$	✓
$2, 8$	$1, 1$	$\left\{\frac{1}{2}\right\}, \left\{\frac{1}{8}, \frac{3}{8}\right\}$	✗
$2, 12$	$1, 1$	$\left\{\frac{1}{2}\right\}, \left\{\frac{1}{12}, \frac{5}{12}\right\}$	✗
3	2	$\left\{\frac{1}{3}, \frac{2}{3}\right\}$	✗
	3	$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$	✓
$3, 4, 6$	$1, 2, 1$	$\left\{\frac{1}{3}\right\}, \left\{\frac{1}{4}, \frac{1}{4}\right\}, \left\{\frac{1}{6}\right\}$	✗
$3, 6$	$1, 2$	$\left\{\frac{2}{3}\right\}, \left\{\frac{1}{6}, \frac{1}{6}\right\}$	✗
	$1, 4$	$\left\{\frac{1}{3}\right\}, \left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$	✓
	$2, 2$	$\left\{\frac{1}{3}, \frac{1}{3}\right\}, \left\{\frac{1}{6}, \frac{1}{6}\right\}$	✗
$3, 12$	$1, 1$	$\left\{\frac{1}{3}\right\}, \left\{\frac{1}{12}, \frac{7}{12}\right\}$	✗
4	2	$\left\{\frac{1}{4}, \frac{3}{4}\right\}$	✗
	4	$\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$	✓
$4, 6$	$2, 3$	$\left\{\frac{1}{4}, \frac{1}{4}\right\}, \left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$	✗
$4, 8$	$1, 1$	$\left\{\frac{1}{4}\right\}, \left\{\frac{1}{8}, \frac{5}{8}\right\}$	✗
	$2, 1$	$\left\{\frac{1}{4}, \frac{1}{4}\right\}, \left\{\frac{1}{8}, \frac{3}{8}\right\}$	✗
$4, 12$	$2, 1$	$\left\{\frac{1}{4}, \frac{1}{4}\right\}, \left\{\frac{1}{12}, \frac{5}{12}\right\}$	✗
$5, 10$	$1, 1$	$\left\{\frac{1}{5}, \frac{2}{5}\right\}, \left\{\frac{1}{10}, \frac{3}{10}\right\}$	✗
6	2	$\left\{\frac{1}{6}, \frac{5}{6}\right\}$	✗
	6	$\left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$	✓
$6, 8$	$3, 1$	$\left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}, \left\{\frac{1}{8}, \frac{3}{8}\right\}$	✗
$6, 12$	$2, 1$	$\left\{\frac{1}{6}, \frac{1}{6}\right\}, \left\{\frac{1}{12}, \frac{7}{12}\right\}$	✗
	$3, 1$	$\left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}, \left\{\frac{1}{12}, \frac{5}{12}\right\}$	✗
7	1	$\left\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right\}$	✓
8	2	$\left\{\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}\right\}$	✓
$8, 12$	$1, 1$	$\left\{\frac{1}{8}, \frac{3}{8}\right\}, \left\{\frac{1}{12}, \frac{5}{12}\right\}$	✗
12	2	$\left\{\frac{1}{12}, \frac{1}{12}, \frac{5}{12}, \frac{5}{12}\right\}$	✓
15	1	$\left\{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}\right\}$	✓
16	1	$\left\{\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}\right\}$	✓
20	1	$\left\{\frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{9}{20}\right\}$	✓
24	1	$\left\{\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}\right\}$	✓

Table 9.3: Possibilities for k parcels of data u_m, α_m, $\frac{1}{d_m}\mathbf{A_m}$ such that

$$\sum_{m=1}^{k} S_{\mathbf{A_m}, d_m}(u_m) = 1$$

We can now prove Proposition 9.2.

Proof of Proposition 9.2. Denote by d the order of the junior element g, by $(e^{2i\pi a_j/d})_{1\leq j\leq n}$ its ranked vector of eigenvalues, and by $P(g)$ the characteristic polynomial of its matrix $M(g)$. As g itself acts freely in codimension 2 and g is junior, it must be that $d \geq 3$.

By Lemma 2.76, there are positive integers k, $(u_m)_{1\leq m\leq k}$ ordered increasingly, and $(\alpha_m)_{1\leq m\leq k}$, such that:

$$\prod_{j=1}^{n}(X - e^{2i\pi a_j/d})(X - \overline{e^{2i\pi a_j/d}}) = P(g)\overline{P(g)} = \prod_{m=1}^{k} {\Phi_{u_m}}^{\alpha_m}. \tag{9.3}$$

Note that $\Phi_{u_m}(e^{2i\pi a_j/d}) = 0$, or equivalently $\Phi_{u_m}(\overline{e^{2i\pi a_j/d}}) = 0$, if and only if $u_m = \frac{d}{d\wedge a_j}$. We define the following partition of $[\![1,n]\!]$

$$\begin{array}{lll} \text{for } m \in [\![1,k]\!], & I_m & := \{j \in [\![1,n]\!] \mid u_m = \frac{d}{d\wedge a_j}\}; \\ & \mathbf{A_m} & := \{\{a_j \mid j \in I_m\}\}, \text{ as a multiset.} \end{array}$$

By Identity 9.3, for $m \in [\![1,k]\!]$ we have:

$$\mathbf{A_m} \cup (d - \mathbf{A_m}) = \{\{r \in [\![1,d-1]\!] \mid u_m = \frac{d}{d\wedge r} = 0\}\}^{*\alpha_m} \tag{9.4}$$

Moreover, since g is junior:

$$1 = \sum_{j=1}^{n} \frac{a_j}{d} = \sum_{m=1}^{k}\sum_{j\in I_m} \frac{a_j}{d} = \sum_{m=1}^{k}\sum_{j\in I_m} \frac{a_j}{u_m(d\wedge a_j)} = \sum_{m=1}^{k} S_{\mathbf{A_m},d}(u_m). \tag{9.5}$$

So, possibly leaving out the data of index 1, if $u_1 = 1$ (which is determined by the multiplicity $\alpha_1 \in \mathbb{N}$, since then $\mathbf{A_1} = \{\{\mathbf{0}_{\alpha_1}\}\}$ and $S_{\mathbf{A_1},d}(u_1) = 0$), Lemma 9.7 applies, showing that there are finitely many possibilities for

$$k,\ (u_m)_{1\leq m\leq k},\ (\alpha_m)_{1\leq m\leq k},\ \left(\frac{1}{d}\mathbf{A_m}\right)_{1\leq m\leq k}$$

and listing them. We exclude by hand a lot of these possibilities using the assumption that $\langle g\rangle$ acts freely in codimension 2 on A, i.e., that for all $\ell \in [\![1,d-1]\!]$, there must be distinct indices $j_1(\ell), j_2(\ell), j_3(\ell) \in [\![1,n]\!]$, such that none of the $\frac{\ell a_{j_i(\ell)}}{d}$ is an integer. What remains then is precisely the list in Table 9.1. $\square$

CHAPTER 10

CYCLICITY OF THE POINTWISE STABILIZERS OF LOCI OF CODIMENSION 3 AND 4

We now know that G is generated by junior elements, which we have classified into twelve different types. However, this is by far insufficient to determine the structure of G. Even locally, for $W \subset A$ a subvariety, the pointwise stabilizer

$$\mathrm{PStab}(W) := \{g \in G \mid \forall w \in W,\, g(w) = w\}$$

could as well be cyclic and generated by one junior element, as it could be more complicated, e.g., if it contained non-commuting junior elements.

In this section, we show that in fact, if W has codimension 3 or 4 in A, $\mathrm{PStab}(W)$ is trivial or cyclic, generated by one junior element. Let us outline the proof. Subsection 10.1 reduces to proving this in the case when W is a point in an abelian variety B of dimension 3 or 4. Up to conjugating the whole group G by a translation, we therefore just work on the case $W = \{0\}$. Assuming $\mathrm{PStab}(W)$ is not trivial, we can then find a junior element $g \in \mathrm{PStab}(W)$, that is of one of the twelve types of Section 9. Subsection 10.2 exhibits a correlation between the type of g and the isogeny type (possibly even isomorphism type) of the abelian variety B on which it acts. A corollary is that if g, $h \in \mathrm{PStab}(W)$ are two junior elements, then they should either have the same type, or one is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$ and the other $(\mathbf{1}_{n-3}, j, j, j)$, or one is of type $(\mathbf{1}_{n-4}, i, i, i, i)$ and the other $(\mathbf{1}_{n-4}, \zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5)$. In particular, if $\mathrm{PStab}(W)$ is cyclic, it must indeed be generated by one junior element. The conclusive Subsection 10.3 is the most technical. For any given abelian three- or fourfold B of one of the types just defined, we classify all finite subgroups of

$$\mathrm{Aut}(B, 0) := \{f \in \mathrm{Aut}(B) \mid f(0) = 0,\ \text{i.e., } T(f) = 0\}$$

that act freely in codimension 2 on B and are generated by junior elements. The main idea is to bound the order of such groups, to scrutinize the finite list arising, and to rule out all but the cyclic case of the list by the assumption on generators.

10.1 Reduction to a 3 or 4-dimensional question.

Definition 10.1. Let A be an abelian variety. An *abelian subvariety* of A is a closed subvariety of A that is also a subgroup of the abelian group $(A, +)$. A *translated abelian subvariety* of A is the image by a translation of an abelian subvariety of A.

We say that two translated abelian subvarieties B and C of A are *complementary* if one of the following equivalent statements hold:

(i) $B \cap C$ is non-empty and, for some $p \in B \cap C$, it holds:

$$H^0(\{p\}, T_B) \oplus H^0(\{p\}, T_C) = H^0(\{p\}, T_A).$$

(ii) The addition map $i : B \times C \to A$ is an isogeny.

Proof. $(i) \Rightarrow (ii)$: as the translation by (p,p), respectively by $2p$, is an isomorphism from $B \times C$ to $(B-p)\times(C-p)$, respectively of A, it is enough to prove the statement for $p = 0$. As $\dim(A) = \dim(B \times C)$ and the varieties are regular, we simply check that i is quasi-finite. Since $B \cap C$ is the intersection of two abelian subvarieties of A satisfying:

$$H^0(\{0\}, T_B) \cap H^0(\{0\}, T_C) = \{0\},$$

the set $B \cap -C$ is discrete in A, hence finite. For $a \in \mathrm{Im}(i)$, say $a = i(a_B, a_C)$, we can express the fiber $i^{-1}(a) = \{(b + a_B, -b + a_C) \mid b \in B \cap -C\}$, so it is finite, and i is indeed quasi-finite.

$(ii) \Rightarrow (i)$: fix $c_0 \in C$. The addition i is onto, so let $(p, c) \in B \times C$ be such that $p + c = 2c_0$. Clearly, $p = 2c_0 - c \in B \cap C$, and as i is locally analytically an isomorphism,

$$H^0(\{p\}, T_B) \oplus H^0(\{p\}, T_C) = H^0(\{2p\}, T_A) = H^0(\{p\}, T_A).$$

$\square$

Remark 10.2. If B and C are complementary translated abelian subvarieties of an abelian variety A, and $t \in A$ is any point, then $B + t$ and C are complementary as well. Our notion of complementarity is weaker than the notion defined for abelian subvarieties in [17, p.125].

Let us now state our reduction result. Note that it applies not only in codimension 3 and 4, but in any higher codimension as well.

Proposition 10.3. *Let A be an abelian variety, G be a finite group acting freely in codimension 2 on A. Suppose that the quotient A/G admits a crepant resolution. Let W be a subvariety of codimension m in A such that $\mathrm{PStab}(W) \neq \{1\}$. Then:*

(1) *For any $t \in W$ there is a translated abelian subvariety B of A which is $\mathrm{PStab}(W)$-stable, contains t, and is complementary to W in A.*

(2) *If t and B are as such, then an element $g \in \mathrm{PStab}(W)$ is junior if and only if $g|_B \in \mathrm{Aut}(B, t)$ is a junior element.*

(3) *The group $\mathrm{PStab}(W) \subset \mathrm{Aut}(B, t)$ is generated by junior elements.*

Proof of Proposition 10.3. Up to conjugating the G-action by the translation by t, we can assume that $t = 0$. Let us establish (1): As G is finite, we can take a G-invariant polarization L on A. We can apply [17, Prop.13.5.1]: there is a unique complementary abelian subvariety $(B, L|_B)$ to $(W, L|_W)$ in (A, L), and it is $\mathrm{PStab}(W)$-stable. By Remark 10.2, B and W are complementary in our sense as well.

We now prove (2): let $g \in \mathrm{PStab}(W)$. As g fixes all points of $B \cap W$, its restriction $g|_B$ has a fixed point. As $g(B) = B$, we have:

$$M(g) = \begin{pmatrix} \mathrm{id}_{\dim(W)} & 0 \\ 0 & M(g|_B) \end{pmatrix},$$

and thus g is indeed junior if and only if $g|_B$ is.

We move on to (3). Take a general point $w \in W$ such that $\mathrm{PStab}(w) = \mathrm{PStab}(W)$. Since $\mathrm{PStab}(w)$ is finite, any analytic neighborhood of w in A contains a contractible analytic neighborhood U of w that is $\mathrm{PStab}(w)$-stable. Up to reducing it even more, we can assume that for any $g \in G \setminus \mathrm{PStab}(w)$, $g(U) \cap U = \varnothing$. So, an analytic neighborhood of $[w] \in A/G$ is biholomorphic to $U/\mathrm{PStab}(w)$. Hence, Proposition 8.4 applies and $\mathrm{PStab}(w)$ is generated by junior elements. □

10.2 The abelian varieties corresponding to the twelve juniors. Let A be an abelian variety of dimension n, G be a finite group acting freely in codimension 2 on A such that A/G has a crepant resolution. By Corollary 8.3, $G \subset \mathrm{Aut}(A)$ must entail a junior element presented in Table 9.1 (up to its translation part, and up to similarity for its linear part). The fact that, in some coordinates, a given matrix of Table 9.1 acts as an automorphism on the abelian variety A imposes some restrictions. Using the theory of abelian varieties with complex multiplication, these restrictions are investigated by Proposition 10.6.

Notation 10.4. Let us defined the following quadratic integers

$$u_7 = \frac{-1+i\sqrt{7}}{2},\ u_8 = i\sqrt{2},\ u_{15} = \frac{1+i\sqrt{15}}{2},\ u_{20} = i\sqrt{5},\ u_{24} = i\sqrt{6},$$

and the following algebraic integers, whose square are quadratic integers

$$u_{16} = i\sqrt{4+2\sqrt{2}},\ v_{16} = i\sqrt{4-2\sqrt{2}}.$$

For $z \in \mathbb{C} \setminus \mathbb{R}$, we define the elliptic curve $E_z := \mathbb{C}/\mathbb{Z} \oplus z\mathbb{Z}$. If z is a quadratic integer, then we denote by $\mathbb{Z}[z]$ the $\mathbb{Z}$-algebra that it generates. It holds $\mathbb{Z}[z] = \mathbb{Z} \oplus z\mathbb{Z} \subset \mathbb{C}$. We also define the simple abelian surface $S_{u_{16},v_{16}} := \mathbb{C}^2/\mathbb{Z}[(u_{16}, v_{16})]$.

Remark 10.5. Note that the simplicity of S_{16} follows from [184, Prop.27].

With these notations, we can state the main result of the subsection.

Proposition 10.6. *Let A be an abelian variety. Suppose that there is a junior element $g \in \mathrm{Aut}(A)$, and that $\langle g \rangle$ acts freely in codimension 2 on A. Denote by W an irreducible component of* $\mathrm{Fix}(g) := \{a \in A \mid g(a) = a\}$. *Let B be a complementary to W in A. Then the isogeny type of B is entirely determined by the type of the junior element g by Table 10.1, unless g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$. Moreover, the isomorphism type of a $\langle g \rangle$-stable complementary B_{st} to W in A is also entirely determined by the type of g, unless g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$ or $(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$.*

Notation 10.7. Let V be a $\mathbb{C}$-vector space, $f : V \to V$ be a linear map. We denote by $\mathrm{EVal}(f)$ the set of eigenvalues of f in $\mathbb{C}$, by $\mathbf{EVal}(f)$ the multiset of eigenvalues of f in $\mathbb{C}$ counted with multiplicities. If $\lambda \in \mathrm{EVal}(f)$, we denote by $E_f(\lambda)$ the eigenspace of f for the eigenvalue λ.
We denote by $Z(\Phi_d) \subset \mathbb{U}_d$ the set of primitive d-th roots of unity in $\mathbb{C}$.

Let us first carry out an important computation, that makes plain where these special types of abelian varieties come from. Let $k \geq 3$ be an integer. There is a natural action of $\zeta_k \otimes 1$ on the algebra $\mathbb{Z}[\zeta_k] \otimes \mathbb{C}$. We compute its eigenvalues.

type of g	isogeny type of B	isomorphism type of B_{st}
$(\mathbf{1}_{n-3}, j, j, j)$	${E_j}^3$	${E_j}^3$
$(\mathbf{1}_{n-4}, i, i, i, i)$	${E_i}^4$	${E_i}^4$
$(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$	$E \times {E_j}^3$, for some elliptic curve E	not determined
$(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$	${E_j}^5$	not determined
$(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$	${E_j}^6$	${E_j}^6$
$(\mathbf{1}_{n-3}, \zeta_7, {\zeta_7}^2, {\zeta_7}^4)$	${E_{u_7}}^3$	${E_{u_7}}^3$
$(\mathbf{1}_{n-4}, \zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3)$	${E_{u_8}}^4$	${E_{u_8}}^4$
$(\mathbf{1}_{n-4}, \zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5)$	${E_i}^4$	${E_i}^4$
$(\mathbf{1}_{n-4}, \zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$	${E_{u_{15}}}^4$	${E_{u_{15}}}^4$
$(\mathbf{1}_{n-4}, \zeta_{15}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7)$	${S_{u_{16},v_{16}}}^2$	${S_{u_{16},v_{16}}}^2$
$(\mathbf{1}_{n-4}, \zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9)$	${E_{u_{20}}}^4$	${E_{u_{20}}}^4$
$(\mathbf{1}_{n-4}, \zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11})$	${E_{u_{24}}}^4$	${E_{u_{24}}}^4$

Table 10.1: Correspondence between types of junior elements and types of abelian varieties.

By definition, $\mathbb{Z}[\zeta_k] \otimes \mathbb{C}$ is the quotient algebra $\mathbb{C}[X]/(\Phi_k)$, multiplication by $\zeta_k \otimes 1$ corresponding to multiplication by the class $X + \Phi_k\mathbb{C}[X]$. So $\xi \in \mathbb{C}$ is an eigenvalue with eigenvector $P + \Phi_k\mathbb{C}[X]$ if and only if $P \notin \mathbb{C}[X]$ and $XP - \xi P \in \Phi_k\mathbb{C}[X]$, or equivalently, ξ is a root of Φ_k and $P \in \frac{\Phi_k}{X-\xi}\mathbb{C}[X]$. Hence the linear decomposition

$$\mathbb{Z}[\zeta_k] \otimes \mathbb{C} = \bigoplus_{\xi \in Z(\Phi_k)} E_{\zeta_k \otimes 1}(\xi) \tag{10.1}$$

Now, consider a subset S_k of $Z(\Phi_k)$ such that $S_k\overline{S_k} = Z(\Phi_k)$. For example, if we let g be a junior element of one of the twelve types in Table 9.1, and we assume that g has an eigenvalue of order k, we could set $S_k = S_k(g) = \mathrm{EVal}(g) \cap Z(\Phi_k)$. This defines a $\mathbb{Z}$-linear inclusion

$$f(S_k) : \mathbb{Z}[\zeta_k] \hookrightarrow \bigoplus_{\xi \in S_k} E_{\zeta_k \otimes 1}(\xi) \;\simeq\; \mathbb{C}^{\phi(k)/2} \tag{10.2}$$

It is worth noting that the $\mathbb{Z}$-linear inclusion $f(S_k) \oplus \overline{f(S_k)}$ corresponds to the natural inclusion of $\mathbb{Z}[\zeta_k]$ in $\mathbb{Z}[\zeta_k] \otimes \mathbb{C}$ given by Identity 10.1.

The following lemma is key.

Lemma 10.8. *If $S_k = S_k(g)$ for a junior element g of Table 9.1, then the corresponding abelian variety $\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))$ is described in Table 10.2.*

Remark 10.9. For $k = 3, 4, 6, 8, 12$, we have $S_k = \{j\}, \{i\}, \{j\}, \{\zeta_8, {\zeta_8}^3\}$, and $\{\zeta_{12}, {\zeta_{12}}^5\}$ respectively, and Lemma 10.8 is [17, Cor.13.3.4, Cor.13.3.6]. In the other cases, the computation relies on the same ideas as [17, Cor.13.3.6], as we will soon see.

k	S_k	$\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))$
3	$\{j\}$	E_j
4	$\{i\}$	E_i
6	$\{\omega\}$	E_j
7	$\{\zeta_7, {\zeta_7}^2, {\zeta_7}^4\}$	${E_{u_7}}^3$
8	$\{\zeta_8, \zeta_8^3\}$	${E_{u_8}}^2$
12	$\{\zeta_{12}, \zeta_{12}^5\}$	${E_i}^2$
15	$\{\zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8\}$	${E_{u_{15}}}^4$
16	$\{\zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7\}$	${S_{u_{16},v_{16}}}^2$
20	$\{\zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9\}$	${E_{u_{20}}}^4$
24	$\{\zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11}\}$	${E_{u_{24}}}^4$

Table 10.2: Computing $\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))$ for given S_k stemming from a junior element.

Proof of Lemma 10.8. Let $F = \mathbb{Q}[\zeta_k]$, $r = \frac{\phi(k)}{2}$. Let us define $\{\varphi_i\}_{1\le i\le r}$: composing $f(S_k)$ defined in Identity 10.2 with the projections on the r eigenspaces, we obtain morphisms of $\mathbb{Z}$-algebras $f_i : \mathbb{Z}[\zeta_k] \to \mathbb{C}$, which we tensor by $\mathbb{Q}$ and normalize to define morphisms of $\mathbb{Q}$-algebras:

$$\varphi_i = \frac{1}{f_i(1)}(f_i \otimes 1) : \mathbb{Q}[\zeta_k] \to \mathbb{C}.$$

By Identities 10.1 and 10.2, the morphisms $\{\varphi_i, \overline{\varphi_i}\}_{1\le i\le r}$ are linearly independent over $\mathbb{Q}$, whereas the morphisms $\{\varphi_i\}_{1\le i\le r}$ define an embedding of F into the $\mathbb{Q}$-algebra of linear endomorphisms of the abelian variety $\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))$. In other words, the abelian variety $\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))$ has CM-type $(F, \{\varphi_i\}_{1\le i\le r})$. This is in fact the sole abelian variety with this CM-type, by [136],[184, Prop.17] remembering that $k \in \{7, 15, 16, 20, 24\}$.

Applying Lemma 2.97 with $K = \mathbb{Q}[u_k]$, we get the wished description of the abelian variety $\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))$, by an easy verification involving that:

- $u_7 = \zeta_7 + \zeta_7^2 + \zeta_7^4$,
- $u_{15} = \zeta_{15} + \zeta_{15}^2 + \zeta_{15}^4 + \zeta_{15}^8$,
- $u_{16} = \zeta_{16} + \zeta_{16}^3 + \zeta_{16}^5 + \zeta_{16}^7$ and $v_{16} = \zeta_{16}^3 + \zeta_{16}^5 + \zeta_{16}^9 + \zeta_{16}^{15}$,
- $u_{20} = \zeta_{20} + \zeta_{20}^3 + \zeta_{20}^7 + \zeta_{20}^9$,
- $u_{24} = \zeta_{24} + \zeta_{24}^5 + \zeta_{24}^7 + \zeta_{24}^{11}$.

□

The next result follows almost effortlessly from the ideas of [155, p.333-334].

Lemma 10.10. *Let B be an abelian variety. Suppose that there is an automorphism g of B whose set of eigenvalues is one of the S_k in Table 10.2. Then B is isomorphic to a power of the abelian variety $\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))$.*

Proof. Let Λ be a lattice in $\mathbb{C}^n$ such that $B = \mathbb{C}^n/\Lambda$. The linear action of g restricting to Λ, it provides it with a $\mathbb{Z}[g]$-module, i.e., a $\mathbb{Z}[\zeta_k]$-module structure, since the minimal polynomial of g is Φ_k. As such, Λ is finitely-generated and torsion-free. But by [136], since $k \in [\![3, 20]\!] \cup \{24\}$, the ring of cyclotomic integers $\mathbb{Z}[\zeta_k]$ is a principal ideal domain. So, by the structure theorem for finitely-generated modules over principal ideal domains, $\Lambda \simeq \mathbb{Z}[\zeta_k]^{2n/\phi(k)}$, and the action of g on Λ identifies with the multiplication by ζ_k on $\mathbb{Z}[\zeta_k]^{2n/\phi(k)}$.

The embedding $\Lambda \hookrightarrow H^0(B, T_B) \simeq \mathbb{C}^n$ can be recovered from the action of g on Λ. Indeed, there is an induced action of $g \oplus \overline{g}$ on $\Lambda \otimes \mathbb{C} = H^0(B, T_{B,\mathbb{R}} \otimes \mathbb{C}) \simeq \mathbb{C}^{2n}$. This action splits into two blocks: g is acting on $H^0(B, T_B)$ and $\overline{g}$ is acting on its supplementary conjugate in $H^0(B, T_{B,\mathbb{R}} \otimes \mathbb{C})$. By the requirement on its set of eigenvalues S_k, g has no eigenvalue in common with $\overline{g}$, and therefore:

$$H^0(B, T_B) = \bigoplus_{\xi \in \mathbf{EVal}(g)} E_{g \oplus \overline{g}}(\xi).$$

Hence, the corresponding embedding $\mathbb{Z}[\zeta_k]^{2n/\phi(k)} \hookrightarrow \mathbb{C}^n$ must similarly be given by:

$$\mathbb{C}^n = \bigoplus_{\xi \in \mathbf{EVal}(g)} E_{\zeta_k \otimes 1}(\xi),$$

where $\zeta_k \otimes 1$ is the action by componentwise multiplication on $\mathbb{Z}[\zeta_k]^{2n/\phi(k)} \otimes \mathbb{C}$. In other words, this embedding is the blockwise embedding $f(S_k)$, repeated on $\frac{2n}{\phi(k)}$ blocks of dimension $\frac{\phi(k)}{2}$ each. So $B \simeq \left(\mathbb{C}^{\phi(k)/2}/\mathrm{Im}(f(S_k))\right)^{2n/\phi(k)}$. ☐

The proof of Proposition 10.6 is now easy.

Proof of Proposition 10.6. By Proposition 10.3, let B_{st} be a $\langle g \rangle$-stable complement to W in A. For any other complement B to W, since $B \times W$ and $B_{\mathrm{st}} \times W$ are isogenous, B and B_{st} are isogenous. Let us determine the isogeny (and if possible isomorphism) type of B_{st}.

On one hand, if g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$ or $(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$, then $g|_{B_{\mathrm{st}}}$ has eigenvalues of two different orders. By [17, Thm.13.2.8], there are then two $\langle g \rangle$-stable complementary translated abelian subvarieties B_1 and B_2 in B_{st}, such that all eigenvalues of $g|_{B_1}$ have order $k_1 = 6$, and all eigenvalues of $g|_{B_2}$ have the same order $k_2 < 6$. By definition, B_{st} is isogenous to $B_1 \times B_2$, and thus its isogeny type can be derived from the isomorphism types of B_1 and B_2, given by Lemma 10.10 if $k_1, k_2 \geq 3$. However, if g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$, then $k_2 = 2$ and B_2 can be any elliptic curve, and that is why the isogeny type of B_{st} is not entirely determined in this case.

On the other hand, if g is of any other type, then all eigenvalues of $g|_{B_{\mathrm{st}}}$ are of the same order $k \geq 3$, and Lemma 10.10 determines the isomorphism type of B_{st}. ☐

10.3 Group theoretical treatment of a point's stabilizer in dimension 3 or 4. We can now establish the following proposition.

Proposition 10.11. *Let A be an abelian variety, $G \subset \mathrm{Aut}(A)$ be a finite group acting freely in codimension 2. Suppose that the quotient A/G admits a crepant resolution. Let W be a subvariety of codimension $m \leq 4$ in A such that $\mathrm{PStab}(W) \neq \{1\}$. Then $\mathrm{PStab}(W)$ is a cyclic group generated by one junior element.*

By Proposition 10.3, it reduces to proving the following result.

Proposition 10.12. *Let B be an abelian variety of dimension $m \leq 4$, $F \subset \mathrm{Aut}(B, 0)$ be a finite group acting freely in codimension 2 and fixing $0 \in B$. Suppose that F is generated by junior elements. Then F is a cyclic group generated by one junior element.*

We refer the reader to [170],[93] for standard facts in finite group theory, and in particular Sylow theory and representation theory. Let us just recall a few notations used in the following.

Notation 10.13. We denote by $C_F(H)$, respectively $N_F(H)$, the centralizer, respectively normalizer, of a subset H of a group F, i.e.,

$$C_F(H) := \{f \in F \mid \forall\, h \in H,\, fh = hf\}$$
$$N_F(H) := \{f \in F \mid fH = Hf\}$$

If H has a single element or is a subgroup of F, then $C_F(H)$ and $N_F(H)$ are subgroups of F.

Notation 10.14. Let F be a finite group, V be a vector space of finite dimension, $\rho : F \to \mathrm{GL}(V)$ be a group morphism, i.e., a faithful representation of F in V. The *character* χ of ρ is the map $\chi : f \in F \to \mathrm{Tr}(\rho(f)) \in \mathbb{C}^*$.

By Schur's lemma, the representation ρ decomposes as a direct sum of irreducible representations:

$$\rho = \rho_1^{\oplus n_1} \oplus \ldots \oplus \rho_k^{\oplus n_k},$$

and, accordingly, if χ_i denotes the character of ρ_i, we have $\chi = n_1\chi_1 + \ldots + n_k\chi_k$. By orthogonality of the irreducible characters,

$$\langle \chi, \chi \rangle = (n_1^2 + \ldots + n_k^2)|F|.$$

We refer to $u = n_1^2 + \ldots + n_k^2$ as the *splitting coefficient* of the representation ρ.

We start proving lemmas towards Proposition 10.12. The first lemma classifies all possible finite order elements in $\mathrm{Aut}(B, 0)$ of determinant one acting freely in codimension 2, when B is an abelian fourfold.

Lemma 10.15. *Let B be an abelian fourfold, and $g \in \mathrm{Aut}(B, 0)$ be a finite order element such that $\langle g \rangle$ acts freely in codimension 2 on B. Then the order of g and the matrix of a generator of $\langle g \rangle$ are given in Table 10.3, together with the restrictions on B, if any.*

Proof. Let ζ be an eigenvalue of g of order u, such that $(\phi(u), u)$ is maximal in $\mathbb{N}^2$ for the lexicographic order. By Lemma 2.76, Φ_u divides the characteristic polynomial $\chi_{g \oplus \overline{g}}$ in $\mathbb{Q}[X]$, so $\phi(u) \leq 2 \dim B = 8$. Let us discuss cases:

(1) If $\phi(u) = 1$, then $u = 1$ or 2. As g acts freely in codimension 2 and has determinant one, $g = \pm \mathrm{id}_B$.

(2) Suppose that $\phi(u) = 8$. Then g has four distinct eigenvalues of order u, and hence has order u. Listing integers of Euler number 8, $u \in \{15, 16, 20, 24, 30\}$. There is a generator g' of $\langle g \rangle$ of which $e^{2i\pi/u}$ is an eigenvalue. Denote its other eigenvalues by $e^{2i\pi a/u}$, $e^{2i\pi b/u}$, $e^{2i\pi c/u}$, with

- $a, b, c \in [\![1, u-1]\!]$ coprime to u
- u divides $1 + a + b + c$
- and

$$\begin{aligned}\Phi_u(X) = &(X - e^{2i\pi/u})(X - e^{2i\pi(u-1)/u})(X - e^{2i\pi a/u})(X - e^{2i\pi(u-a)/u}) \\ &(X - e^{2i\pi b/u})(X - e^{2i\pi(u-b)/u})(X - e^{2i\pi c/u})(X - e^{2i\pi(u-c)/u})\end{aligned}$$

We check by hand the solutions to this system and plug them in Table 10.3. For example, this is how we add $\mathrm{diag}(\zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$.

(3) Suppose that $\phi(u) = 6$. Then g has three distinct eigenvalues of order u, and one eigenvalue of order v, with $\phi(v) = 1$ or 2. Since g^u has three trivial eigenvalues and $\langle g \rangle$ acts freely in codimension 2, $g^u = \mathrm{id}_B$, so g has order u and v divides u. Listing the integers of Euler number 6, $u \in \{7, 9, 14, 18\}$. Using that $\chi_{g \oplus \overline{g}} = \Phi_u \Phi_v$ or $\Phi_u {\Phi_v}^2$, g has determinant 1 and $\langle g \rangle$ acts freely in codimension 2, we work out all possibilities by hand and add them to the table. One example falling in this case is $\mathrm{diag}(1, \zeta_7, {\zeta_7}^2, {\zeta_7}^4)$.

(4) Suppose that $\phi(u) = 4$. Then g has two distinct eigenvalues of order u, and two remaining eigenvalues of respective order $v_1 \leq v_2$. As $\langle g \rangle$ acts freely in codimension 2, g^u, which has two trivial eigenvalues, must be trivial, so g has order u and v_1 and v_2 divide u. Similarly, $g^{\mathrm{lcm}(v_1, v_2)} = \mathrm{id}_B$, so u divides $\mathrm{lcm}(v_1, v_2)$. Listing integers of Euler number 4, $u \in \{5, 8, 10, 12\}$.

   (a) If v_1 divides v_2, then $v_2 = u$. We investigate all possibilities of determinant 1 satisfying Lemma 2.76 by hand and add them to the table. One of them is $\mathrm{diag}(\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$.

   (b) If v_1 does not divide v_2, then by Lemma 2.76 again, $\phi(v_1) + \phi(v_2) \leq 4$. Listing possibilities by hand, we see that $(v_1, v_2) \in \{(2, 3), (3, 4), (4, 6)\}$. From the divisibility relations between v_1, v_2 and u, we obtain that $u = 12$, and in fact, $(v_1, v_2) = (3, 4)$ or $(4, 6)$. In particular, g has order 12, so $g^6 = -\mathrm{id}_B$, and so g^3 has four eigenvalues of order 4. But since $v_1 = 3$ or $v_2 = 6$, this can not be the case. Contradiction!

(5) The last case is when $\phi(u) = 2$, i.e., $u = 3, 4$, or 6. In that case, each eigenvalue of g has order $1, 2, 3, 4$, or 6. As $\langle g \rangle$ acts freely in codimension 2, g has at most one eigenvalue of order 1 or 2.

   (a) Suppose that g has an eigenvalue of order 4. As it has determinant 1, it has an even number of eigenvalues of order 4, so at least two of them. Hence, by freeness in codimension 2, $g^4 = \mathrm{id}_B$, and so $g^2 = -\mathrm{id}_B$, i.e., all eigenvalues of g have order 4. There is a generator of $\langle g \rangle$ similar to either $\mathrm{diag}(i, i, i, i)$, or $\mathrm{diag}(i, -i, i, -i)$.

(b) Suppose that $u = 3$. Then as $(\phi(v), v) \leq (\phi(u), u)$ for any order v of another eigenvalue of g, the other eigenvalues have order $1, 2$, or 3. Hence, there are at least three eigenvalues of order 3, and thus by freeness in codimension 2, $g^3 = \mathrm{id}_B$. So g has order 3 and there is a generator of $\langle g \rangle$ similar to either $\mathrm{diag}(1, j, j, j)$, or $\mathrm{diag}(j, j^2, j, j^2)$.

(c) Suppose finally that $u = 6$ and g has no eigenvalue of order 4: Then g has order 6, so $g^3 = -\mathrm{id}_B$. All eigenvalues of g thus have order 2 or 6, so g has at least three eigenvalues of order 6. As g has determinant 1, we only have two possibilities: There is a generator of $\langle g \rangle$ similar to $\mathrm{diag}(-1, \omega, \omega, \omega)$, or $\mathrm{diag}(\omega, \omega^5, \omega, \omega^5)$.

This discussion constructs the first two columns of the table. The restrictions on B given in the third column are given by the same arguments as in the proof of Lemmas 10.8, 10.10. □

Corollary 10.16. *Let B be an abelian fourfold, and let $g, h \in \mathrm{Aut}(B, 0)$ be junior elements such that $\langle g \rangle$ and $\langle h \rangle$ act freely in codimension 2, with $\mathrm{ord}(g) \leq \mathrm{ord}(h)$. Then there are three possibilities:*

- *g and h are similar, in particular have the same order;*
- *g is similar to $\mathrm{diag}(1, j, j, j)$, h is similar to $\mathrm{diag}(-1, \omega, \omega, \omega)$, and B is isogenous to $E \times {E_j}^3$ for some elliptic curve E;*
- *$g = i\mathrm{id}_B$, h is similar to $\mathrm{diag}(\zeta_{12}, \zeta_{12}^5, \zeta_{12}, \zeta_{12}^5)$ and B is isomorphic to ${E_i}^4$.*

Proof. If g has order 7, then by Lemma 10.15, B is isogenous to $E \times {E_{u_7}}^3$ for some elliptic curve E. By uniqueness in the Poincaré decomposition of B [17, Thm.5.3.7], B is not isogenous to any of the other special abelian varieties appearing in Lemma 10.15. So, by Lemma 10.15 again, h being junior must have order 7. By Proposition 9.2, any junior element k of order 7 acting on a fourfold with $\langle k \rangle$ acting freely in codimension 2 are similar to $\mathrm{diag}(1, \zeta_7, {\zeta_7}^2, {\zeta_7}^4)$. So g and h are similar.

The same argument works if g has order $8, 15, 16, 20, 24$.

If g has order 3 or 6, then by Lemma 10.15, B is isogenous to $E \times {E_j}^3$ for some elliptic curve E. By uniqueness in the Poincaré decomposition of B [17, Thm.5.3.7], B is not isogenous to any of the other special abelian varieties appearing in Lemma 10.15. So, by Lemma 10.15 again, h being junior must have order 3 or 6. As we assumed $\mathrm{ord}(g) \leq \mathrm{ord}(h)$, the only strict inequality is when g has order 3 and h has order 6. In this case, by Proposition 9.2, g is similar to $\mathrm{diag}(1, j, j, j)$ and h to $\mathrm{diag}(-1, \omega, \omega, \omega)$.

The same argument works if g has order 4 or 12. □

We can now prove cyclicity of F when it contains a junior element of order 3.

Proposition 10.17. *Let B be an abelian fourfold, and let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements. Suppose that F contains an element similar to $\mathrm{diag}(1, j, j, j)$. Then F is cyclic and generated by one junior element.*

Proof. By Corollary 10.16, B is isogenous to $E \times {E_j}^3$ for some elliptic curve E, and any junior element in $\mathrm{Aut}(B, 0)$ is similar to $\mathrm{diag}(1, j, j, j)$, or $\mathrm{diag}(-1, \omega, \omega, \omega)$.

Suppose by contradiction that F is not generated by one junior element. Then there are two junior elements $g, h \in F$ such that $\langle g \rangle \not\subseteq \langle h \rangle$ and $\langle h \rangle \not\subseteq \langle g \rangle$. Up to possibly replacing them by their square, we have $\tilde{g}$ and $\tilde{h}$ both similar to $\mathrm{diag}(1, j, j, j)$. Their eigenspaces satisfy $\dim E_{\tilde{g}}(j) \cap E_{\tilde{h}}(j) = 2 \leq \dim E_{\tilde{g}\tilde{h}^{-1}}(1)$. As $\langle \tilde{g}, \tilde{h} \rangle \subset F$ acts freely in codimension 2, $\tilde{g} = \tilde{h}$. Since $\langle \tilde{g} \rangle \subset \langle h \rangle$, $\tilde{g} \neq g$, so $\tilde{g} = g^2$. Similarly, $\tilde{h} = h^2$. Since $g^3 = h^3 = -\mathrm{id}$, it nonetheless yields $g = h$, contradiction. $\square$

Let us now present our general strategy to prove that F is cyclic. By Lemma 10.15, the prime divisors of $|F|$ are $2, 3, 5$, and 7. Hence, $|F| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta$. Since 2^α (respectively 3^β, etc.) is the order of a 2 (respectively 3, etc.)-Sylow subgroup of F, we can rely on Sylow theory to bound $|F|$, as in the following result.

Proposition 10.18. *Let B be an abelian fourfold, and let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *acting freely in codimension* 2*, generated by junior elements, containing no junior element of order 3. Then*

$$|F| \textit{ divides } 2^4 \cdot 3 \cdot 5 \cdot 7 = 1680.$$

The proof of this proposition relies on the following two lemmas.

Lemma 10.19. *Let B be an abelian fourfold, and let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *acting freely in codimension* 2*, containing no junior element of order 3. Let $p = 3, 5,$ or 7 divide $|F|$. Then a p-Sylow subgroup S of F is cyclic of order p.*

Proof. As S is a p-group, its center $Z(S)$ is non-trivial. Hence, it contains an element g of order p. Let $h \neq \mathrm{id} \in S$. By Lemma 10.15, F has no element of order p^2, so h has order p. Since g and h commute, they are codiagonalizable. Let v, w be two non-colinear common eigenvectors of them associated with eigenvalues other than 1. Let $\tilde{g} \in \langle g \rangle$ and $\tilde{h} \in \langle h \rangle$ satisfy $\tilde{g}(v) = \tilde{h}(v) = \zeta_p v$.

If $p = 3$ or 5, Lemma 10.15 shows that $E_g(1) = E_h(1) = \{0\}$, so $\tilde{g}\tilde{h}^{-1}$ can not have 1 as an eigenvalue and be of order p. So it is trivial, i.e., $\tilde{g} = \tilde{h}$, and $h \in \langle g \rangle$.

Suppose $p = 7$. If $\tilde{g}(w) \neq \tilde{h}(w)$, then by Lemma 10.15, $\{\tilde{g}(w), \tilde{h}(w)\} = \{{\zeta_7}^2 w, {\zeta_7}^4 w\}$. So $\tilde{g}\tilde{h}^2$ has eigenvalue ${\zeta_7}^3$ on v, and ζ_7 or ${\zeta_7}^3$ on w, which in either case contradicts Lemma 10.15. So $\tilde{g}(w) = \tilde{h}(w)$, i.e., $\tilde{g}\tilde{h}^{-1}$ has eigenvalue 1 with multiplicity two. By freeness in codimension 2, $\tilde{g} = \tilde{h}$, hence $h \in \langle g \rangle$. $\square$

Lemma 10.20. *Let B be an abelian fourfold, and let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *acting freely in codimension* 2*. If not trivial, a 2-Sylow subgroup S of F is cyclic or a generalized quaternion group, and its order divides 16.*

Proof. By Lemma 10.15, the element of order 2 in F is unique: it is $-\mathrm{id}_B$. By [170, 5.3.6], S is hence either cyclic or a generalized quaternion group. Moreover, by Lemma 10.15, S has no element of order 32. Hence, the only case where the order of S does not divide 16, is when S is isomorphic to Q_{32}. Let us however show that this is impossible.

Indeed, Q_{32} contains an element h of order 16 and an element s of order 4 such that $shs^{-1} = h^{-1}$ [170, pp.140-141]. However, if $h \in S$ is an element of order 16, it can not be conjugated in S to h^{-1}, because by Lemma 10.15 they have distinct eigenvalues. $\square$

Proof of Proposition 10.18. It is straightforward from Lemma 10.19 and Lemma 10.20. $\square$

The following Lemma and Proposition show that if 7 divides $|F|$, i.e., if F contains a junior element of order 7, then F is cyclic generated by one junior element of order 7.

Lemma 10.21. *Let B be an abelian fourfold, and let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *acting freely in codimension* 2*, containing no junior element of order* 3*. Suppose that* 7 *divide* $|F|$*. Let S be a* 7*-Sylow subgroup of F. Then there is a normal subgroup N of F such that $F = N \rtimes S$.*

Proof. By Burnside's normal complement theorem [172, Theorem 7.50], it is enough to show that $N_F(S) = C_F(S)$.

Let g be a generator of S. By Lemma 10.15, if $f \in N_F(S)$, then $fgf^{-1} \in \{g, g^2, g^4\}$, because they are the only elements with the same set of eigenvalues as g. So $f^3 \in C_F(S)$. Let us show by contradiction that $f \in N_F(S)$ can not have order 3. Looking at the action of f on the eigenspaces of g in coordinates diagonalizing g,

$$f = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & y \\ 0 & x & 0 & 0 \end{pmatrix},$$

with $xyz = \bar{t}$, and so $\chi_f = (X - t)(X^3 - \bar{t})$. But by Lemma 10.15, elements of order 3 in F (which by assumption cannot be junior) have characteristic polynomial $(X^2 + X + 1)^2$, contradiction. So $N_F(S)$ has no element of order 3. To sum up, if $f \in N_F(S)$, then $f^3 \in C_F(S)$ and 3 is prime to the order of f, so $f \in C_F(S)$. □

Proposition 10.22. *Let B be an abelian fourfold, and let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *acting freely in codimension* 2*, generated by junior elements, containing no junior element of order 3. Suppose that* 7 *divides* $|F|$*. Then F is cyclic and generated by one junior element.*

Proof. Let S be a 7-Sylow subgroup of F. By Lemma 10.21, $F = N \rtimes S$, where N is a normal subgroup of F, and by Proposition 10.18, $|N|$ divides 240. A simple `GAP` program in the appendix checks that a group of order dividing 240 cannot have an automorphism of order 7. So S acts trivially on N, i.e., $F = N \times S$. But F is generated by its junior elements, which all have order 7 by Corollary 10.16. So N is trivial, and $F = S$ is cyclic of order 7. □

Now we can focus on the case when F contains no junior element of order 3 or 7. We start by showing that, provided F is cyclic, it is generated by one junior element.

Lemma 10.23. *Let F be a cyclic group. If E is a set of generators of F and all elements of E have the same order, then any element of E actually generates F.*

Proof. Suppose $F = \mathbb{Z}_d$ and every element of E has order k dividing d. Then E is actually a subset of $\mathbb{Z}_k \subset \mathbb{Z}_d$, and since E must generate $\mathbb{Z}_d$, it must be $k = d$. So any element $e \in E$ satisfies $\langle e \rangle = \mathbb{Z}_d = F$. □

Corollary 10.24. *Let B be an abelian fourfold, and let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *acting freely in codimension* 2*, generated by junior elements, containing no junior element of order* 3 *or* 7*. If F is cyclic, then F is generated by one junior element.*

Proof. Assume that F is cyclic. If F contains one junior element of order $8, 15, 16, 20$, or 24, then by Corollary 10.16, all junior elements have the same order and we use Lemma 10.23 to conclude.

Else, the junior elements of F each have order 4 or 12. If there are no junior elements of order 12, Lemma 10.23 concludes again. If there is a junior element g of order 12, then a quick computation from Lemma 10.15 shows that g^3 is the only junior element of order 4 in F, and thus the junior elements of order 12 actually generate F too, so we conclude by Lemma 10.23. $\square$

These versions of Lemma 10.21 for 3- and 5-Sylow subgroups will be useful too.

Lemma 10.25. *Let B be an abelian fourfold, and let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements. Suppose that $p \in \{3, 5\}$ divides $|F|$. Let S be a p-Sylow subgroup of F. Then $N_F(S)/C_F(S)$ is isomorphic to a subgroup of $(\mathbb{Z}_p)^\times$.*

Proof. The quotient $N_F(S)/C_F(S)$ acts faithfully by conjugation on S, and therefore embeds in $\mathrm{Aut}(S)$, which by Lemma 10.19 is isomorphic to $(\mathbb{Z}_p)^\times$. $\square$

Lemma 10.26. *Let B be an abelian fourfold, and let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements. Suppose that 5 divides $|F|$. Let S be a 5-Sylow subgroup of F. Then, if $f \in N_F(S)$ is a junior element of order 8, $[f] \in N_F(S)/C_F(S)$ cannot have order 4.*

Proof. Let $f \in N_F(S)$ be a junior element of order 8 such that $[f] \in N_F(S)/C_F(S)$ has order 4, and let g be a generator of S. Looking at the action of f on the eigenspaces of g in coordinates diagonalizing g,

$$f = \begin{pmatrix} 0 & 0 & 0 & t \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \end{pmatrix},$$

with $xyzt = -1$, and so $\chi_f = X^4 + 1$. By Lemma 10.15, no junior element of order 8 has this characteristic polynomial, contradiction. $\square$

We finally prove the following two key propositions, which imply Proposition 10.12.

Proposition 10.27. *Let B be an abelian fourfold, and let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements, containing no junior element of order 3 or 7. Then a 2-Sylow subgroup of F is either trivial, or cyclic.*

Proposition 10.28. *Let B be an abelian fourfold, and let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements, containing no junior element of order 3 or 7. Suppose that a 2-Sylow subgroup of F is trivial or cyclic. Then F is cyclic.*

Proof of Proposition 10.28. Let us write $|F| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma$ with $\alpha \in [\![0, 4]\!]$, $\beta, \gamma \in [\![0, 1]\!]$. By Lemma 10.19 and by assumption, the Sylow subgroups of F are cyclic, so [170, pp.290-291] applies and F is a semidirect product: $F \simeq (\mathbb{Z}_{5^\gamma} \rtimes \mathbb{Z}_{3^\beta}) \rtimes \mathbb{Z}_{2^\alpha}$. Since 3^β is coprime to $\phi(5^\gamma)$, the group $\mathbb{Z}_{5^\gamma}$ has no automorphism of order 3, and thus the first semidirect product is direct:

$$F \simeq (\mathbb{Z}_{5^\gamma} \times \mathbb{Z}_{3^\beta}) \rtimes \mathbb{Z}_{2^\alpha}.$$

If $\beta = \gamma = 1$, the group F contains an element of order 15, so by Lemma 10.15, B is isomorphic to ${E_{u_{15}}}^4$ and all junior elements of F have order 15. However, since $F \simeq \mathbb{Z}_{15} \rtimes \mathbb{Z}_{2^\alpha}$, and since F is generated by its junior elements, we must have $\alpha = 0$, and so $F \simeq \mathbb{Z}_{15}$ is cyclic and generated by one junior element.

If $\beta = \gamma = 0$, then $F \simeq \mathbb{Z}_{2^\alpha}$ is cyclic.

Else, write $p = 3^\beta 5^\gamma$ and $F \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_{2^\alpha}$. Note that $\mathbb{Z}_p \rtimes \mathbb{Z}_{2^{\alpha-1}}$ is a proper subgroup of F containing all elements whose order divides $2^{\alpha-1}p$. As F is generated by its junior elements, their orders cannot all divide $2^{\alpha-1}p$: There is a junior element $g \in F$ of order 2^α or $2^\alpha p$. If g has order $2^\alpha p$, $\langle g \rangle = F$ and so F is cyclic. If g has order 2^α, we can write $F \simeq \langle u \rangle \rtimes \langle g \rangle$, where u is an element of F of order p. The discussion now depends on α and p.

(1) By Lemma 10.15, if g has order 4, then $g = \imath\mathrm{id}$ commutes with every element of F, so the semidirect product is direct and F is cyclic.

(2) If $p = 5$ and g has order 8, by Lemma 10.26, g^2 and u commute, so $g^2 u$ has order 20. Since g is junior of order 8, by Lemma 10.15, B is isomorphic to ${E_{u_8}}^4$. So by Lemma 10.15 again, B has no automorphism of order 20, contradiction.

(3) If $p = 5$ and g has order 16, by Lemma 10.25, g^4 and u commute, so $g^4 u$ has order 20. But since g is junior of order 16, by Lemma 10.15, B has no automorphism of order 20, contradiction.

(4) If $p = 3$ and g has order 16, by Lemma 10.25, $g^2 u$ has order 24. But since g is junior of order 16, by Lemma 10.15, B has no automorphism of order 24, contradiction.

(5) If $p = 3$ and g has order 8, then $F \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_8$. With `GAP`, we check in the Appendix that:

   - The irreducible representations of F have rank 1 or 2.
   - No irreducible character of F takes value j or j^2, so $F \subset \mathrm{Aut}(B, 0)$ has no irreducible subrepresentation of rank 1.
   - The only two irreducible representations of F of rank 2 sending $-\mathrm{id} \in F$ to $-\mathrm{id}$ indeed are complex conjugates, so all elements of $F \subset \mathrm{Aut}(B, 0)$ have characteristic polynomials in $\mathbb{Q}[X]$.

   However, $g \in F$ is a junior element of order 8, which by Lemma 10.15 has a non-rational characteristic polynomial, contradiction.

□

We prove Proposition 10.27 by contradiction.

Proof of Proposition 10.27. Suppose that 2 divides $|F|$ and that a 2-Sylow subgroup of F is not cyclic. We first show that any junior element in F has order 15, 20 or 24.

By contradiction and by Proposition 9.2, let $g \in F$ be a junior element of order $4, 8, 12$, or 16. If g has order 12, then $g^3 \in F$ is a junior element of order 4, and F thus contains a junior element $\tilde{g}$ of order $4, 8$, or 16. Let S be a 2-Sylow subgroup containing that junior element. By assumption, S is not cyclic, so by Lemma 10.20, S is isomorphic to Q_8 or to Q_{16}. Clearly, Q_8 and Q_{16} have no element of order 16, and no element of order 4 in their centers, so $\tilde{g}$ has order 8. As Q_8 has no element of order 8, S is isomorphic to Q_{16}. But we easily check with `GAP` that:

- The irreducible representations of Q_{16} have rank 1 or 2.
- The only irreducible representations of Q_{16} of rank r sending the unique element of order 2 to $-\mathrm{id}_r$ are two complex conjugates representations with $r = 2$, so all elements of $S \subset \mathrm{Aut}(B, 0)$ have characteristic polynomials in $\mathbb{Q}[X]$.

However, $\tilde{g} \in S$ is a junior element of order 8, which by Lemma 10.15 has a non-rational characteristic polynomial, contradiction.

So any junior element in F has order 15, 20 or 24. We also know that:

- F has exactly one element of order 2, by Lemma 10.15.
- A 2-Sylow subgroup of F is isomorphic to Q_8 or Q_{16}, by Lemma 10.20.
- $|F|$ divides 240, by Proposition 10.18.
- F has no element of order 60 or 40, by Lemma 10.15.
- If F has elements of orders $o, o' \in \{15, 20, 24\}$, then $o = o'$, by Lemma 10.15.

We check with `GAP` that there are only five groups satisfying all these properties, namely the groups indexed `(40,4)`,`(40,11)`,`(80,18)`,`(48,8)`, and `(48,27)` in the `SmallGroup` library. The function `StructureDescription` then shows that they are respectively of the form $\mathbb{Z}_5 \rtimes Q_8$, $\mathbb{Z}_5 \times Q_8$, $\mathbb{Z}_5 \rtimes Q_{16}$, $\mathbb{Z}_3 \rtimes Q_{16}$, and $\mathbb{Z}_3 \times Q_{16}$. Note that only $\mathbb{Z}_5 \times Q_8$, $\mathbb{Z}_5 \rtimes Q_{16}$ are generated indeed by their elements of orders $(15, 24,$ or$)$ 20. Checking the irreducible character tables of these two cadidates with `GAP` shows that they have no appropriate four-dimensional representation (see Appendix for programs supporting this discussion.)

This concludes the proof of Proposition 10.27. □

Proof of Proposition 10.12. If F contains a junior element of order 3, then Proposition 10.17 applies and shows that F is cyclic generated by one junior element. If F contains no junior element of order 3, but one of order 7, then Proposition 10.22 applies and shows that F is cyclic generated by one junior element. Finally, if F contains no junior element of order 3 or 7, Proposition 10.27 shows that its 2-Sylow subgroups are cyclic or trivial, Proposition 10.28 deduces that F is cyclic and Corollary 10.24 proves that F is generated by one junior element. □

order of g	a generator of $\langle g \rangle$ up to similarity	restrictions on B
1	id	B arbitrary
2	$-$id	
3	$\mathrm{diag}(j, j^2, j, j^2)$	
4	$\mathrm{diag}(i, -i, i, -i)$	
5	$\mathrm{diag}(\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$	
6	$\mathrm{diag}(\omega, \omega^5, \omega, \omega^5)$	
8	$\mathrm{diag}(\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7)$	
10	$\mathrm{diag}(\zeta_{10}, \zeta_{10}^3, \zeta_{10}^7, \zeta_{10}^9)$	
12	$\mathrm{diag}(\zeta_{12}, \zeta_{12}^5, \zeta_{12}^7, \zeta_{12}^{11})$	
3	$\mathrm{diag}(1, j, j, j)$	$B \sim E \times {E_j}^3$
6	$\mathrm{diag}(-1, \omega, \omega, \omega)$	
9	$\mathrm{diag}(j^2, \zeta_9, \zeta_9^4, \zeta_9^7)$	$B \simeq {E_j}^4$
18	$\mathrm{diag}(\omega^5, \zeta_{18}, \zeta_{18}^7, \zeta_{18}^{13})$	
4	$i\mathrm{id}$	$B \simeq {E_i}^4$
12	$\mathrm{diag}(\zeta_{12}, \zeta_{12}^5, \zeta_{12}, \zeta_{12}^5)$	
20	$\mathrm{diag}(\zeta_{20}, \zeta_{20}^9, \zeta_{20}^{13}, \zeta_{20}^{17})$	
7	$\mathrm{diag}(1, \zeta_7, {\zeta_7}^2, {\zeta_7}^4)$	$B \sim E \times {E_{u_7}}^3$
14	$\mathrm{diag}(-1, \zeta_{14}, \zeta_{14}^9, \zeta_{14}^{11})$	
8	$\mathrm{diag}(\zeta_8, \zeta_8^3, \zeta_8, \zeta_8^3)$	$B \simeq {E_{u_8}}^4$
24	$\mathrm{diag}(\zeta_{24}, \zeta_{24}^{11}, \zeta_{24}^{17}, \zeta_{24}^{19})$	
15	$\mathrm{diag}(\zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$	$B \simeq {E_{u_{15}}}^4$
30	$\mathrm{diag}(\zeta_{30}, \zeta_{30}^{17}, \zeta_{30}^{19}, \zeta_{30}^{23})$	
16	$\mathrm{diag}(\zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7)$	$B \simeq {S_{u_{16}, v_{16}}}^2$
	$\mathrm{diag}(\zeta_{16}, \zeta_{16}^7, \zeta_{16}^{11}, \zeta_{16}^{13})$	
20	$\mathrm{diag}(\zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9)$	$B \simeq {E_{u_{20}}}^4$
24	$\mathrm{diag}(\zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11})$	$B \simeq {E_{u_{24}}}^4$

Table 10.3: Classification of possible elements of g in $\mathrm{Aut}(B, 0)$, with colored junior elements.

CHAPTER 11

RULING OUT JUNIOR ELEMENTS IN CODIMENSION 4

The aim of this section is to rule out eight out of the twelve types of junior elements presented in Proposition 9.2, namely those which fix pointwise at least one subvariety of codimension 4, but no subvariety of codimension 3.

Proposition 11.1. *Let A be an abelian variety of dimension n, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X. Then, if $g \in G$ is a junior element, the matrix $M(g)$ cannot have eigenvalue 1 with multiplicity exactly $n-4$.*

Remark 11.2. Whether the local affine quotients corresponding to these eight types of junior elements admit a crepant resolution is actually settled by toric geometry in [176]. In fact, by [176, Thm.3.1],

$$\mathbb{C}^4/\langle \imath \mathrm{id}\rangle, \quad \mathbb{C}^4/\langle \mathrm{diag}(\omega,\omega,\omega,-1)\rangle, \quad \mathbb{C}^4/\langle \mathrm{diag}(\zeta_8,\zeta_8,\zeta_8^3,\zeta_8^3)\rangle,$$
$$\mathbb{C}^4/\langle \mathrm{diag}(\zeta_{12},\zeta_{12},\zeta_{12}^5,\zeta_{12}^5)\rangle, \quad \mathbb{C}^4/\langle \mathrm{diag}(\zeta_{15},\zeta_{15}^2,\zeta_{15}^4,\zeta_{15}^8)\rangle$$

have a crepant Fujiki-Oka resolution, and by [176, Prop.3.9],

$$\mathbb{C}^4/\langle \mathrm{diag}(\zeta_{16},\zeta_{16}^3,\zeta_{16}^5,\zeta_{16}^9)\rangle, \ \mathbb{C}^4/\langle \mathrm{diag}(\zeta_{20},\zeta_{20}^3,\zeta_{20}^7,\zeta_{20}^9)\rangle, \ \mathbb{C}^4/\langle \mathrm{diag}(\zeta_{24},\zeta_{24}^5,\zeta_{24}^7,\zeta_{15}^{11})\rangle$$

admit no toric crepant resolution. They could nevertheless have a non-toric crepant resolution.

In light of this remark, the proof of Proposition 11.1 must crucially involve global arguments.

11.1 Ruling our junior elements of order 4,8,12,15,16,20,24. In this subsection, we rule out the seven types of junior elements or order other than $3,6,7$.

Proposition 11.3. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X. Then any junior element of G has order $3,6,$ or 7.*

Remark 11.4. Let A be an abelian variety, G be a group acting freely in codimension 2 on A. As translations in G form a normal subgroup G_0, we can write:

$$(A/G_0)/(G/G_0) \simeq A/G.$$

Clearly, A/G_0 is isogenous to A and G/G_0 still acts freely in codimension 2 on it, except that it contains no translation. Hence, we can assume without loss of generality that G contains no translation (other than id). In particular, any element of G has the same finite order as its matrix.

Proof of Proposition 11.3. By contradiction, suppose that $g \in G$ is a junior element of order $d \in \{4, 8, 12, 15, 16, 20, 24\}$, of minimal order among the junior elements of G of such orders. Up to conjugating the whole group G by an appropriate translation, we may assume that g fixes $0 \in A$. In particular, g fixes pointwise an abelian subvariety W of A of codimension 4, so Propositions 10.11 and 10.3 show that $\mathrm{PStab}(W) = \langle g \rangle$, and define a $\langle g \rangle$-stable complementary abelian subvariety B to W in A. The key to the proof is that a well-chosen power g^α of g has strictly more fixed points in B than g, as many distinct eigenvalues as g, but is not be a junior element. Indeed, we set α depending on d as follows, and check with Proposition 9.2 that g^α is not junior and has as many distinct eigenvalues as g. As for fixed points, applying [17, Prop.13.2.5(c)] shows that $(g^\alpha)|_B$ has strictly more of them than $g|_B$ in B.

d	4	8	12	15	16	20	24
α	2	2	4	3	2	4	3

Table 11.1: Definition of a certain $\alpha \in [\![0, d-1]\!]$ depending on d

Let $\tau \in B$ be a fixed point of g^α that is not fixed by g. Note that $W + \tau$ is pointwise fixed by g^α. By Proposition 10.11, $\mathrm{PStab}(W + \tau) = \langle h \rangle$ for some junior element h.

By Proposition 10.3, there is an $\langle h \rangle$-stable translated abelian subvariety B' of A containing τ such that B' and $W + \tau$ are complementary. By uniqueness in Poincaré's complete reducibility theorem [17, Thm.5.3.7], the abelian varieties B and $B' - \tau$ are isogenous, hence determined by the order of g and h respectively, by Lemma 10.15.

Let us discuss the special case when $B \simeq {E_i}^4$, i.e., when junior elements of order both 4 and 12 exist in $\mathrm{Aut}_{\mathbb{Q}}(B, 0) = \mathrm{Aut}_{\mathbb{Q}}(B', 0)$. If g or h has order 4, then by the minimality assumption on g, g has order 4, and by Lemma 10.15, either $g = h$ or $g^3 = h$. So $g \in \langle h \rangle$, and thus $g(\tau) = \tau$, contradiction!

By Corollary 10.16, we can now assume that g and h have the same order $d \in \{8, 12, 15, 16, 20, 24\}$, and similar matrices. Recall that $g^\alpha \in \langle h \rangle$. Since g and h have the same order, it implies $\langle g^\alpha \rangle = \langle h^\alpha \rangle$, i.e., $g^\alpha = h^{u\alpha}$ for some u coprime to $\frac{d}{\alpha}$. Since g and g^α, and h and $h^{u\alpha}$ have the same number of distinct eigenvalues, it follows from $g^\alpha = h^{u\alpha}$ that the eigenspaces of g and h are the same, i.e., g and h commute. We discuss two cases separately.

(1) If $d = 8$ or 12, then in appropriate coordinates, we have:

$$M(g) = \mathrm{diag}(\mathbf{1}_{n-4}, \zeta_d, \zeta_d, {\zeta_d}^m, {\zeta_d}^m)$$
$$M(h) = \mathrm{diag}(\mathbf{1}_{n-4}, {\zeta_d}^m, {\zeta_d}^m, \zeta_d, \zeta_d)$$

for some integer $m \in [\![2, d-1]\!]$ such that $2 + 2m = d$. In particular, $m^2 \equiv 1 \bmod d$, so $g = h^m \in \langle h \rangle$, contradiction!

(2) Else, $d = 15, 16, 20$, or 24. There is an integer u' coprime to d such that, in appropriate coordinates,

$$M(g) = \mathrm{diag}(\mathbf{1}_{n-4}, \zeta_d, {\zeta_d}^a, {\zeta_d}^b, {\zeta_d}^c)$$
$$M(h^{u'}) = \mathrm{diag}(\mathbf{1}_{n-4}, \zeta_d, {\zeta_d}^{\sigma(a)}, {\zeta_d}^{\sigma(b)}, {\zeta_d}^{\sigma(c)})$$

for some distinct integers $a, b, c \in [\![2, d-1]\!]$ coprime to d, and permutation σ of $\{a, b, c\}$. If $\sigma = \mathrm{id}$, then $g = h^{u'} \in \langle h \rangle$, contradiction! Nevertheless, let us prove that $\sigma = \mathrm{id}$. Note that

$$(h^{u'-u})^{\alpha} = (h^{u'} g^{-1})^{\alpha} (h^{-u} g)^{\alpha} = \mathrm{diag}(\mathbf{1}_{n-3}, {\zeta_d}^{(\sigma(a)-a)\alpha}, {\zeta_d}^{(\sigma(b)-b)\alpha}, {\zeta_d}^{(\sigma(c)-c)\alpha}),$$

and thus $(h^{u'-u})^{\alpha}$ fixes a translated abelian variety $W' \supset W + \tau$ of codimension at most 3. By Proposition 10.11, $\mathrm{PStab}(W')$ is trivial, or cyclic and generated by one junior element k of order 3 or 7. In the second case, as $k \in \mathrm{PStab}(W + \tau)$, k restricts to an automorphism of the fourfold B', which also has h junior of order $d \neq 3, 6, 7$ acting on it. This contradicts Corollary 10.16. Hence, $(h^{u'-u})^{\alpha} \in \mathrm{PStab}(W') = \{\mathrm{id}\}$, so for any $\ell \in \{a, b, c\}$, $(\sigma(\ell) - \ell)\alpha$ is a multiple of d. However, α was chosen so that g^{α} and g have the same number of distinct eigenvalues, i.e., $a\alpha, b\alpha, c\alpha$ are distinct modulo d. In particular, $\sigma(\ell)\alpha = \ell\alpha$ modulo d if and only if $\sigma(\ell) = \ell$. So $\sigma = \mathrm{id}$, contradiction!

□

11.2 Ruling out junior elements of order 6 with four non-trivial eigenvalues. In this subsection, we conclude the proof of Proposition 11.1 by ruling out the one remaining type of junior element fixing at least one subvariety of codimension 4, but no subvariety of codimension 3. It is the type of junior element of order 6, and matrix similar to $\mathrm{diag}(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$.

Proposition 11.5. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X. Then there is no junior element of G with matrix similar to* $\mathrm{diag}(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$.

The proof involves general arguments which we will use later, hence we factor it into a general lemma.

Lemma 11.6. *Let A be an abelian variety of dimension n, G a group acting freely in codimension 2 on A without translations such that A/G has a simply-connected crepant resolution X. Suppose that $g \in G$ fixes $0 \in A$ and has order d. Let W be the abelian subvariety of codimension k in A that g fixes pointwise, and denote by G_W the subgroup of G generated by*

$$G_{\mathrm{gen}} = {G_{\mathrm{gen}}}^{-1} = \{h \in G \mid \exists \tau \in A \text{ such that } h \in \mathrm{PStab}(W + \tau)\}.$$

Then

(1) *There is an $M(G_W)$-stable complementary abelian subvariety B to W, which induces a representation $\rho : G_W \to \mathrm{Aut}(B, 0)$ by $\rho(h) := M(h)|_B$.*

(2) *If we denote by pr_W, pr_B the projections induced by the splitting of the tangent space, then, for any $h \in G_W$,*

- *$M(h) = pr_W + \rho(h) pr_B$*
- *$pr_W(T(h)) = 0$, i.e., $T(h) \in B$*

(3) *The representation ρ is faithful and takes values in $\mathrm{SL}(H^0(T_B))$.*

(4) *The abelian subvariety B is in fact G_W-stable.*

(5) *Every $h \in G_W$ that fixes a point $\tau \in A$ fixes the point $pr_B(\tau) \in B$.*

(6) *Moreover, if we assume additionally that there is an integer $\alpha \in [\![1, d-1]\!]$ such that $M(g^\alpha)$ is similar to $\mathrm{diag}(\mathbf{1}_{n-k}, -\mathbf{1}_k)$, then, for any $h \in G_W$, h and g^α commute and*

   - *either there is a point $\tau \in A$ such that $h \in \mathrm{PStab}(W+\tau) \cup g^\alpha \mathrm{PStab}(W+\tau)$;*
   - *or there is no such point, and 1 and -1 are eigenvalues of $\rho(h)$.*

(7) *Same assumption. The translation part $T(h)$ of h is a 2-torsion point of B.*

(8) *Same assumption. If h has even order and fixes a point in A, all fixed points of h in B are of 2-torsion.*

(9) *Same assumption. If h is a junior element of order 3, then h fixes a 2-torsion point in B.*

Proof. (1) follows immediately from [17, Prop.13.5.1], since $M(G_W)$ is a finite group of group automorphisms of the abelian variety A, and W is $M(G_W)$-stable.

(2) is proven by induction on the number of generators used to write $h \in G_W$. First, if $h \in G_W$ is in G_{gen}, there is a point $\tau \in A$ such that $h \in \mathrm{PStab}(W+\tau)$. In particular, for $w \in W$ and $b \in B$;

$$M(h)(w+b) = h(w+\tau) - h(\tau) + M(h)(b) = w + \rho(h)(b),$$

as wished. Moreover, $T(h) = (\mathrm{id} - M(h))(\tau)$, so $pr_W(T(h)) = 0$.

Second, if $h_1, h_2 \in G_W$ satisfy (2), then

$$M(h_1h_2) = M(h_1)M(h_2) = pr_W + \rho(h_1h_2)pr_B,$$

since ρ is a group morphism and $pr_W pr_B = pr_B pr_W = 0$. Moreover, $T(h_1h_2) = T(h_1) + M(h_1)T(h_2)$, and the fact that $pr_W(T(h_1h_2)) = 0$ easily follows from the induction assumption, notably using $pr_W(\mathrm{id} - M(h_1)) = 0$.

For (3), let $h \in G_W$ and note that $\rho(h) = \mathrm{id}_B$ if and only if $M(h) = pr_W + pr_B = \mathrm{id}_A$, so ρ is faithful since M is. Note that by Proposition 8.4 and Lemma 8.6, M takes values in $\mathrm{SL}(H^0(T_A))$. Hence, by Item 1 of (2), ρ takes values in $\mathrm{SL}(H^0(T_B))$.

Regarding (4) we note that, for $h \in G_W$, $h(B) = M(h)B + T(h) = B + T(h) = B$ by Item 2 of (2).

(5) is clear from Item 1 of (2).

We now prove (6). Note that $\rho(g^\alpha) = -\mathrm{id}_B$ commutes with any element of $\rho(G_W)$, and thus, as ρ is faithful, g^α is in the center of G_W.

Let $h \in G_W$ and assume that there is no point $\tau \in A$ fixed by h or $g^\alpha h$. In other words, neither $T(h)$ is in $\mathrm{Im}(\mathrm{id} - M(h))$, nor $T(g^\alpha h)$ is in $\mathrm{Im}(id - M(g^\alpha h|_B))$. By Item 2 of (2), $T(h)$ and $T(g^\alpha h)$ belongs to B. Hence, the images $\mathrm{Im}(\mathrm{id}_B - M(h)|_B)$ and $\mathrm{Im}(id_B - M(g^\alpha h)|_B) = \mathrm{Im}(id + M(h)|_B)$ must be proper subvarieties of B, so 1 and -1 must be eigenvalues of $\rho(h) = M(h)|_B$.

For (7), we use that h commutes with g^α by (6), that $g(0) = 0$, that $T(h) \in B$ by Item 2 of (2), and that $g^\alpha|_B = -\mathrm{id}_B$. It yields

$$0 = g^\alpha(h(0)) - h(g^\alpha(0)) = g^\alpha(T(h)) - T(h) = -2T(h),$$

so $T(h)$ is of 2-torsion.

For (8), assume that h fixes a point τ in A and has even order. For some β, h^β has order 2, and thus equals g^α: So, every fixed point of h is a fixed point of g^α. The points fixed by g^α are all of the form $w+\tau$, with $w \in W$, and $\tau \in B$ a 2-torsion point. But such a point $w+\tau$ being fixed by h, we have that $W + w + \tau = W + \tau$ is pointwise fixed by h, and in particular, the 2-torsion point $\tau \in B$ is a fixed point of h.

For (9), assume that h is a junior element of order 3. By (5), it fixes a point $\tau \in B$, and a translated abelian subvariety $W' + \tau$, where W' is an abelian subvariety of codimension 3 in A. Let B' be a $\langle h \rangle$-stable complementary to $W' \cap B$ in B. We write $\tau = w' + b'$, with $w' \in W' \cap B$ and $b' \in B'$: It gives $h(b') = h(\tau - w') = \tau - w' = b'$, i.e., h fixes $b' \in B'$. Moreover, since $h|_{B'} = j\mathrm{id}_{B'}$, it holds

$$0 = h(b') - b' = (j-1)b' + T(h).$$

Multiplying by $2(j^2-1)$, we see that $3b'$ is a point of 2-torsion of B'. Since $h(b') = b'$ and $3T(h) = T(h)$, this point $3b'$ is fixed by h. □

We can now come back to our Proposition.

Proof of Proposition 11.5. By Remark 11.4, we can assume that G contains no translation other than id_A. By contradiction, suppose that there is an element $g \in G$ such that $g(0) = 0$ and, in some coordinates,

$$M(g) = \mathrm{diag}(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1).$$

We import the notations of Lemma 11.6, whose hypotheses are satisfied by g for $k = 4$, $d = 6$, $\alpha = 3$. The proof of the proposition now goes in three steps. First, we show that every element of $\rho(G_W)$ is similar to an element of $\langle \rho(g) \rangle \simeq \langle \mathrm{diag}(\omega, \omega, \omega, -1) \rangle$. Second, we deduce that $G_W = \langle g \rangle$. Third, we use global considerations on fixed loci to derive a contradiction from this description of G_W.

Step 1: By Lemma 11.6 (1) and (4), there is a G_W-stable complementary B to W. As $\rho(g)$ acts on it, B is isogenous to $E \times {E_j}^3$ for some elliptic curve E. By Proposition 10.11, for any τ in A, the group $\mathrm{PStab}(W + \tau)$ is trivial, or cyclic generated by one junior element k, and by Corollary 10.16, $\rho(k)$ is similar to $\rho(g)$ (if of order 6) or to $\rho(g^2)$ (if of order 3) in $\mathrm{GL}(H^0(T_B))$. By Lemma 11.6 Item 1 of (2), $M(k)$ is therefore similar to $M(g)$ or $M(g^2)$ in $\{\mathrm{id}_W\} \times \mathrm{GL}(H^0(T_B))$. As g^3 commutes with such conjugation matrices, any element of $\langle k \rangle \cup \langle g^3 k \rangle = \mathrm{PStab}(W+\tau) \cup g^3\mathrm{PStab}(W+\tau)$ is similar to a power of g.

Now, assume that $h \in G_W$ is not similar to a power of g. Then Lemma 11.6 (6) shows that 1 and -1 are eigenvalues of $\rho(h)$. Applying Lemma 11.6 (6) again to h^2, we see that either h^2 is similar to a power of g, or 1 and -1 are eigenvalues of $\rho(h^2)$.

If 1 and -1 are eigenvalues of $\rho(h^2)$, $\rho(h)$, which has determinant 1, is similar to $\mathrm{diag}(1, -1, i, i)$, or to $\mathrm{diag}(1, -1, -i, -i)$. Moreover, $\rho(h)$ defines an automorphism of B, and by [17, Thm.13.2.8, Thm.13.3.2], B must thus be isogenous to $S \times {E_i}^2$ for some abelian surface S. We already know that B is isogenous to $E\times {E_j}^3$, but this contradicts the uniqueness of the Poincaré decomposition of B up to isogeny [17, Thm.5.3.7].

Hence, h^2 is similar to a power of g, and as 1 is an eigenvalue of multiplicity at least 2 for it, $\rho(h^2) = \mathrm{id}_B$. Hence, $\rho(h)$ is similar to $\mathrm{diag}(1, 1, -1, -1)$.

We just proved that if $h \in G_W$ is not similar to a power of g, then $\rho(h)$ is similar to $\mathrm{diag}(1, 1, -1, -1)$. However, if $\rho(h)$ is similar to $\mathrm{diag}(1, 1, -1, -1)$, then $\rho(hg)$ has ω and $-\omega$ as eigenvalues, and thus is neither similar to a power of g, nor to $\mathrm{diag}(1, 1, -1, -1)$, contradiction. This concludes Step 1.

Step 2: By Step 1 and since ρ is faithful, we know that every element of G_W has order $1, 2, 3$, or 6. Moreover, there is exactly one element of order 2, namely g^3, so $|G_W| = 2\cdot 3^\beta$ for some $\beta \geq 1$. Let S be a 3-Sylow subgroup of G_W, and $s \in Z(S)$ of order 3. Let $s' \in S \setminus \{\mathrm{id}_A\}$. By Step 1, every element of $\rho(S)$ other than id_B is similar to $\mathrm{diag}(1, j, j, j)$, or to $\mathrm{diag}(1, j^2, j^2, j^2)$, in particular, this is the case of s and s', and cannot be both the case of ss' and s^2s', since they commute. Hence, $s' \in \langle s\rangle$. So $S = \langle s\rangle \simeq \mathbb{Z}_3$, and thus $\beta = 1$.

So $G_W \supset \langle g\rangle$ has order 6: Hence $G_W = \langle g\rangle$.

Step 3: By [17, Cor.13.2.4, Prop.13.2.5(c)], the number of fixed points of g and g^3 on B are respectively 4 and 256. Let τ be a point of B fixed by g^3 but not by g.

By Proposition 10.11, there is a junior element h generating the cyclic group $\mathrm{PStab}(W + \tau)$. By Step 2, $\langle h\rangle \subset G_W = \langle g\rangle$. Moreover, as $g^3 \in \mathrm{PStab}(W + \tau) = \langle h\rangle$, we know that h has even order, hence order 6 by Proposition 11.3. So $\langle h\rangle = \langle g\rangle$, and as both g and h are the only junior elements of order 6 in their generated cyclic groups, $g = h$. But h fixes τ and g does not, contradiction. □

Proof of Proposition 11.1. It is straightforward from Propositions 11.3 and 11.5. □

CHAPTER 12

THE ISOGENY TYPE OF A

This section proves the first part of Theorem 7.6, namely the following proposition, inspired by [152, Proof of Lem.3.4].

Proposition 12.1. *Let A be an abelian variety of dimension n, G be a finite group acting freely in codimension 2 on A. Suppose that A/G has a crepant resolution X which is a Calabi-Yau manifold. Then either A is isogenous to ${E_j}^n$ and G is generated by junior elements of order 3 and 6, or A is isogenous to ${E_{u_7}}^n$ and G is generated by junior elements of order 7.*

Proof. By the $M(G)$-equivariant Poincaré's complete reducibility theorem [17, Thm.13.5.2, Prop.13.5.4, and the paragraph before], there are $M(G)$-stable abelian subvarieties $Y_1, \dots, Y_s$ of A such that:

(1) For any $i \in [\![1, s]\!]$, Y_i is isogenous to a power of a $M(G)$-stable $M(G)$-simple abelian subvariety of A. In particular, by [17, Prop.13.5.5], there is a simple abelian subvariety Z_i of Y_i such that Y_i is isogenous to a power of Z_i.

(2) For each $i \neq j$, the set of $M(G)$-equivariant homomorphisms satisfies

$$\mathrm{Hom}_{M(G)}(Y_i, Y_j) = \{0\}.$$

(3) The addition map $Y_1 \times \ldots \times Y_s \to A$ is an $M(G)$-equivariant isogeny.

We define

$$\begin{aligned} Y_I &= \prod_{i \in I} Y_i, \text{ where } I = \{i \in [\![1, s]\!] \mid Z_i \sim E_j\} \\ Y_J &= \prod_{j \in J} Y_j, \text{ where } J = \{j \in [\![1, s]\!] \mid Z_j \sim E_{u_7}\} \\ Y_K &= \prod_{k \in K} Y_k, \text{ where } K = [\![1, s]\!] \setminus (I \cup J). \end{aligned}$$

The action of $M(G)$ on $Y_I \times Y_J \times Y_K$ is diagonal by (2), and there is a proper surjective finite morphism $A/M(G) \to Y_I/M(G) \times Y_J/M(G) \times Y_K/M(G)$ induced by the G-equivariant addition by (3). Composing with projections, we get proper surjective morphisms f_I, f_J, f_K from $A/M(G)$ to $Y_I/M(G)$, to $Y_J/M(G)$, and to $Y_K/M(G)$.

Let $g \in G$ be a junior element. By Propositions 11.3 and 11.5, g has order 3, or 7, or 6 and then five or six non-trivial eigenvalues. By Proposition 10.6, A thus contains an abelian subvariety isogenous to ${E_j}^3$, or to ${E_{u_7}}^3$. Hence, $\dim Y_I + \dim Y_J \geq 3$, so one of the two quotients $Y_I/M(G)$, $Y_J/M(G)$ has positive dimension. Moreover, by

Proposition 10.6 again, if g has order 3 or 6, $M(g)$ acts trivially on Y_J and Y_K, and if g as order 7, it acts trivially on Y_I and Y_K. Hence, $M(g)$ acts with determinant 1 on each of the three factors.

But G is generated by its junior elements by Lemma 8.6 and Proposition 8.4. By [103, 199], $Y_I/M(G)$, $Y_J/M(G)$ and $Y_K/M(G)$ are thus normal Gorenstein varieties.

We can now pullback the volume form of $Y_I/M(G)$ if it has positive dimension $y_I = y$, of $Y_J/M(G)$ of dimension $y_J = y$ else, to an $M(G)$-invariant non-zero global holomorphic y-form on A. Note that the sections of $\Omega^{\cdot}_A$ are invariant by translations of A, so that we in fact have a G-invariant non-zero global holomorphic y-form on A. It pulls back to X, which is a Calabi-Yau variety. Hence $y = n$, and either $A \sim {E_j}^n$ or $A \sim {E_{u_7}}^n$. The order of junior elements generating G is given accordingly by Propositions 10.6, 11.3. $\square$

CHAPTER 13

JUNIOR ELEMENTS AND POINTWISE STABILIZERS IN CODIMENSION 5

In this section, we extend the results of Chapters 10 and 11 to codimension $k = 5$. In the first subsection, we exclude the one type of junior element with exactly five non-trivial eigenvalues. In the second subsection, we prove the following result.

Proposition 13.1. *Let A be an abelian variety on which a finite group G acts freely in codimension 2. Suppose that A/G has a crepant resolution X. Let W be a translated abelian subvariety of codimension $k \leq 5$ in A such that $\{1\} \neq \mathrm{PStab}(W) < G$. Then $\mathrm{PStab}(W)$ is a cyclic group, generated by one junior element g of order 3 or 7.*

13.1 Ruling out junior elements of order 6 with five non-trivial eigenvalues.

Proposition 13.2. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X. Then there is no junior element of G whose matrix is similar to $\mathrm{diag}(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$.*

Proof. Suppose by contradiction that there is an element $g \in G$ such that $g(0) = 0$ and, in some coordinates,

$$M(g) = \mathrm{diag}(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j).$$

Then there is an abelian subvariety W of codimension 4 in A which is pointwise fixed by g^3. By Proposition 10.11, $\mathrm{PStab}(W)$ is cyclic, generated by one junior element h. As $g^3 \in \langle h \rangle$, h has even order. However, by Propositions 11.3 and 11.5, it must have order 3 or 7, contradiction! □

13.2 The pointwise stabilizer for loci of codimension 5. For proving Proposition 13.1, it is enough to establish the following result.

Proposition 13.3. *Let B be an abelian fivefold isogenous to either ${E_j}^5$ or ${E_{u_7}}^5$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ generated by junior elements of order p, and such that any subgroup of it acting not freely in codimension 4 is cyclic and generated by one junior element of order p. Then F is itself cyclic.*

Proof of Proposition 13.1 admitting Proposition 13.3. Let W be a translated abelian subvariety of codimension $k \leq 5$ in A such that $\{1\} \neq \mathrm{PStab}(W) < G$. Propositions

10.11, 11.1 show that if $k \leq 4$, then $k = 3$ and $\mathrm{PStab}(W)$ is cyclic, generated by one junior element. By Proposition 9.2, the junior generator thus has order 3 or 7.

So we can assume $k = 5$. Up to conjugating the whole group G by a translation, we can assume that $0 \in W$, and apply Proposition 10.3 to obtain a $\mathrm{PStab}(W)$-stable complementary abelian fivefold B to W. Let $F = \mathrm{PStab}(W) \subset \mathrm{Aut}(B, 0)$. It is generated by junior elements by Proposition 10.3 (3), which have order 3 or 7 by Propositions 11.3, 11.5, 13.2. Let F' be a non-trivial subgroup of F acting not freely in codimension 4: There is an abelian variety $W' \supsetneq W$ of codimension at most 4 such that $F' \subset \mathrm{PStab}(W')$. By Propositions 10.11, 11.3, 11.5, $\mathrm{PStab}(W')$ is cyclic of prime order, so $F' = \mathrm{PStab}(W')$ is cyclic generated by one junior element of order 3 or 7.

Note that, by uniqueness of the Poincaré decomposition of B [17, Thm.5.3.7], the group $\mathrm{Aut}(B, 0)$ cannot contain both a junior element of order 3 and a junior element of order 7. Hence, if $F = \mathrm{PStab}(W)$ is cyclic, Lemma 10.23 shows that it is generated by one junior element of order $p = 3$ or $p = 7$.

To conclude the proof of Proposition 13.1, we thus show by contradiction that F is not cyclic. If F is not cyclic, there are two junior elements $g, h \in F$ such that $\langle g, h\rangle$ is not cyclic, hence acts freely in codimension 4 on B. Let B_g and B_h be the abelian subvarieties of dimension 3 fixed pointwise by g and h in B. Note that $B_g \sim B_h \sim {E_j}^3$ if g and h have order $p = 3$, or $B_g \sim B_h \sim {E_{u_7}}^3$ if g and h have order $p = 7$. Hence, B is accordingly isogenous to ${E_j}^5$ or to ${E_{u_7}}^5$. So the assumptions of Proposition 13.3 are satisfied, whence F is cyclic, contradiction! □

To establish Proposition 13.3, we start with a lemma.

Lemma 13.4. *Let B be an abelian fivefold isogenous to either ${E_j}^5$ or ${E_{u_7}}^5$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *generated by junior elements of order p, and such that any subgroup of it acting not freely in codimension 4 is cyclic and generated by one junior element of order p. Let g be an element of F of prime order q. Then $p = q$.*

Proof. If 1 is an eigenvalue of g, then $\langle g\rangle$ acts not freely in codimension 4, so it is cyclic of order p, and $p = q$.

Suppose that 1 is not an eigenvalue of g. As g has prime order, and by Lemma 2.76, the characteristic polynomial $\chi_{g\oplus\overline{g}}$ is a power of the cyclotomic polynomial Φ_q. Hence, $\deg(\Phi_q) = q - 1$ divides 10, so $q \in \{2, 3, 11\}$. But:

- Since g has determinant 1 and no 1 among its eigenvalues, $q \neq 2$.
- If $q = 11$, since $\Phi_{11} = \chi_g\overline{\chi_g}$, it is reducible over $\mathbb{Q}[j]$ (if $p = 3$) or $\mathbb{Q}[u_7]$ (if $p = 7$). But by [197, Prop.2.4] Φ_{11} is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction!
- If $q = 3$, then ${\Phi_3}^5 = \chi_g\overline{\chi_g}$, so Φ_3 is reducible over $\mathbb{Q}[j]$ (if $p = 3$) or $\mathbb{Q}[u_7]$ (if $p = 7$). But by [197, Prop.2.4] Φ_3 is irreducible over $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, so $p = q = 3$.

□

Proof of Proposition 13.3. In the notations of Proposition 13.3, Lemma 13.4 proves that F is a p-group. Hence, there is an element $g \in Z(F)$ of order p. Let $h \in F \setminus \langle g\rangle$ have order p too. Since $\langle g, h\rangle$ is not cyclic, it must act freely in codimension 4, i.e., $E_g(1) \cap E_h(1) = \{0\}$, or equivalently the trivial representation is not a subrepresentation of $\langle g, h\rangle \subset \mathrm{Aut}(B, 0)$. As g and h commute, they are codiagonalizable.

If $p = 7$, this yields that gh has four or five eigenvalues of order 7, and thus the characteristic polynomial $\chi_{gh\oplus\overline{gh}}$ has exactly eight or ten common roots with Φ_7, which contradicts its rationality (Lemma 2.76).

If $p = 3$, the elements of order p in F are each similar to one of the following:

$$\operatorname{diag}(1,1,j,j,j),\ \operatorname{diag}(1,1,j^2,j^2,j^2),\ \operatorname{diag}(j,j,j,j,j^2),\ \operatorname{diag}(j,j^2,j^2,j^2,j^2).$$

Most importantly, $\operatorname{diag}(1,j,j,j^2,j^2)$ is forbidden because it is neither a power of a junior element, nor acting freely in codimension 4. Let χ be the character of the representation $\langle g,h\rangle \subset \operatorname{Aut}(B,0)$, and a be the number of elements of $\langle g,h\rangle$ similar to $\operatorname{diag}(1,1,j,j,j)$. As $\langle g,h\rangle \simeq \mathbb{Z}_3\times\mathbb{Z}_3$, it then has $4-a$ elements similar to $\operatorname{diag}(j,j,j,j,j^2)$. Hence,

$$0 = \langle \chi, \mathbf{1}\rangle = \chi(\mathrm{id}) + a(2+3j+2+3j^2) + (4-a)(4j+j^2+4j^2+j) = -15+6a,$$

contradiction!

Hence, $\langle g\rangle$ is the only cyclic subgroup of order p in F, so by [170, 5.3.6], F is cyclic. $\square$

CHAPTER 14

JUNIOR ELEMENTS AND POINTWISE STABILIZERS IN CODIMENSION 6

The goal of this section is to extend the results of Chapters 10, 11, 13 to codimension $k = 6$. For the first time in our study of pointwise stabilizers, and for the second time in this paper after Section 12, we need to assume the existence of a Calabi-Yau resolution, and not just a crepant (or even simply-connected crepant) resolution of the singular quotient A/G. Indeed, in dimension 6, products of the two examples of [155] yield non-Calabi-Yau simply-connected crepant resolutions of certain singular quotients A/G.

We start by proving the following partial classification of pointwise stabilizers in codimension 6 in Subsection 14.1.

Proposition 14.1. *Let A be an abelian variety on which a finite group G acts freely in codimension 2. Suppose that A/G has a crepant resolution X which is a Calabi-Yau manifold. Let W be a translated abelian subvariety of codimension $k \leq 6$ in A such that $\{1\} \neq \mathrm{PStab}(W) < G$ contains no junior element of type $\mathrm{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. Then $\mathrm{PStab}(W)$ is one of the following.*

- *A cyclic group generated by one junior element of order 3 or 7.*
- *An abelian group generated by two junior elements g and h of order both 3 or both 7, satisfying $E_g(1) \cap E_h(1) = H^0(W, T_W)$.*
- $\mathrm{SL}_2(\mathbb{F}_3)$*, and the representation $M : \mathrm{PStab}(W) \hookrightarrow \mathrm{Aut}(A, 0)$ decomposes as $1^{\oplus n-6} \oplus \sigma^{\oplus 3}$, where σ is the unique irreducible 2-dimensional faithful representation of $\mathrm{SL}_2(\mathbb{F}_3)$ over the splitting field $\mathbb{Q}[j]$.*

We then use this result to rule out the existence of junior elements with six non-trivial eigenvalues in Subsection 14.2 by a mix of local and global arguments, and finally refine Proposition 14.1 in Subsection 14.3 to the following result.

Proposition 14.2. *Let A be an abelian variety on which a finite group G acts freely in codimension 2. Suppose that A/G has a crepant resolution X which is a Calabi-Yau manifold. Let W be a translated abelian subvariety of codimension $k \leq 6$ in A such that $\{1\} \neq \mathrm{PStab}(W) < G$. Then $\mathrm{PStab}(W)$ is one of the following.*

- *A cyclic group generated by one junior element of order 3 or 7.*
- *An abelian group generated by two junior elements g and h of order both 3 or both 7, satisfying $E_g(1) \cap E_h(1) = H^0(W, T_W)$.*

14.1 The pointwise stabilizers for loci of codimension 6.

For proving Proposition 14.1, it is enough to establish the following result.

Proposition 14.3. *Let B be an abelian sixfold isogenous to either ${E_j}^6$ or ${E_{u_7}}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ generated by junior elements of order p, such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p. Suppose that $\omega\mathrm{id}_B \notin F$. Then F is one of the following.*

- *A cyclic group generated by one junior element of order p.*
- *An abelian group generated by two junior elements g and h of order p satisfying $E_1(g) \cap E_1(h) = H^0(W, T_W)$.*
- $\mathrm{SL}_2(\mathbb{F}_3)$*, and the representation $M : \mathrm{PStab}(W) \hookrightarrow \mathrm{Aut}(B, 0)$ decomposes as $\sigma^{\oplus 3}$, where σ is the unique irreducible 2-dimensional faithful representation of* $\mathrm{SL}_2(\mathbb{F}_3)$ *over the splitting field $\mathbb{Q}[j]$. In this case, $p = 3$.*

Proof of Proposition 14.1 admitting Proposition 14.3. Let W be a translated abelian subvariety of codimension $k \leq 6$ in A such that $\{1\} \neq \mathrm{PStab}(W) < G$ contains no junior element of type $\mathrm{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. Proposition 13.1 settles the cases when $k \leq 5$, so we can assume $k = 6$. Up to conjugating the whole group G by a translation, we can assume that $0 \in W$, and apply Proposition 10.3 to obtain a $\mathrm{PStab}(W)$-stable complementary abelian sixfold B to W. By Proposition 12.1 and as an abelian subvariety of A, B is isogenous to either ${E_j}^6$ or ${E_{u_7}}^6$.

Let $F = \mathrm{PStab}(W) \subset \mathrm{Aut}(B, 0)$. It is generated by junior elements by Proposition 10.3 (3), which have order 3 or 7 by Propositions 11.1, 11.5, 13.2 and since, by assumption, $\omega\mathrm{id}_B \notin F$. Let F' be a subgroup of F acting not freely in codimension 5: then there is an abelian variety $W' \supsetneq W$ of codimension at most 5 such that $F' \subset \mathrm{PStab}(W')$. By Proposition 13.1, $\mathrm{PStab}(W')$ is cyclic of prime order, so $F' = \mathrm{PStab}(W')$ is cyclic generated by one junior element of order 3 or 7.

So Proposition 14.3 applies, and proves Proposition 14.1. □

To establish Proposition 14.3, we need numerous lemmas.

Lemma 14.4. *Let B be an abelian sixfold isogenous to either ${E_j}^6$ or ${E_{u_7}}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let $g \in \mathrm{Aut}(B, 0)$ be an element of prime order q. Suppose that, in case $\langle g \rangle$ acts non-freely in codimension 5, it is cyclic generated by one junior element of order p. We have $q \in \{2, 3, 7\}$.*

Proof. If 1 is an eigenvalue of g, then g has order $q = p$, as wished.

Suppose that 1 is not an eigenvalue of g. By Lemma 2.76, the characteristic polynomial $\chi_{g\oplus\overline{g}}$ is thus a power of Φ_q, so $q - 1$ divides 12, so $q \in \{2, 3, 5, 7, 13\}$.

- If $q = 13$, then $\Phi_{13} = \chi_g\overline{\chi_g}$. But by [197, Prop.2.4], Φ_{13} is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction.
- If $q = 5$, then $\Phi_5^3 = \chi_g\overline{\chi_g}$. But by [197, Prop.2.4], the cyclotomic polynomial Φ_5 is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction.

□

Let us describe the 2-, 3-, and 7-Sylow subgroups of F.

Lemma 14.5. *Let B be an abelian sixfold isogenous to either ${E_j}^6$ or ${E_{u_7}}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *generated by junior elements of order p, such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p. If 2 divides $|F|$, a 2-Sylow subgroup S of F is isomorphic to Q_8.*

Proof. Since $-\mathrm{id}_B$ is the unique element of order 2 that can belong to F, by [170, 5.3.6], S is cyclic or a generalized quaternion group. Let us show that S has no element of order 8. By contradiction, let $s \in S$ be of order 8. Since $s^4 = -\mathrm{id}_B$, all eigenvalues of s have order 8, so the characteristic polynomial $\chi_{s\oplus\bar{s}}$ is a power of Φ_8. Comparing degrees yields ${\Phi_8}^3 = \chi_s\overline{\chi_s}$. But by [197, Prop.2.4], Φ_8 is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction! So S is isomorphic to $\mathbb{Z}_2, \mathbb{Z}_4$, or Q_8.

If S is cyclic, then by [170, 10.1.9], there is a normal subgroup N of F such that $F = N \rtimes S$. But all junior elements of F have odd order, so they belong to N and cannot generate F, contradiction! So S is isomorphic to Q_8. □

Lemma 14.6. *Let B be an abelian sixfold. Let $g \in \mathrm{Aut}(B, 0)$ be an element of finite order. Then g cannot have order $27, 49$, or 63.*

Proof. It is an immediate consequence of Lemma 2.76. □

Lemma 14.7. *Let B be an abelian sixfold isogenous to either ${E_j}^6$ or ${E_{u_7}}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *generated by junior elements of order p, such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p. Let $q = 7$ if $p = 3$, $q = 3$ if $p = 7$. If q divides $|F|$, a q-Sylow subgroup S of F is cyclic and has order $3, 7$, or 9.*

Proof. As S is a q-group, there is an element $g \in Z(S)$ of order q. Let $h \in S \setminus \langle g\rangle$ be another element of order q. Because $q \notin \{2, p\}$, g, h can not be powers of junior elements, and so 1 is not an eigenvalue of them. By Lemma 2.76, g and h are similar to

$$\mathrm{diag}(j, j, j, j^2, j^2, j^2) \text{ if } q = 3$$
$$\mathrm{diag}(\zeta_7, {\zeta_7}^2, \zeta_7^3, {\zeta_7}^4, \zeta_7^5, \zeta_7^6) \text{ if } q = 7$$

One can then find a on-trivial element of $\langle g, h\rangle$ with 1 as an eigenvalue. But as g and h commute, it has order $q \notin \{2, p\}$, contradiction. So $\langle g\rangle$ is the unique subgroup of order p in S. By [170, 5.3.6], S is thus cyclic, and its order is given by Lemma 14.6. □

Lemma 14.8. *Let B be an abelian sixfold isogenous to either ${E_j}^6$ or ${E_{u_7}}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of* $\mathrm{Aut}(B, 0)$ *generated by junior elements of order p, such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p. Then a p-Sylow subgroup S of F is either cyclic, or the direct product of two cyclic groups. It can be*

$$\mathbb{Z}_3,\ \mathbb{Z}_9,\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ \textit{or } \mathbb{Z}_3 \times \mathbb{Z}_9 \textit{ if } p = 3$$
$$\mathbb{Z}_7,\ \textit{or } \mathbb{Z}_7 \times \mathbb{Z}_7 \textit{ if } p = 7$$

Proof. Let $g \in Z(S)$ be an element of order p. If $\langle g \rangle$ is the only subgroup of order p in S, then by [170, 5.3.6], S is cyclic. Control on its order follows from Lemma 14.6. Else, let $[h], [k] \in S/\langle g \rangle$ have order p, $[h]$ belonging to the center of this p-group. Let us prove that $\langle [h] \rangle = \langle [k] \rangle$. If it is the case, then by [170, 5.3.6] again, $S/\langle g \rangle$ is cyclic. A fortiori, $S/Z(S)$ is cyclic, so S is abelian, and $S \simeq \langle g \rangle \times C$ for a cyclic group C containing $\langle h \rangle$. Control on the factors' orders follows from Lemma 14.6, and then concludes the proof.

If $p = 7$, then g has an eigenvalue ζ of order 7 with corresponding eigenspace $E_g(\zeta)$ of dimension 1. By Lemma 14.6, h and k have order 7 in S. As g commutes with h and k, we can thus choose $h' \in [h], k' \in [k]$ which both have 1 as an eigenvalue on $E_g(\zeta)$. Hence, the group $\langle h', k' \rangle$ does not act freely in codimension 5 on B, so it is cyclic generated by one junior element, and $\langle h' \rangle = \langle k' \rangle$ as wished.

If $p = 3$, let us show that $j\mathrm{id}_B \in S$. By contradiction, suppose that elements of order 3 in S are all similar to one of the following matrices

$$\mathrm{diag}(1,1,1,j,j,j),\ \mathrm{diag}(1,1,1,j^2,j^2,j^2),\ \mathrm{diag}(j,j,j,j^2,j^2,j^2).$$

Take $s \in S \setminus \langle g \rangle$. As g and s commute, a simple computation shows that one of the products gs, g^2s, gs^2, g^2s^2 will not fall under these three similarity classes, contradiction.

Hence, we can take $g = j\mathrm{id}_B$. A fortunate consequence of that choice, of Lemma 2.76, and of the fact that matrices in S all have determinant 1 is that g has no cubic root in S, i.e., every element of order 9 in S has a class of order 9 in $S/\langle g \rangle$. Hence, h and k above have order 3. Moreover, recall that $hkh^{-1}k^{-1} \in \langle g \rangle = \langle j\mathrm{id}_B \rangle$. If k is conjugated to jk or j^2k, then $1, j$, and j^2 each are eigenvalues of k, contradiction! Hence, $hkh^{-1} = k$, i.e., h and k commute. They commute with g as well, and thus we can find some non-trivial elements in $[h]$ and $[k]$ with a common eigenvector of eigenvalue 1. So $\langle [h] \rangle = \langle [k] \rangle$. $\square$

Proof of Proposition 14.3. We now run (see Appendix) a `GAP` search through all groups with such 2, 3, and 7-Sylow subgroups, which have at most an element of order 2, and no element of order 63. Among the ninety-four of them, only $\mathbb{Z}_7$ and $\mathbb{Z}_7 \times \mathbb{Z}_7$ can be generated by their elements of order 7, whereas $\mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathrm{SL}_2(\mathbb{F}_3), Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$, and $\mathbb{Z}_3 \times (Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ can be generated by their elements of order 3. However, it is easy to check that $Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$, and $\mathbb{Z}_3 \times (Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ have elements of order 28, which by Lemma 2.76 and [197, Prop.2.4] cannot occur in $\mathrm{Aut}_{\mathbb{Q}}(E_j{}^6, 0)$.

The representation theoretic description is easily obtained from `GAP` for $\mathrm{SL}_2(\mathbb{F}_3)$, and follows from the condition about freeness in codimension 5 for $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_7 \times \mathbb{Z}_7$. $\square$

14.2 Ruling out junior elements of order 6 with six non-trivial eigenvalues.

Proposition 14.9. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X. Then there is no junior element of G with matrix similar to* $\mathrm{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$.

In order to prove this, we first reduce to a 6-dimensional situation, where a lot of local information is given by Proposition 14.3.

Lemma 14.10. *Let A be an abelian variety, G a group acting freely in codimension 2 on A without translations such that A/G has a crepant resolution X. Suppose that there is an element $g \in G$ such that $g(0) = 0$, and with matrix similar to* $\operatorname{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. *Then there are complementary $\langle g \rangle$-stable abelian subvarieties B and W in A such that $g|_B = \omega \mathrm{id}_B$ and $g|_W = \mathrm{id}_W$. For any $\tau \in B$, it holds* $\operatorname{PStab}(W + \tau) \subset \operatorname{PStab}(W)$, *and if τ is a non-zero 2-torsion point of B, we have* $\operatorname{PStab}(W + \tau) \simeq \mathrm{SL}_2(\mathbb{F}_3)$.

Proof. The existence of W and B follows from [17, Thm.13.2.8]. The fact that $\omega \mathrm{id}_B \in \operatorname{Aut}(B, 0)$ implies that B is isogenous to ${E_j}^6$, by Proposition 10.6. By Schur's lemma, there is an $M(G)$-stable supplementary S to $H^0(T_W)$ in $H^0(T_A)$ (which is not necessarily $H^0(T_B)$), since $M(G)$ is a larger group than $\operatorname{PStab}(W)$).

Let $\tau \in B$. Let $h \in \operatorname{PStab}(W + \tau)$. The matrices of both g^3 and h split into blocks with respect to the decomposition $H^0(T_A) = H^0(T_W) \oplus S$, so g^3 commutes with h. As the matrices of g and g^3 have the same eigenspaces (with possibly different eigenvalues), the matrices of g and h commute too, and since G contains no translation, g and h commute themselves. In particular, $g(T(h)) = T(h)$. Let us decompose then $T(h) = w + b$ with $w \in W$, $b \in B$:

$$0 = g(T(h)) - T(h) = g(w + b) - w - b = g(b) - b = (\omega - 1)b.$$

As by [17, Cor.13.2.4], $\omega \mathrm{id}_B$ has exactly one fixed point on B, namely 0, we have $b = 0$, i.e., $T(h) \in W$. But h has a fixed point, so $T(h) \in \operatorname{Im}(\mathrm{id}_A - M(h))$. These two constraints yield $T(h) = 0$, whence $h \in \operatorname{PStab}(W)$.

Suppose now that τ is an non-zero 2-torsion point. As $g^3|_B = -\mathrm{id}_B$, g^3 fixes τ, i.e., $g^3 \in \operatorname{PStab}(W + \tau)$. Since G contains no translation and contains g, no element with matrix similar to $\operatorname{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$ belongs to $\operatorname{PStab}(W + \tau)$. Proposition 14.3 therefore applies to $\operatorname{PStab}(W + \tau)$, implying that it is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$ (as it contains the element g^3 of order 2). □

Remark 14.11. This notably shows that, if G contains a junior element g of type $\operatorname{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$ such that $g(0) = 0$, and W is the maximal abelian subvariety of A fixed by g, then the group G_W defined in Lemma 11.6 coincides with $\operatorname{PStab}(W)$.

This description of the pointwise stabilizers of the translations of W by 2-torsion points yields the following description of the much larger group $\operatorname{PStab}(W)$.

Lemma 14.12. *Let A be an abelian variety, G a group acting freely in codimension 2 on A without translations such that A/G has a crepant resolution X. Suppose that there is an element $g \in G$ such that $g(0) = 0$, and with matrix similar to* $\operatorname{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. *Let B, W be as in Lemma 14.10. Then there is an element $h \in \operatorname{PStab}(W)$ of prime order p if and only if $p = 2$ or 3. Moreover, a 2-Sylow subgroup S_2 of $\operatorname{PStab}(W)$ is isomorphic to Q_8, and a 3-Sylow subgroup S_3 contains an even number of junior elements (of order 3). The group $\operatorname{PStab}(W)$ contains exactly 260 junior elements.*

Proof. The group $\operatorname{PStab}(W)$ contains a unique element g^3 of order 2, so by [170, 5.3.6], its 2-Sylow subgroup S_2 is cyclic or a generalized quaternion group. Moreover, $\operatorname{PStab}(W)$ acts on a complementary abelian variety to W, which is isomorphic to ${E_j}^6$ by Proposition 10.6, and the only elements of $\operatorname{PStab}(W)$ with 1 as an eigenvalue are powers of junior elements. Hence, $\operatorname{PStab}(W) \subset \mathrm{SL}_6(\mathbb{Q}[j])$ has no element of order 8, i.e., S_2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$, or Q_8. But by Lemma 14.10, a copy of $Q_8 \subset \mathrm{SL}_2(\mathbb{F}_3)$ embeds in $\mathrm{PStab}(W)$, and therefore $S_2 \simeq Q_8$.

The group $\mathrm{PStab}(W)$ contains g^2, which has order 3. Note that g^2 commutes with all elements of $\mathrm{PStab}W$, and thus belongs to any 3-Sylow subgroup of it, in particular S_3. Now, the map $h \in S_3 \mapsto g^2h^2 \in S_3$ sends a junior element of order 3 to a junior element of order 3, and is a fixed-point-free involution. Hence, S_3 contains an even number of junior elements (of order 3).

We can also count the number of junior elements in $\mathrm{PStab}(W)$ easily: each of them fixes exactly $2^6 - 1$ non-zero 2-torsion points of B, and every non-zero 2-torsion point of B is fixed by exactly 4 junior elements by Lemma 14.4. Since B has $2^{12}-1$ non-zero 2-torsion points, the number of junior elements in $\mathrm{PStab}(W)$ is $\frac{(2^{12}-1)\cdot 4}{2^6-1} = 260$.

At last, let $h \in \mathrm{PStab}(W)$ have prime order p. Suppose by contradiction that $p \neq 2, 3$. By Lemma 14.4, we have $p = 7$, and since $\mathrm{SL}_6(\mathbb{Q}[j])$ has no junior element of order 7, 1 is not an eigenvalue of h. Hence, all six eigenvalues of h have order 7. Note that h acts by conjugation on the set of junior elements of $\mathrm{PStab}(W)$, whose cardinal, which we just computed, is not divisible by 7. Hence, h commutes with a junior element $k \in \mathrm{PStab}(W)$, so $hk \in \mathrm{PStab}(W)$ has order 21, and three eigenvalues of order 7, three eigenvalues of order 21. By Lemma 2.76, $\Phi_7\Phi_{21}$ thus divides the characteristic polynomial of $hk \oplus \overline{hk}$, but they have respective degrees $\phi(7)+\phi(21) = 18$ and 12, contradiction! $\square$

This result has the following consequence.

Corollary 14.13. *Let A be an abelian variety, G a group acting freely in codimension 2 on A without translations such that A/G has a crepant resolution X. Suppose that there is an element $g \in G$ such that $g(0) = 0$, and with matrix similar to* $\mathrm{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. *Let B, W be as in Lemma 14.10. Then the group* $\mathrm{PStab}(W)$ *has exactly four 3-Sylow subgroups S, T, U and V. There is no junior element in the intersection $S \cap T$, and thus S contains exactly* 65 *junior elements of order 3.*

Proof. By Lemma 14.12, there is a positive integer β such that

$$|\mathrm{PStab}(W)| = 8 \cdot 3^\beta.$$

The number n_3 of 3-Sylow subgroups in $\mathrm{PStab}(W)$ is thus either 1, or 4.

Let $\tau \neq 0$ be a 2-torsion point in B. By Lemma 14.10, there are exactly four junior elements s, t, u, v of order 3 of $\mathrm{PStab}(W)$ fixing τ. We can check in the multiplication table of $\mathrm{SL}_2(\mathbb{F}_3)$ that the product of any two distinct elements of $\{s, t, u, v\}$ has order 6. Hence, each 3-Sylow subgroup of $\mathrm{PStab}(W)$ contains at most one element of $\{s, t, u, v\}$. So $n_3 \geq 4$, hence $n_3 = 4$. Denote by S, T, U, and V the four 3-Sylow subgroups of $\mathrm{PStab}(W)$.

Suppose by contradiction that $S \cap T$ contains a junior element h (of order 3). Let $\tau \neq 0$ be a non-zero 2-torsion point in B fixed by h. Again, there are exactly four junior elements s, t, u, v of order 3 in $\mathrm{PStab}(W + \tau)$, and no two of them belong to the same 3-Sylow subgroup of $\mathrm{PStab}(W)$: In particular, t, u, v belong to either U or V, but that is three elements to fit into two 3-Sylow subgroups, contradiction!

Finally, the junior elements of S, T, U, V, partition the set of junior elements of $\mathrm{PStab}(W)$. By the second Sylow theorem, these four partitioning pieces are in bijection, so S has $\frac{260}{4} = 65$ junior elements. $\square$

Proof of Proposition 14.9. By contradiction, suppose that G contains a junior element g of type $\mathrm{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. By Remark 11.4, we can assume that G contains no translation other than id_A, and up to conjugating the whole group by a translation, we can assume that $g(0) = 0$. Now, Lemma 14.12 and Corollary 14.13 apply, but since 65 is odd, they contradict one another. $\square$

14.3 Ruling out the pointwise stabilizer $\mathrm{SL}_2(\mathbb{F}_3)$. In this subsection, we prove Proposition 14.2. By Proposition 14.1, it is enough to show the following:

Lemma 14.14. *Let A be an abelian variety on which a finite group G acts freely in codimension 2 without translations. Suppose that A/G has a simply-connected crepant resolution X. Then there is no abelian subvariety W of codimension 6 in A such that $\mathrm{PStab}(W) \simeq \mathrm{SL}_2(\mathbb{F}_3) < G$, with representation $M = 1^{\oplus n-6} \oplus \sigma^{\oplus 3}$ as in Proposition 14.1.*

This result resembles [5, Sec.6.1], although working under a different set of assumptions and in dimension 6.

Proof. We prove it by contradiction, using global arguments. Consider such an abelian subvariety W, and apply Lemma 11.6, defining the group G_W and a G_W-stable complementary B to W. The peculiar features of the representation $\sigma^{\oplus 3} : \mathrm{SL}_2(\mathbb{F}_3) < G_W \to \mathrm{Aut}(B, 0)$ yield that B is isogenous to ${E_j}^6$. Let $g \in \mathrm{PStab}(W) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ be the unique element of order 2. Recall that $g|_B = -\mathrm{id}_B$.

<u>*Step 1:*</u> If $h \in G_W$ fixes no point, then h has even order.

Proof. Indeed, by Lemma 11.6 (6), either hg fixes a point τ, or 1 and -1 are eigenvalues of h. Clearly, h has even order in the second case. In the first case, hg actually is in $\mathrm{PStab}(W+\tau)$, and Propositions 14.1, 14.9 yield that $\mathrm{PStab}(W+\tau)$ is isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathrm{SL}_2(\mathbb{F}_3)$. So either hg has order 3, in which case h has even order 6, or $hg \in \mathrm{PStab}(W+\tau) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ has order $2, 4$, or 6. But then, $g \in \mathrm{PStab}(W+\tau)$ since G_W contains no translation. So $h \in \mathrm{PStab}(W+\tau)$ fixes points, contradiction! $\square$

<u>*Step 2:*</u> If $h \in G_W$ has prime order p, then $p \in \{2, 3\}$. Moreover, if $p = 3$, h is a junior element or has junior square.

Proof. By Step 1, $p = 2$ if h fixes no point. By Proposition 14.1 in the case $B \sim {E_j}^6$, $p \in \{2, 3\}$ if h fixes a point.

Hence, in the case when $p = 3$, we have $h \in \mathrm{PStab}(W+\tau)$ for some $\tau \in A$. Apply Proposition 14.1 to $\mathrm{PStab}(W+\tau)$. Note that by Proposition 14.9, $\omega\mathrm{id}_B$ does not appear in $\rho(G_W)$, and as $g|_B = -\mathrm{id}_B$ does, $j\mathrm{id}_B$ does not. In particular, $\mathrm{PStab}(W+\tau)$ can not be Item 2 (i.e., $\mathbb{Z}_3 \times \mathbb{Z}_3$) of Proposition 14.1. In the remaining Items 1 and 3 of that proposition, every order 3 element of $\mathrm{PStab}(W+\tau)$ is junior or has junior square, and so is h. $\square$

<u>*Step 3:*</u> A 3-Sylow subgroup S of G_W is isomorphic to $\mathbb{Z}_3$, generated by one junior element.

Proof. Let $h \in S$ be a non-trivial element. It has odd order, hence it fixes a point by Step 1, and thus it has order 3 by Proposition 14.1. By Step 2, it is thus junior or a square of a junior element.

Let $s \in Z(S)$ be non-trivial, hence again (the square of) a junior element. Let us show that $h \in \langle s \rangle$. As h and s commute, either they have the same eigenspace for the eigenvalue 1, in which case $h \in \langle s \rangle$ as wished, or $E_{s|_B}(1)$ and $E_{h|_B}(1)$ are in direct sum, in which case $j\mathrm{id}_B \in \langle s|_B, h|_B \rangle$, and so $\omega\mathrm{id}_B \in \rho(G_W)$, which contradicts Proposition 14.9. Hence, $h \in \langle s \rangle$ and thus $S = \langle s \rangle \simeq \mathbb{Z}_3$. $\square$

Step 4: If S_2, S_3 are 2 and 3-Sylow subgroups of G_W, then $G_W = S_2 \rtimes S_3$.

Proof. By Step 3, no two elements of S_3 are conjugated in G_W, so $N_{G_W}(S_3) = C_{G_W}(S_3)$, and by Burnside's normal complement theorem [170, 10.1.8], there is a normal subgroup $N \triangleleft G_W$ such that $G_W = N \rtimes S_3$. By Step 2, N is a 2-group, and it is clearly maximal. As it is normal, it is the unique 2-Sylow subgroup of G, so $N = S_2$. $\square$

Step 5: S_2 has order 2^9.

Proof. We first count the number of junior elements in G_W. By Lemma 11.6 (9), every junior element in G_W fixes at least one 2-torsion point in B. Since it acts trivially on a 3-dimensional translated abelian subvariety of B, it fixes precisely 2^6 of the 2-torsion points in B. Each 2-torsion point τ in B is besides fixed by the four junior elements of $\mathrm{PStab}(W+\tau) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ (by Proposition 14.1 and since g of order 2 belongs to $\mathrm{PStab}(W+\tau)$). Hence, there are $\frac{2^{12}\times 4}{2^6} = 2^8$ junior elements in G_W.

Now, note that by Step 3, the number n_3 of 3-Sylow subgroups of G_W equals the number of junior elements in G_W. Hence, denoting by S_3 a 3-Sylow subgroup of G_W,

$$3|S_2| = |G_W| = n_3|N_{G_W}(S_3)| = n_3|C_{G_W}(S_3)| = 2^9 \cdot 3,$$

since it is easily checked that $C_{G_W}(S_3) = \langle g, S_3 \rangle \simeq \mathbb{Z}_6 < \mathrm{SL}_2(\mathbb{F}_3)$. $\square$

Step 6: Denote by m_2, m_4 the number of elements of order 2 and 4 in S_2. Then $m_2 = 6 \cdot 61 + 1$ and $m_4 = 144$.

Proof. We first describe the order and trace of elements $h \in S_2$ different from id_A and g. By Lemma 2.76, since $B \sim {E_j}^6$, and by [197, Prop.2.4], the characteristic polynomial of $\rho(h) = M(h|_B)$ satisfies

$$\chi_{\rho(h)} = (X-1)^{\alpha}(X+1)^{\beta}\Phi_4(X)^{\gamma}\Phi_8(X)^{\delta},$$

with $\alpha, \beta, \gamma, \delta \geq 0$, β being even because of the determinant and $\alpha + \beta + 2\gamma + 4\delta = 6$ because of the dimension. Hence, α is even too. If $\alpha\beta = 0$, then by Lemma 11.6, there is $\tau \in A$ such that $h \in \mathrm{PStab}(W+\tau) \cup g\mathrm{PStab}(W+\tau)$, so by Proposition 14.1, the only possibility for h other than id and g satisfies $\chi_{\rho(h)} = {\Phi_4}^3$, hence $\alpha = \beta = 0$. Else, α and β are positive. So, $(\alpha, \beta, \gamma, \delta)$ can be $(0,0,3,0)$,$(2,2,1,0)$,$(2,4,0,0)$, or $(4,2,0,0)$. In particular, h has order 2 or 4, with order 4 if and only if $\mathrm{Tr}(h|_B) = 0$, and order 2 if and only if $\mathrm{Tr}(h|_B) \in \{-2, 2\}$.

Decomposing the representation $\rho|_{S_2}$ into irreducible subrepresentations yields a splitting coefficient $u \in \mathbb{N}$ such that $u|S_2| = 72 + 4(m_2 - 1)$, where m_2 is the number of elements of order 2 in S_2. Denoting by m_4 the number of elements of order 4 in S_2 ans using Step 5, we rewrite $(u-4) \cdot 2^9 + 4m_4 = 64$. So $u \leq 4$.

Note that $h \in G_W$ junior of order 3 acts by conjugation on the set of elements of order 2 of the normal subgroup S_2, and the only fixed point is the element $g \in C_{G_W}(\langle h \rangle)$. Hence, $m_2 - 1$ is divisible by 3. So u is divisible by 3, and thus $u = 3$, and $m_2 = 6 \cdot 61 + 1$, and $m_4 = 144$. $\square$

<u>*Step 7:*</u> But $m_4 \geq 6 \cdot 2^6$, contradiction!

Proof. Let us show that the number of elements of G_W of order 4 fixing a point is exactly $6 \cdot 2^6$. By Lemma 11.6 (8), if $h \in G_W$ has order 4 and fixes a point, then all its 2^6 fixed points in B are 2-torsion points of B. Moreover, by Proposition 14.1, for any $\tau \in B$ of 2-torsion, $\mathrm{PStab}(W + \tau) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ contains exactly six elements of order 4. Hence the count of $\frac{2^{12} \cdot 6}{2^6} = 6 \cdot 2^6$ elements of order 4 fixing a point in G_W. $\square$

And with this contradiction ends the proof of Lemma 14.14. $\square$

Remark 14.15. Local information would not have been enough to rule out $\mathrm{SL}_2(\mathbb{F}_3)$. Indeed, considering a simply-connected neighborhood $U \subset \mathbb{C}^6$ of 0, which is stable by the action of $\rho^{\oplus 3} : SL_2(\mathbb{F}_3) \hookrightarrow SL_6(\mathbb{Q}[j])$, the quotient $U/SL_2(\mathbb{F}_3)$ admits a crepant resolution. Let us construct it.

Under the action of $SL_2(\mathbb{F}_3)$ on $\mathbb{C}^6$, exactly four 3-dimensional linear subspaces Z_1, Z_2, Z_3, Z_4 have non-trivial point-wise stabilizers $\langle g_1 \rangle, \langle g_2 \rangle, \langle g_3 \rangle, \langle g_4 \rangle \simeq \mathbb{Z}_3$, where g_1, g_2, g_3, g_4 are the four junior elements of $\mathrm{SL}_2(\mathbb{F}_3)$. Using `Macaulay2`, a quick computation shows that the blow-up:

$$\varepsilon : B := Bl_{\mathcal{I}_{Z_1} \cap \mathcal{I}_{Z_2} \cap \mathcal{I}_{Z_3} \cap \mathcal{I}_{Z_4}}(\mathbb{C}^6) \to \mathbb{C}^6$$

is a smooth quasiprojective variety with a four-dimensional central fiber $\varepsilon^{-1}(0)$. In particular, B contains exactly four prime exceptional divisors, one above each Z_i.

By the universal property of the blow-up, the action of $\mathrm{SL}_2(\mathbb{F}_3)$ on $\mathbb{C}^6$ lifts to an action on B. The lifted automorphism $\tilde{g}_i$ fixes the exceptional divisor $\varepsilon^{-1}(Z_i)$ pointwise: hence, locally, for any $x \in B$, $\mathrm{PStab}(x)$ is generated by pseudoreflections. Hence by Chevalley-Shepherd-Todd theorem, the quotient $X := B/\mathrm{SL}_2(\mathbb{F}_3)$ is smooth.

We are going to prove that the resolution $X \to \mathbb{C}^6/\mathrm{SL}_2(\mathbb{F}_3)$ is crepant. As $\mathrm{SL}_2(\mathbb{F}_3) \subset \mathrm{GL}_6(\mathbb{C})$ has one conjugacy class of junior elements, by Theorem 2.62, there is exactly one crepant divisor above $\mathbb{C}^6/\mathrm{SL}(2,3)$: A smooth resolution must contain this crepant divisor, and is thus crepant if and only if it contains exactly one exceptional divisor. This is clearly the case for X, since the action of $Q_8 \subset \mathrm{SL}_2(\mathbb{F}_3)$ on B is transitive on the set of the four prime exceptional divisors in B.

CHAPTER 15

PROOF OF THEOREM 7.4 AND THEOREM 7.6

15.1 Proof of Theorem 7.4. This is now straightforward.

Proof of Theorem 7.4. By Lemma 8.6, there is an element $g \in G$ that admits a fixed point $a \in A$. As $\mathrm{PStab}(a)$ is non-trivial, by Proposition 10.3, it contains a junior element. By Propositions 11.3, 11.5, 13.2, 14.9, this junior element has eigenvalue 1 with multiplicity $\dim(A) - 3$, i.e., it stabilizes a translated abelian subvariety of A of codimension 3. But G acts freely in codimension 3, contradiction. □

15.2 Concluding the proof of Theorem 7.6. Let us assemble the parts of the previous sections to prove Theorem 7.6.

Proof of Theorem 7.6. Let A be an abelian variety of dimension n, and let G be a finite group acting freely in codimension 2 on A, such that A/G has a resolution X that is a Calabi-Yau manifold. By Proposition 12.1, either A is isogenous to ${E_j}^n$ and G is generated by junior elements of order 3 and 6, or A is isogenous to ${E_{u_7}}^n$ and G is generated by junior elements of order 7. In particular, G is generated by its elements admitting fixed points. Also note that G contains no junior element of order 6 by Propositions 11.5, 13.2, 14.9.

Let us show that for any translated abelian subvariety $W \subset A$, the pointwise stabilizer $\mathrm{PStab}(W)$ is abelian. It is generated by junior elements by Proposition 10.3. Let g, h be two junior elements in $\mathrm{PStab}(W)$. As g and h both fix abelian varieties of codimension 3, their intersection W' has codimension $3, 4, 5$, or 6 in A. By Proposition 14.2, $\mathrm{PStab}(W')$ is thus abelian, and therefore g and h commute.

Moreover, any two junior elements g and h in $\mathrm{PStab}(W)$ have the same order (3 if $A \sim {E_j}^n$, 7 if $A \sim {E_{u_7}}^n$). Hence, using the structure theorem for finite abelian groups, $\mathrm{PStab}(W)$ is isomorphic to ${\mathbb{Z}_3}^k$ for some k if $A \sim {E_j}^n$, to ${\mathbb{Z}_7}^k$ for some k if $A \sim {E_{u_7}}^n$. Finally, if $g, h \in \mathrm{PStab}(W)$ are junior elements, then their eigenspaces with eigenvalues other than 1 are in direct sum by Proposition 14.2. An induction using that all junior elements of $\mathrm{PStab}(W)$ are codiagonalizable then yields Item 3 in Theorem 7.6. □

CHAPTER 16

PROOF OF THEOREM 7.5

In this section, we proceed to the proof of Theorem 7.5, which in fact splits into two pieces. The first piece describes a slight generalization of the situation in dimension 3 [155]. It notably gives an alternative proof of [155, Key Claim 2], replacing the discussion on invariant cohomology and topological Euler characteristics inherent to [155, §3] with group theory and a geometric fixed loci argument ruling out the special linear group $\mathrm{SL}_3(\mathbb{F}_2)$.

Theorem 16.1. *Let A be an abelian variety on which a finite group G acts freely in codimension 2 without translations. Suppose that A/G has a resolution X which is a Calabi-Yau manifold. Then, for any two junior elements $g, h \in G$ such that $\langle g\rangle \neq \langle h\rangle$, the intersection of eigenspaces $E_{M(g)}(1) \cap E_{M(h)}(1)$ has codimension $k \neq 3$ in $H^0(A, T_A)$.*

The second piece is rather specific to dimension 4.

Theorem 16.2. *Let A be an abelian variety on which a finite group G acts freely in codimension 2 without translations. Suppose that A/G has a resolution X which is a Calabi-Yau manifold. Then, for any two junior elements $g, h \in G$ such that $\langle g\rangle \neq \langle h\rangle$, the intersection of eigenspaces $E_{M(g)}(1) \cap E_{M(h)}(1)$ has codimension $k \neq 4$ in $H^0(A, T_A)$.*

Let us show how these two results imply Theorem 7.5.

Proof of Theorem 7.5, using Theorems 16.1, 16.2. Suppose by contradiction that A has dimension 4, and that A/G admits a simply-connected crepant resolution X. Then by [149, Thm, Cor.1], X can not be holomorphic symplectic. Hence, by the smooth Beauville-Bogomolov decomposition theorem, X is a Calabi-Yau fourfold. Up to replacing A by an isogenous variety, we can assume that G contains no translation.

If G entails two junior elements g, h such that $\langle g\rangle \neq \langle h\rangle$, then Theorems 16.1 and 16.2 show that the eigenspaces $E_{M(g)}(1)$ and $E_{M(h)}(1)$ are in direct sum. But they are 3-dimensional subspaces of the 4-dimensional vector space $H^0(T_A)$, contradiction!

So G has all of its junior elements contained in $\langle g\rangle$, and thus by Item 1 in Theorem 7.6, $G = \langle g\rangle$ and g has order 3 or 7, and admits 1 as an eigenvalue of multiplicity one. Up to conjugating the whole group G by a translation, we can assume $g(0) = 0$. Let $E \subset A$ be the elliptic curve containing 0 and fixed pointwise by g, and B be its $\langle g\rangle$-stable supplementary. Hence, G acts diagonally on $E \times B$ by $\{\mathrm{id}_E\} \times \langle g|_B\rangle$, and the addition map $E \times B \to A$ is a G-equivariant isogeny by [17, Thm.13.2.8]. The volume form on E thus pulls back to a G-invariant 1-form on A, and thus to a non-zero global holomorphic 1-form on the Calabi-Yau resolution X of A/G, contradiction. □

16.1 Proof of Theorem 16.1. By Theorem 7.6, the proof reduces to the following two cases. The first one is simple.

Proposition 16.3. *Let A be an abelian variety isogenous to ${E_j}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 3 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $g = h$.*

Proof. Recall that $M : \mathrm{Aut}(A) \to \mathrm{Aut}(A, 0)$ which, to any automorphism of A, associates its matrix, induces a representation of $\langle g, h\rangle$. As $\langle g, h\rangle$ contains no translation, M is faithful. Applying Maschke's theorem to the invariant subspace $E_{M(g)}(1) = E_{M(h)}(1)$ in $H^0(T_A)$ yields an $\langle M(g), M(h)\rangle$-stable supplementary S to it. Let ρ be the faithful representation of $\langle g, h\rangle$ obtained by restricting M to S. By the classification of junior elements in Proposition 9.2, $\rho(g) = \rho(h) = j\mathrm{id}_B$. But ρ is faithful, and thus $g = h$. □

The second case is the following result.

Proposition 16.4. *Let A be an abelian variety isogenous to ${E_{u_7}}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $\langle g\rangle = \langle h\rangle$.*

Its proof relies on two lemmas.

Lemma 16.5. *Let A be an abelian variety isogenous to ${E_{u_7}}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $\langle g, h\rangle$ is isomorphic to $\mathbb{Z}_7$ or $\mathrm{SL}_3(\mathbb{F}_2)$.*

Proof. By Maschke's theorem, there is an $\langle M(g), M(h)\rangle$-stable supplementary S to $E_{M(g)}(1) = E_{M(h)}(1)$ in $H^0(T_A)$. Consider the faithful representation ρ of $\langle g, h\rangle$ given by restricting M to S, with character χ.

Let $k \in \langle g, h\rangle$. If k has a fixed point in A, then k is junior of order 7. Else, 1 is an eigenvalue of $\rho(k)$. Since $\rho(k)$ has determinant 1, by Lemma 2.76 and [197, Prop.2.4], the characteristic polynomial of $\rho(k)$ in $\mathbb{Q}[u_7]$ is one of the following:

$$\Phi_1{}^3,\ \Phi_1\Phi_2{}^2,\ \Phi_1\Phi_3,\ \Phi_1\Phi_4,\ \Phi_1\Phi_6,$$

$$X^3 - \overline{u_7}X^2 + u_7X - 1,\ X^3 - u_7X^2 + \overline{u_7}X - 1.$$

So, possible prime divisors of $|\langle g, h\rangle|$ belong to $\{2, 3, 7\}$.

Let S_2 be a 2-Sylow subgroup of $\langle g, h\rangle$, it inherits the restricted representation $\rho|_{S_2}$ with character $\chi|_{S_2}$, and splitting coefficient v_2. As S_2 has a non-trivial center, $v_2 \geq 2$, so

$$9 + |S_2| - 1 = \langle \chi|_{S_2}, \chi|_{S_2}\rangle = v_2|S_2| \geq 2|S_2|$$

yielding that $|S_2|$ divides 8. Let S_3, S_7 be 3 and 7-Sylow subgroups of $\langle g, h\rangle$: Similarly, we obtain $|S_3| = 3$ and $|S_7| = 7$. Hence, the order $|\langle g, h\rangle|$ is a divisor of $8 \cdot 3 \cdot 7 = 168$. A `GAP` search (see Appendix) through all groups of such order, which have no element of order 12, 14, or 21, and either none or a non-cyclic 2-Sylow subgroup [170, 10.1.9] yields three candidates: $\mathbb{Z}_7$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $\mathrm{SL}_3(\mathbb{F}_2)$. We exclude the second candidate as it is not generated by its elements of order 7. □

We exclude $\mathrm{SL}_3(\mathbb{F}_2)$ by a geometric argument.

Lemma 16.6. *Let A be an abelian variety isogenous to ${E_{u_7}}^3$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $\langle g, h\rangle$ cannot be isomorphic to $\mathrm{SL}_3(\mathbb{F}_2)$.*

Proof. The multiplication table of $\mathrm{SL}_3(\mathbb{F}_2)$ shows that

$$C_{\langle g,h\rangle}(\langle g\rangle) = \langle g\rangle \text{ and } N_{\langle g,h\rangle}(\langle g\rangle)/C_{\langle g,h\rangle}(\langle g\rangle) \simeq \mathbb{Z}_3.$$

Take $k \in N_{\langle g,h\rangle}(\langle g\rangle)$ of order 3. Denote by $W_1, \dots, W_7$ the seven disjoint translated abelian subvarieties of codimension 3 in A that g fixes pointwise. Then

$$k\left(\bigsqcup_{i=1}^{7} W_i\right) = \bigsqcup_{i=1}^{7} W_i,$$

and since 3 and 7 are coprime, there is some $1 \le i \le 7$ such that $k(W_i) = W_i$. Up to conjugating the whole group $\langle g, h\rangle$, we can assume that $0 \in W_i$. We apply Lemma 11.6 (2) to g, noting that $W = W_i$ and $k \in \langle g, h\rangle < G_W$. It shows that for any $w \in W_i$, one has $k(w) = w + T(k)$, and $pr_{W_i}(T(k)) = 0$. As $k(W_i) = W_i$, we obtain $T(k) = 0$, so k has fixed points and order 3. In particular, it is not a power of a junior element, contradiction. □

Proof of Proposition 16.4. By Lemmas 16.5 and 16.6, we have $\langle g, h\rangle \simeq \mathbb{Z}_7$. But $\mathbb{Z}_7$ has no proper subgroup, so $\langle g\rangle = \langle h\rangle$. □

16.2 Proof of Theorem 16.2.

By Theorem 7.6, the proof reduces to the following two cases.

Proposition 16.7. *Let A be an abelian variety isogenous to ${E_{u_7}}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points. Then $E_{M(g)}(1) \cap E_{M(h)}(1)$ cannot have codimension 4 in $H^0(T_A)$.*

Proposition 16.8. *Let A be an abelian variety isogenous to ${E_j}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 3 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points. Then $E_{M(g)}(1) \cap E_{M(h)}(1)$ cannot have codimension 4 in $H^0(T_A)$.*

Both propositions are proved by classifying matrices of elements in $\langle g, h\rangle$, and using representation theory to infer contradictory properties of $\langle g, h\rangle$. We start with one lemma used in the proof of Proposition 16.7.

Lemma 16.9. *Let A be an abelian variety isogenous to ${E_{u_7}}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) \cap E_{M(h)}(1)$ has codimension at most 4 in $H^0(T_A)$. Then for every $k \in \langle g, h\rangle$, the trace of $M(k) \oplus \overline{M(k)}$ is at least $2n - 8$, and equals $2n - 7$ if k is junior of order 7.*

Proof. By Maschke's theorem, there is an $\langle M(g), M(h)\rangle$-stable supplementary S to $E_{M(g)}(1) = E_{M(h)}(1)$ in $H^0(T_A)$. Consider the faithful representation ρ of $\langle g, h\rangle$ given by restricting M to S, with character χ.

Let $k \in \langle g, h\rangle$. If k has a fixed point in A, then k is junior of order 7, and it is clear from Proposition 9.2 that the trace of $M(k) \oplus \overline{M(k)}$ equals $2n-7$. Else, 1 is an eigenvalue of $\rho(k)$, and we check as in Lemma 16.5 that its characteristic polynomial is one of the following:

$$\Phi_1{}^4,\ \Phi_1{}^2\Phi_2{}^2,\ \Phi_1{}^2\Phi_3,\ \Phi_1{}^2\Phi_4,\ \Phi_1{}^2\Phi_6,$$

$$(X^3-\overline{u_7}X^2+u_7X-1)\Phi_1,\ (X^3-u_7X^2+\overline{u_7}X-1)\Phi_1.$$

The consequence is that $\rho(k) \oplus \overline{\rho(k)}$ has non-negative trace, which concludes. □

From this lemma follows a reduction to codimension 3 that concludes the proof of Proposition 16.7.

Proof of Proposition 16.7. Denote by $\mathbf{1}$ both the trivial representation of $\langle g, h\rangle$ and its character. We have

$$\langle M|_{\langle g,h\rangle}, \mathbf{1}\rangle = \sum_{k\in\langle g,h\rangle} \operatorname{Tr} M(k) = \frac{1}{2}\sum_{k\in\langle g,h\rangle} \operatorname{Tr} M(k) + \operatorname{Tr}\overline{M(k)} > (n-4)|\langle g,h\rangle|,$$

by Lemma 16.9, the inequality being strict since $\langle g, h\rangle$ contains at least one junior element of order 7. Hence, $\mathbf{1}$ has multiplicity at least $n-3$ as a subrepresentation of M, i.e., $E_1(M(g)) \cap E_1(M(h))$ has codimension at most 3 in $H^0(T_A)$. □

We now prove an auxiliary lemma for Proposition 16.8.

Lemma 16.10. *Let A be an abelian variety isogenous to $E_j{}^n$. Let $g, h \in \operatorname{Aut}(A)$ be two junior elements of order 3 such that $\langle g, h\rangle$ contains no translation and no non-junior element fixing points, and $E_1(M(g)) \cap E_1(M(h))$ has codimension 4 in $H^0(A, T_A)$. Then each non-trivial element of $\langle g, h\rangle$ has order 3.*

Proof. By Maschke's theorem, there is an $\langle M(g), M(h)\rangle$-stable supplementary S to $E_{M(g)}(1) + E_{M(h)}(1)$ in $H^0(T_A)$, and it has dimension 4. Consider the faithful representation ρ of $\langle g, h\rangle$ given by restricting M to S, with character χ.

Let $k \in \langle g, h\rangle$. If k has a fixed point in A, then k is junior of order 3. Else, 1 is an eigenvalue of $\rho(k)$, and since the intersection $E_{\rho(g)}(j) \cap E_{\rho(h)}(j)$ has dimension 2, it must be that $1, j$, or j^2 is an eigenvalue of multiplicity 2 of $\rho(k)$. By Lemma 2.76, [197, Prop.2.4], and as $\rho(k)$ has determinant one, the characteristic polynomial of $\rho(k)$ in $\mathbb{Q}[j]$ is one of the following:

$$\Phi_1{}^4,\ \Phi_1{}^2\Phi_2{}^2,\ \Phi_1{}^2\Phi_3,\ \Phi_1{}^2\Phi_4,\ \Phi_1{}^2\Phi_6,\ (X-j)^3\Phi_1,\ (X-j^2)^3\Phi_1.$$

So the order of k is 1, 3, or an even number.

To conclude, it is enough to show that k cannot have order 2. We prove it by contradiction: Suppose that $\rho(k)$ is similar to $\operatorname{diag}(1,1,-1,-1)$. As the eigenspace $E_{\rho(g)}(j)$ is a hyperplane in S, $\rho(gk)$ has j and $-j$ as eigenvalues. In particular, it is not junior and thus it fixes no point. But its characteristic polynomial should be one of the polynomials listed above, contradiction. □

Proof of Proposition 16.8. By Lemma 16.10, $\rho(\langle g, h\rangle)$ contains id_S and elements similar to

$$\operatorname{diag}(1,j,j,j),\ \operatorname{diag}(1,j^2,j^2,j^2),\ \text{or}\ \operatorname{diag}(1,1,j,j^2). \tag{16.1}$$

Note in particular that $\mathrm{diag}(j, j, j^2, j^2)$ is not an option.

As $\langle g, h\rangle$ is a 3-group, we can set $k \in Z(\langle g, h\rangle)$ to be an element of order 3. Up to exchanging g and h, we can assume $k \notin \langle g\rangle$. If $\rho(k)$ is similar to $\mathrm{diag}(1, j, j, j)$ or $\mathrm{diag}(1, j^2, j^2, j^2)$, then respectively $\rho(gk)$ or $\rho(g^2k)$ has no 1 as an eigenvalue, which contradicts (16.1). Else, $\rho(k)$ is similar to $\mathrm{diag}(1, 1, j, j^2)$. As $E_{\rho(g}(j) \cap E_{\rho(h)}(j)$ has dimension 2, it is the eigenspace for the eigenvalue 1 of $\rho(k)$. Again, either $\rho(gk)$ or $\rho(g^2k)$ has no 1 as an eigenvalue, which contradicts (16.1). $\square$

APPENDIX

Groups of order dividing 240 with an automorphism of order 7

```
order_list := [];
nb_groups_of_order_list := [];

for a in [0..4] do
   for b in [0..1] do
      for c in [0..1] do
         n := (2^a)*(3^b)*(5^c);
         Add(order_list, n);
         Add(nb_groups_of_order_list, NumberSmallGroups(n));
      od;
   od;
od;

have_aut7 := [];

for i in [1..Length(order_list)] do
   n := order_list[i];
   for v in [1..nb_groups_of_order_list[i]] do
      g := SmallGroup(n,v);
      s := SylowSubgroup(g,2);
      if StructureDescription(s) = "Q16" or StructureDescription(s) = "Q8"
      or StructureDescription(s) = "C16" or StructureDescription(s) = "C8"
      or StructureDescription(s) = "C4" or StructureDescription(s) = "C2"
      or StructureDescription(s) = "1" then
         h := AutomorphismGroup(g);
         if Order(h) mod 7 = 0 then
            Add(l,(n,v));
         fi;
      fi;
   od;
od;
```

Representations of $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$

```
for v in [1..NumberSmallGroups(24)] do
   g := SmallGroup(24, v);
```

```
   if StructureDescription(g) = "C3␣:␣C8"
   then Add(groups_checked, v);
        tbl_conjcl := ConjugacyClasses(g);
        nb_conjcl := Size(tbl_conjcl);
#locating the unique element of order 2
#among conjugacy classes of g
        index_2 := 0;
        for j in [1..nb_conjcl] do
            o := Order(Representative(tbl_conjcl[j]));
            if o = 2 then index_2 := j; fi;
        od;
#only keeping irreducible characters sending
#the unique element of order 2 to -id
        T := Irr(g);
        Tbis := [];
        for k in [1..nb_conjcl] do
           if T[k][index_2] + T[k][1] = 0
           then Add(Tbis,T[k]);
           fi;
        od;
        Print(Tbis);
        Print("\n\n");
   fi;
od;
```

Proposition 10.27: Five candidates for F

```
order_list := [];
nb_groups_of_order_list := [];

for a in [3..4] do
   for b in [0..1] do
      for c in [0..1] do
         n := (2^a)*(3^b)*(5^c);
         Add(order_list, n);
         Add(nb_groups_of_order_list, NumberSmallGroups(n));
      od;
   od;
od;

right_sylows := [];
right_sylows_and_orders := [];

for i in [1..Length(order_list)] do
   n := order_list[i];
   for v in [1..nb_groups_of_order_list[i]] do
      g := SmallGroup(n,v);
      s := SylowSubgroup(g,2);
      if StructureDescription(s) = "Q16" or StructureDescription(s) = "Q8" t
```

```
         Add(right_sylows, [n,v]);
         Add(right_sylows_and_orders, [n,v]);
         tbl_conjcl := ConjugacyClassesByOrbits(g);
         nb_conjcl := Size(tbl_conjcl);
         remove_once_only := 0;
         is_15 := 0;
         is_20 := 0;
         is_24 := 0;
         for i in [1..nb_conjcl] do
            o := Order(Representative(tbl_conjcl[i]));
            if o = 15 then
               is_15 := 1;
            fi;
            if o = 20 then
               is_20 := 1;
            fi;
            if o = 24 then
               is_24 := 1;
            fi;
            s := Size(tbl_conjcl[i]);
            if remove_once_only = 0 and
            ((o = 2 and s > 1) or (o mod 60 = 0) or (o mod 40 = 0)
            or (is_20 = 1 and o mod 15 = 0) or (is_24 = 1 and o mod 15 = 0)
            or (is_15 = 1 and o mod 20 = 0) or (is_24 = 1 and o mod 20 = 0)
            or (is_15 = 1 and o mod 24 = 0) or (is_20 = 1 and o mod 24 = 0))
               Remove(right_sylows_and_orders);
               remove_once_only := 1;
            fi;
         od;
         if remove_once_only = 0 and (1 - is_15)*(1 - is_20)*(1 - is_24) = 1
            Remove(right_sylows_and_orders);
            remove_once_only := 1;
         fi;
      fi;
   od;
od;
```

Proposition 10.27: Two candidates generated by elements of the right order

```
testing := [[48,8],[48,27]];

for i in [1..2] do
g := SmallGroup(testing[i][1],testing[i][2]);
Print(StructureDescription(g));
tbl_conjcl := ConjugacyClasses(g);
nb_conjcl := Size(tbl_conjcl);
   nb_elts_order_24 := 0;
   for j in [1..nb_conjcl] do
      o := Order(Representative(tbl_conjcl[j]));
```

```
      s := Size(tbl_conjcl[j]);
      if o = 24 then
         nb_elts_order_24 := nb_elts_order_24 + s;
      fi;
   od;
Print("number␣of␣elements␣of␣order␣24:");
Print(nb_elts_order_24);
Print("␣");
Print("\n");
od;

Print("\n\n");

testing := [[40,4],[40,11],[80,18]];

for i in [1..3] do
g := SmallGroup(testing[i][1],testing[i][2]);
Print(StructureDescription(g));
tbl_conjcl := ConjugacyClasses(g);
nb_conjcl := Size(tbl_conjcl);
   nb_elts_order_20 := 0;
   for j in [1..nb_conjcl] do
      o := Order(Representative(tbl_conjcl[j]));
      s := Size(tbl_conjcl[j]);
      if o = 20 then
         nb_elts_order_20 := nb_elts_order_20 + s;
      fi;
   od;
Print("␣number␣of␣elements␣of␣order␣20:␣");
Print(nb_elts_order_20);
Print("␣");
Print("\n");
od;
```

Proposition 10.27: None admitting the right representation

```
tables_char_irr := [[],[]];
indices_20 := [[],[]];
testing := [[40,11],[80,18]];

for i in [1..2] do
   g := SmallGroup(testing[i][1], testing[i][2]);
   tbl_conjcl := ConjugacyClasses(g);
   nb_conjcl := Size(tbl_conjcl);
#locating the unique element of order 2
#among conjugacy classes of g
   index_2 := 0;
   for j in [1..nb_conjcl] do
      o := Order(Representative(tbl_conjcl[j]));
```

```
      if o = 2 then
         index_2 := j;
      fi;
   od;
#only keeping irreducible characters sending
#the unique element of order 2 to -id
   T := Irr(g);
   Tbis := [];
   for k in [1..nb_conjcl] do
      if T[k][index_2] + T[k][1] = 0 then
         Add(Tbis,T[k]);
      fi;
   od;
   Add(tables_char_irr[i], Tbis);
   Print(StructureDescription(g));
   Print("␣possible␣irreducible␣representations␣have␣characters:␣");
   Print(tables_char_irr[i]);
   Print("\n\n");
od;
```

Pointwise stabilizers in codimension 6 as in Subsection 14.1

```
order_list := [];
nb_groups_of_order_list := [];

for a in [0,3] do
   for b in [0..3] do
      for c in [0..2] do
         if b <= 1 or c <= 1 then
            n := (2^a)*(3^b)*(7^c);
            Add(order_list, n);
            Add(nb_groups_of_order_list, NumberSmallGroups(n));
         fi;
      od;
   od;
od;

right_sylows := [];
right_sylows_and_orders := [];

for i in [1..Length(order_list)] do
   n := order_list[i];
   for v in [1..nb_groups_of_order_list[i]] do
      g := SmallGroup(n,v);
      s := SylowSubgroup(g,2);
      t := SylowSubgroup(g,3);
      u := SylowSubgroup(g,7);
      if (StructureDescription(s) = "1" or StructureDescription(s) = "Q8")
      and (StructureDescription(t) = "1" or StructureDescription(t) = "C3"
```

```
            or StructureDescription(t) = "C9"
            or StructureDescription(t) = "C3␣x␣C3"
            or StructureDescription(t) = "C3␣x␣C9")
       and (StructureDescription(u) = "1" or StructureDescription(u) = "C7"
            or StructureDescription(u) = "C7␣x␣C7")
       then Add(right_sylows, [n,v]);
            Add(right_sylows_and_orders, [n,v]);
#we now remove of the list right_sylows_and_orders candidates with elements
#of inappropriate order 63, or with several elements of order 2
            tbl_conjcl := ConjugacyClassesByOrbits(g);
            nb_conjcl := Size(tbl_conjcl);
            remove_once_only := 0;
            for i in [1..nb_conjcl] do
               o := Order(Representative(tbl_conjcl[i]));
               s := Size(tbl_conjcl[i]);
               if remove_once_only = 0 and
               ((o = 2 and s > 1) or (o mod 63 = 0))
               then Remove(right_sylows_and_orders);
                    remove_once_only := 1;
               fi;
            od;
       fi;
   od;
od;


describe := [];
for i in [1..Length(right_sylows_and_orders)] do
   g := SmallGroup(right_sylows_and_orders[i][1], right_sylows_and_orders[i]
   Add(describe, StructureDescription(g));
   Print(StructureDescription(g));
   Print("\n\n");
od;
```

Groups of order dividing 168 as in Lemma 16.5

```
order_list := [];
nb_groups_of_order_list := [];

for a in [0..3] do
   for b in [0..1] do
      n := (2^a)*(3^b)*7;
      Add(order_list, n);
      Add(nb_groups_of_order_list, NumberSmallGroups(n));
   od;
od;

right_sylow := [];
right_sylow_description := [];
```

```
for i in [1..Length(order_list)] do
   n := order_list[i];
   for v in [1..nb_groups_of_order_list[i]] do
      g := SmallGroup(n,v);
      h := SylowSubgroup(g, 2);
      if StructureDescription(h) = "Q8" or StructureDescription(h) = "D8"
      or StructureDescription(h) = "1"
      then Add(right_sylow, [n, v]);
           Add(right_sylow_description, StructureDescription(g));
      fi;
   od;
od;

right_sylow_and_orders := [];
right_sylow_and_orders_description := [];
for element in right_sylow do
    n := element[1];
    v := element[2];
    g := SmallGroup(n,v);
   tbl_conjcl := ConjugacyClassesByOrbits(g);
   nb_conjcl := Size(tbl_conjcl);
   v_to_discard := 0;
   for i in [1..nb_conjcl] do
      o := Order(Representative(tbl_conjcl[i]));
      if (o = 14 or o = 21 or o = 12) and v_to_discard = 0
      then v_to_discard := 1;
      fi;
   od;
   if v_to_discard = 0
   then Add(right_sylow_and_orders, [n, v]);
        Add(right_sylow_and_orders_description, StructureDescription(g));
   fi;
od;

Print(right_sylow_and_orders);
Print("\n");
Print(right_sylow_and_orders_description);
```

PART III

NEF CONE OF FIBER PRODUCTS OVER CURVES AND AN APPLICATION TO THE CONE CONJECTURE

CHAPTER 17
INTRODUCTION

Cone Conjecture. To understand the geometry of a smooth projective variety X, studying the Mori cone of curves $\overline{\mathrm{NE}}(X)$ and its dual, the nef cone $\mathrm{Nef}(X)$, is central, especially from the viewpoint of the minimal model program (MMP).

An important part of the relationship between the Mori cone and the MMP is captured by the Cone Theorem and the Contraction Theorem. These theorems assert that the K_X-negative part of the Mori cone of a smooth projective variety X is rational polyhedral away from the K_X-trivial hyperplane, and the extremal rays of the K_X-negative part correspond to some morphisms from X, involved in the MMP. In particular, when X is a Fano variety (namely, $-K_X$ is ample), the cone $\mathrm{Nef}(X)$ is a rational polyhedral cone, and its extremal rays are generated by semiample classes. In general, however, it is difficult to describe the whole Mori cone, or dually the whole nef cone, even under the slightly weaker assumption that $-K_X$ is semiample. For instance, if X is the blowup of $\mathbb{P}^2$ at the base points of a general pencil of cubic curves in $\mathbb{P}^2$, then $-K_X$ is semiample but $\mathrm{Nef}(X)$ is not rational polyhedral.

When X is K-trivial, we expect nevertheless that some essential parts of the nef cone of X are rational polyhedral, up to the action of $\mathrm{Aut}(X)$. A precise statement, known as the Cone Conjecture, was first formulated by Morrison [143] and Kawamata [101]. It was later generalized by Totaro [192] to klt Calabi–Yau pairs (X, Δ) (see Section 18.2 for a definition), thus including much more examples, already in dimension 2.

In this work, we study the Cone Conjecture for certain Calabi–Yau pairs. Let us recall the statement of the Cone Conjecture formulated by Totaro in [192, Conjecture 2.1] (in the absolute situation). For a pair (X, Δ), we define

$$\mathrm{Aut}(X, \Delta) := \{f \in \mathrm{Aut}(X) \mid f(\mathrm{supp}(\Delta)) = \mathrm{supp}(\Delta)\}.$$

We also define the *nef effective cone* $\mathrm{Nef}^e(X)$ as

$$\mathrm{Nef}^e(X) := \mathrm{Nef}(X) \cap \mathrm{Eff}(X),$$

where $\mathrm{Eff}(X)$ is the effective cone of X.

Conjecture 17.1 (Kawamata–Morrison–Totaro Cone Conjecture)**.** *Let (X, Δ) be a klt Calabi–Yau pair. There exists a rational polyhedral cone Π in $\mathrm{Nef}^e(X)$ which is a fundamental domain for the action of $\mathrm{Aut}(X, \Delta)$ on $\mathrm{Nef}^e(X)$, in the sense that*

$$\mathrm{Nef}^e(X) = \bigcup_{g \in \mathrm{Aut}(X,\Delta)} g^*\Pi,$$

and $\Pi^\circ \cap (g^*\Pi)^\circ = \varnothing$ unless $g^* = \mathrm{id}$.

An important prediction of the Cone Conjecture to the Minimal Model Program is that the number of $\mathrm{Aut}(X, \Delta)$-equivalence classes of faces of the nef effective cone $\mathrm{Nef}^e(X)$ corresponding to birational contractions or fiber space structures is finite (see e.g. [192, p.243]).

Nef cones of fiber products. The starting point of this work is a decomposition theorem for the nef cone of a fiber product over a curve.

It begins with the following general question. Let W_1 and W_2 be smooth projective varieties and let $\phi_1 : W_1 \to B$ and $\phi_2 : W_2 \to B$ be surjective morphisms with connected fibers over a smooth base B. Assume that the fiber product $W := W_1 \times_B W_2$ is smooth.

Question 17.2. Let $p_i : W \to W_i$ be the projection. When do we have

$$p_1^*\mathrm{Nef}(W_1) + p_2^*\mathrm{Nef}(W_2) = \mathrm{Nef}(W)? \tag{17.1}$$

As the nef cone $\mathrm{Nef}(X)$ of a smooth projective variety X spans the whole space $N^1(X)_{\mathbb{R}}$ of numerical classes of $\mathbb{R}$-divisors, such a decomposition exists only if

$$p_1^*N^1(W_1)_{\mathbb{R}} + p_2^*N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}}. \tag{17.2}$$

We may then ask which fiber products satisfying the decomposition (17.2) also have the decomposition (17.1).

When B is a point, a simple application of the projection formula shows that (17.2) implies (17.1). When B is $\mathbb{P}^1$ and the varieties W_i are certain rational elliptic surfaces, the decomposition (17.1) was proven in [68, Proposition 3.1]. We show that the implication (17.2) $\Rightarrow$ (17.1) continues to hold for an arbitrary fiber product over a curve.

Theorem 17.3. *For $i = 1, 2$, let $\phi_i : W_i \to B$ be a surjective morphism with connected fibers from a smooth projective variety to a smooth projective curve B. Assume that*

1 the variety $W = W_1 \times_B W_2$ is smooth;

2 we have

$$p_1^*N^1(W_1)_{\mathbb{R}} + p_2^*N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}}.$$

Then

$$p_1^*\mathrm{Nef}(W_1) + p_2^*\mathrm{Nef}(W_2) = \mathrm{Nef}(W).$$

As a consequence, we also have $p_1^*\mathrm{Amp}(W_1) + p_2^*\mathrm{Amp}(W_2) = \mathrm{Amp}(W)$.

In Examples 19.5 and 19.6, we construct explicit examples of fiber products over bases of dimension at least 2, that fail the implication (17.2) $\Rightarrow$ (17.1).

Theorem 17.3 has the following corollary.

Corollary 17.4. *In the setting of Theorem 17.3, assume moreover that for $i = 1, 2$, $\overline{\mathrm{NE}}(W_i)$ is generated by classes of curves. Let $E \in \mathrm{Nef}(W_i)$. Then $E \in \mathrm{Nef}(W_i)$ is extremal, if and only if $p_i^*E \in \mathrm{Nef}(W)$ is extremal. As a consequence, $\mathrm{Nef}(W)$ is rational polyhedral if and only if both $\mathrm{Nef}(W_1)$ and $\mathrm{Nef}(W_2)$ are rational polyhedral.*

It provides a way of constructing fiber products (over curves) whose nef cones are not rational polyhedral.

Cone Conjecture for Schoen varieties. The main goal of this paper is to apply Theorem 17.3 to a certain type of varieties with globally generated anticanonical bundle, which we call *Schoen varieties*. This is the higher dimensional generalization of C. Schoen's construction of Calabi–Yau threefolds [180] as fiber products over $\mathbb{P}^1$.

Let us first summarize the construction of Schoen varieties; we refer to Subsections 20.1 and 20.2 for more details. Let Z_1 and Z_2 be Fano manifolds of dimension at least two. For $i = 1, 2$, let D_i be an ample and globally generated divisor on Z_i such that $-(K_{Z_i} + D_i)$ is globally generated. Let $W_i \subset \mathbb{P}^1 \times Z_i$ be a general member in the linear system $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_i}(D_i)|$. We have a fibration $\phi_i : W_i \to \mathbb{P}^1$. Consider the fiber product over $\mathbb{P}^1$:

$$\phi : X := W_1 \times_{\mathbb{P}^1} W_2 \to \mathbb{P}^1.$$

When X is smooth, such a variety X is called a *Schoen variety*. It is easy to check that $-K_X$ is globally generated, and hence we can define a *Schoen pair* $(X, \Delta_{m,X})$ as in Example 18.1.

We prove the following result.

Theorem 17.5. *Let X be a Schoen variety, and let $(X, \Delta_{m,X})$ be a Calabi–Yau pair associated to it as in Example 18.1. Then there exists a rational polyhedral fundamental domain for the action of* $\mathrm{Aut}(X, \Delta_{m,X})$ *on* $\mathrm{Nef}^e(X) = \mathrm{Nef}(X)$.

Note that, by Corollary 17.4, the cone $\mathrm{Nef}(X)$ is not rational polyhedral as long as one of $\mathrm{Nef}(W_1)$ and $\mathrm{Nef}(W_2)$ is not. This is the case when there exists i such that $Z_i = \mathbb{P}^2$ and $D_i = -K_{Z_i}$ (in which case W_i is a rational elliptic surface). In particular, our construction provides the first series of strict Calabi–Yau manifolds, and also Calabi–Yau pairs in arbitrary dimension, for which the Cone Conjecture holds and whose nef cones are not rational polyhedral (see Example 21.6). We also note that X is a complete intersection of two hypersurfaces, which are nef but not ample, in the Fano manifold $\mathbb{P}^1 \times Z_1 \times Z_2$. That the cone $\mathrm{Nef}(X)$ may admit infinitely many faces resonates with Theorem 17.6 below.

As direct corollaries, we obtain the finite presentation of the discrete group of components $\pi_0\mathrm{Aut}(X)$ and the finiteness of real structures on X up to equivalence.

Historical remarks. Let us first discuss the state of the art of the Cone Conjecture without the boundary divisor ($\Delta = 0$). The Cone Conjecture was verified for K3 surfaces by Sterk [186], and for Enriques surfaces by Namikawa [150] using the Torelli theorem. In [166], Prendergast-Smith proved the Cone Conjecture for abelian varieties. A version of this conjecture was also proven for the two main families of projective hyperkähler manifolds in [135], shortly before the general proof by Amerik–Verbitsky came out in [2].

Very little is known about the Cone Conjecture for strict Calabi–Yau manifolds (see Definition 18.2), even in dimension three. The most general result might be the following, due to Kollár [20].

Theorem 17.6. *Let D be a smooth anticanonical hypersurface in a Fano manifold Y of dimension at least 4. Then the natural restriction map* $\mathrm{Nef}(D) \to \mathrm{Nef}(Y)$ *is an isomorphism. In particular,* $\mathrm{Nef}(D)$ *is a rational polyhedral cone.*

Among the strict Calabi–Yau manifolds whose nef cones are not rational polyhedral, to our knowledge the Cone Conjecture is known so far for only two special cases.

One of them is the desingularized Horrocks–Mumford quintics, studied by Borcea in [21] (see also [61]), and the other is the fiber product of two general rational elliptic surfaces with sections over $\mathbb{P}^1$, investigated by Schoen in [180], by Namikawa in [151], by Grassi and Morrison in [68]. Both examples are of dimension three.

Some partial results for Calabi–Yau manifolds with Picard number two are due to Lazić, Oguiso and Peternell ([126, 156]). Further evidence supporting the Cone Conjecture for Calabi–Yau manifolds includes results obtained by Filipazzi–Hacon–Svaldi [60], Kawamata [101], Li–Zhao [130], Oguiso–Peternell [158], Oguiso–Sakurai [160], Szendrői [189] and Uehara [194]; see also the recent survey [125].

Let us now mention some known cases where the Cone Conjecture holds for Calabi–Yau pairs with a boundary divisor $\Delta \neq 0$. Totaro proved the Cone Conjecture for arbitrary klt Calabi–Yau pairs of dimension two in [192]. Prendergast-Smith proved the Cone Conjecture for certain rational elliptic threefolds in [167]. The Cone Conjecture was also verified for some Calabi–Yau pairs arising from blow-ups of Fano manifolds of index $n-1$ and $n-2$ by Coskun and Prendergast-Smith in [37, 38]. We should also notice that for the Calabi–Yau pairs in [167, 37, 38], the nef cone is rational polyhedral. Kopper verified the Cone Conjecture for Calabi–Yau pairs arising from Hilbert schemes of points on certain rational elliptic surfaces in [113]. In this case, the nef cone may admit infinitely many faces (while the dimension of these varieties are always even).

As we said, the fiber product of two general rational elliptic surfaces with sections over $\mathbb{P}^1$ was investigated by Schoen and others [180, 151, 68]. It recently came back to light as Suzuki considered a certain higher-dimensional generalization of Schoen's construction and studied its arithmetic properties in [187], and as Sano used similar ideas to construct non-Kähler Calabi–Yau manifolds with arbitrarily large second Betti number in [174].

Structure of the paper. We prove Theorem 17.3 in Section 19, and Theorem 17.5 in Section 21. A crucial result by Looijenga, together with preliminaries, is recalled in Section 18. Section 20 describes the construction of Schoen pairs in some detail.

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CHAPTER 18

PRELIMINARIES

18.1 Notation. The group of automorphisms of X is denoted by $\mathrm{Aut}(X)$, and acts on $N^1(X)$ by pullback. This action

$$\rho : \mathrm{Aut}(X) \to \mathrm{GL}(N^1(X))$$

linearly extends to $N^1(X)_{\mathbb{R}}$, preserving the cones $\mathrm{Nef}^e(X)$ and $\mathrm{Nef}^+(X)$. The connected component of the identity in $\mathrm{Aut}(X)$ is a normal subgroup $\mathrm{Aut}^0(X)$, which acts trivially on $N^1(X)$ [23, Lemma 2.8]. This induces an action of the discrete group of components

$$\pi_0\mathrm{Aut}(X) := \mathrm{Aut}(X)/\mathrm{Aut}^0(X)$$

on $N^1(X)$, that we denote by

$$\overline{\rho} : \pi_0\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)).$$

18.2 Klt Calabi–Yau pairs. The definition of a klt Calabi-Yau pair was given in Section 2.2.

Example 18.1. Let X be a smooth projective variety with $-K_X$ semiample. Let m be a positive integer m such that $-mK_X$ is globally generated. Then we can always define a Calabi–Yau pair $(X, \Delta_{m,X})$ by taking

$$\Delta_{m,X} = \frac{1}{m}\Delta'_{m,X}, \quad \text{where } \Delta'_{m,X} \in |-mK_X|.$$

Moreover, if $m \geq 2$ and $\Delta'_{m,X}$ is general in its linear system, then the associated pair $(X, \Delta_{m,X})$ is a klt Calabi–Yau pair.

In this part, we use the following terminology.

Definition 18.2. Let X be a smooth projective variety. We say that X is a *Calabi–Yau manifold* if the canonical line bundle K_X is trivial and $h^i(X, \mathcal{O}_X) = 0$ for any $0 < i < \dim X$. If in addition, X is simply-connected, it is called a *strict Calabi–Yau manifold*.

18.3 Looijenga's result. We will use the following crucial result in this paper.

Proposition 18.3. *Let X be a normal projective variety and let $H \leq \mathrm{Aut}(X)$ be a subgroup. Assume that there is a rational polyhedral cone $\Pi \subset \mathrm{Nef}^+(X)$ such that $\mathrm{Amp}(X) \subset H \cdot \Pi$. Then*

1 *There is a rational polyhedral fundamental domain for the action of $\rho(H)$ on $\mathrm{Nef}^+(X)$.*

2 *The group $\rho(H)$ is finitely presented.*

Such a result and related statements are well-known to experts. We include a proof for the sake of completeness. It relies on the fundamental results due to Looijenga [132, Application 4.14 and Corollary 4.15], which we extract and formulate here as Lemma 18.4. Recall that a cone $C \subset N_{\mathbb{R}}$ in a finite dimensional $\mathbb{R}$-vector space $N_{\mathbb{R}}$ is called *strict* if its closure $\overline{C} \subset N_{\mathbb{R}}$ contains no line.

Lemma 18.4. *Let N be a finitely generated free $\mathbb{Z}$-module, and let C be a strict convex open cone in the $\mathbb{R}$-vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Let C^+ be the convex hull of $\overline{C} \cap N_{\mathbb{Q}}$. Let $(C^\vee)^\circ \subset N_{\mathbb{R}}^\vee$ be the interior of the dual cone of C. Let Γ be a subgroup of $\mathrm{GL}(N)$ which preserves the cone C. Suppose that:*

- *there is a rational polyhedral cone $\Pi \subset C^+$ such that $C \subset \Gamma \cdot \Pi$;*
- *there exists an element $\xi \in (C^\vee)^\circ \cap N_{\mathbb{Q}}^\vee$ whose stabilizer in Γ (with respect to the dual action $\Gamma \circlearrowleft N_{\mathbb{Q}}^\vee$) is trivial.*

Then the Γ-action on C^+ has a rational polyhedral fundamental domain, and the group Γ is finitely presented.

Proof of Proposition 18.3. In Lemma 18.4, now set $N = N^1(X)$, $C = \mathrm{Amp}(X)$, and $\Gamma = \rho(H)$. Let us construct an element $\xi \in (C^\vee)^\circ \cap N_{\mathbb{Q}}^\vee$ whose stabilizer is trivial. If we have such an element, then by Lemma 18.4, Proposition 18.3 follows.

Claim 18.5. *There exists an ample class $\eta_0 \in N^1(X)$ such that Γ_{η_0} is trivial.*

Proof. Our proof is inspired by the argument of [123, Proposition 6.5].

By Fujiki–Liebermann's theorem [23, Theorem 2.10], the action of Γ on $C \cap N$ has finite stabilizers. Take an element $\eta \in C \cap N_{\mathbb{Q}}$ such that the order of the stabilizer Γ_η is minimal. Since the Γ-action on $N_{\mathbb{R}}$ preserves N, we can find an open neighborhood $U \subset C$ of η, such that $\gamma U \cap U = \varnothing$ for every $\gamma \notin \Gamma_\eta$. Thus, for every $\eta' \in U \cap N_{\mathbb{Q}}$, we have $\Gamma_{\eta'} \subset \Gamma_\eta$, which then implies $\Gamma_{\eta'} = \Gamma_\eta$ by the minimality of Γ_η. It follows that every $\gamma \in \Gamma_\eta$ satisfies $\gamma|_{U \cap N_{\mathbb{Q}}} = \mathrm{id}_{U \cap N_{\mathbb{Q}}}$, and since γ acts linearly, necessarily $\gamma = \mathrm{id}$. This proves that $\eta \in C \cap N_{\mathbb{Q}}$ has trivial stabilizer, and so do some positive multiple $\eta_0 \in C \cap N$ of η. ◻

Now choose any $\xi \in (C^\vee)^\circ$. Since $\xi(x) > 0$ for any $x \in \overline{C} \backslash \{0\}$, the subset

$$\{x \in \overline{C} \mid \xi(x) \le r\} \subset V$$

is bounded, so compact for any $r > 0$. Since $C \cap N$ is discrete, among

$$\Sigma := \{\eta \in C \cap N \mid \Gamma_\eta \text{ is trivial}\} \neq \varnothing$$

there are only finitely many $\eta \in \Sigma$ minimizing $\xi|_\Sigma$.

Again, as $C \cap N$ is discrete, we can perturb ξ and obtain $\xi_0 \in (C^\vee)^\circ \cap N_{\mathbb{Q}}^\vee$ such that there is a unique $\eta \in \Sigma$ minimizing $\xi_0|_\Sigma$. As Σ is Γ-invariant, we have

$$(\gamma\xi_0)(\eta) = \xi_0(\gamma\eta) > \xi_0(\eta)$$

for every $\gamma \notin \Gamma_\eta$. Since $\eta \in \Sigma$, the stabilizer Γ_η is trivial, so the stabilizer of ξ_0 in Γ is trivial as well. ◻

CHAPTER 19

THE NEF CONE OF A FIBER PRODUCT OVER A CURVE

We now prove Theorem 17.3 about the decomposition of the nef cone.

For $i = 1, 2$, recall that $\phi_i : W_i \to B$ is a surjective morphism with connected fibers from a smooth projective variety to a smooth projective curve B. We consider the fiber product

$$\begin{tikzcd} & W = W_1 \times_B W_2 \arrow[dl, "p_1"'] \arrow[d, "p"] \arrow[dr, "p_2"] & \\ W_1 \arrow[dr, "\phi_1"'] & B & W_2 \arrow[dl, "\phi_2"] \end{tikzcd}$$

and work under the following assumptions:

1 the variety $W = W_1 \times_B W_2$ is smooth;

2 for every $D \in N^1(W)_{\mathbb{R}}$, there exist $D_1 \in N^1(W_1)_{\mathbb{R}}$ and $D_2 \in N^1(W_2)_{\mathbb{R}}$ such that

$$D = p_1^* D_1 + p_2^* D_2.$$

Proof of Theorem 17.3. Let $D \in \mathrm{Nef}(W)$ and let

$$D = p_1^* D_1 + p_2^* D_2 \in N^1(W)_{\mathbb{R}}$$

be a decomposition as in (2). First, note the following simple fact.

Lemma 19.1. *Let $C_i \subset W_i$ be an irreducible curve. If $\phi_i(C_i)$ is a point, then $D_i \cdot C_i \geq 0$.*

Proof. We may only consider $i = 1$. Choose any point $s \in \phi_2^{-1}(\phi_1(C_1))$ and let $\widetilde{C_1} := C_1 \times_B \{s\} \subset W$. We have

$$0 \leq D \cdot \widetilde{C_1} = (p_1^* D_1 + p_2^* D_2) \cdot \widetilde{C_1} = D_1 \cdot p_{1*}\widetilde{C_1} + D_2 \cdot p_{2*}\widetilde{C_1} = D_1 \cdot C_1.$$

This proves the assertion. □

We use this fact to prove the following two lemmas.

Lemma 19.2. *Either D_1 or D_2 is nef.*

Proof. Assume by contradiction that both D_1 and D_2 are not nef. Then for each i, there exists an irreducible curve $C_i \subset W_i$ such that $D_i \cdot C_i < 0$. By Lemma 19.1, we have $\phi_i(C_i) = B$, so $\widetilde{C} := C_1 \times_B C_2$ is a curve. Let $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$ be such that $p_{i*}\widetilde{C} = \beta_i C_i$. Then

$$0 > \beta_1 D_1 \cdot C_1 + \beta_2 D_2 \cdot C_2 = (p_1^* D_1 + p_2^* D_2) \cdot \widetilde{C} = D \cdot \widetilde{C} \geq 0,$$

which is a contradiction. □

Now we fix a point $b \in B$.

Lemma 19.3. *For $i = 1, 2$, there exists $N_i \in \mathbb{R}$ such that the divisor $D_i + n\phi_i^* b$ is nef if $n \geq N_i$.*

Proof. We may only consider the case when $i = 2$.
Let $C_1 \subset W_1$ be an irreducible curve such that $\phi_1(C_1) = B$. Define

$$D_1' := D_1 - N_2 \phi_1^* b \quad \text{and} \quad D_2' := D_2 + N_2 \phi_2^* b$$

where

$$N_2 := \frac{D_1 \cdot C_1}{\deg(C_1 \xrightarrow{\phi_1} B)}.$$

By construction, we have

$$D_1' \cdot C_1 = 0 \text{ and } D = p_1^* D_1' + p_2^* D_2'.$$

Let us show that D_2' is nef. Let $C_2 \subset W_2$ be an irreducible curve. If $\phi_2(C_2)$ is a point, then $D_2' \cdot C_2 \geq 0$ by Lemma 19.1. Suppose now that $\phi_2(C_2) = B$. Set $\widetilde{C} := C_1 \times_B C_2$ and define $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$ such that $p_{i*}\widetilde{C} = \beta_i C_i$. We have

$$\begin{aligned} \beta_2 D_2' \cdot C_2 &= \beta_1 D_1' \cdot C_1 + \beta_2 D_2' \cdot C_2 \\ &= (p_1^* D_1' + p_2^* D_2') \cdot \widetilde{C} \\ &= D \cdot \widetilde{C} \geq 0. \end{aligned}$$

This shows that D_2' is nef. Hence, for $n \geq N_2$, the divisor

$$D_2 + n\phi_2^* b = D_2' + (n - N_2)\phi_2^* b$$

is nef. □

We can now resume the proof of Theorem 17.3. For any $t \in \mathbb{R}$, let

$$D_1(t) := D_1 - t\phi_1^* b \quad \text{and} \quad D_2(t) := D_2 + t\phi_2^* b.$$

By Lemma 19.3, there exist

$$I_1 =]-\infty, -N_{1,\min}] \quad \text{and} \quad I_2 = [N_{2,\min}, +\infty[$$

such that $D_i(t)$ is nef if and only if $t \in I_i$. Since we have

$$D = p_1^* D_1(t) + p_2^* D_2(t),$$

Lemma 19.2 shows that either $D_1(t)$ or $D_2(t)$ is nef, namely, $I_1 \cup I_2 = \mathbb{R}$. Thus, $I_1 \cap I_2$ is non-empty. As both $D_1(t)$ and $D_2(t)$ are nef whenever $t \in I_1 \cap I_2$, this gives a desired decomposition.

The last statement about the decomposition of the ample cone follows from [171, Corollary 6.6.2]. □

Remark 19.4. In the setup of Theorem 17.3, we also have the decomposition of the relative nef cone

$$\mathrm{Nef}(W/B) = p_1^*\mathrm{Nef}(W_1/B) + p_2^*\mathrm{Nef}(W_2/B)$$

by the projection formula – this is exactly Lemma 19.1.

Now we prove Corollary 17.4.

Proof of Corollary 17.4. We may assume $i = 1$.

First assume that p_1^*E is extremal. Let $E = F + F'$ be a decomposition with $F, F' \in \mathrm{Nef}(W_1)$. Then $p_1^*E = p_1^*F + p_1^*F'$ with $p_1^*F, p_1^*F' \in \mathrm{Nef}(W)$, and thus, p_1^*F and p_1^*F' are proportional by assumption. Since $p_1^* : N^1(W_1)_{\mathbb{R}} \to N^1(W)_{\mathbb{R}}$ is injective, F and F' are proportional as well. This shows that E is extremal.

Next assume that $E \in \mathrm{Nef}(W_1)$ is extremal. Let $p_1^*E = D + D'$ be a decomposition with $D, D' \in \mathrm{Nef}(W)$. Up to adding terms to D', we can assume that D is extremal. By Theorem 17.3, we can write

$$D = p_1^*D_1 + p_2^*D_2, \quad \text{and} \quad D' = p_1^*D_1' + p_2^*D_2'$$

with $D_i, D_i' \in \mathrm{Nef}(W_i)$. As D is extremal, the divisors D, $p_1^*D_1$ and $p_2^*D_2$ are proportional. Moreover $p_1^*(E - D_1 - D_1') = p_2^*(D_2 + D_2') \in \mathrm{Nef}(W)$. Hence, by the projection formula, $E - D_1 - D_1'$ is nef. But E is extremal in the cone $\mathrm{Nef}(W_1)$, so E, D_1, and D_1' are proportional. In particular, $p_1^*E, p_1^*D_1, p_1^*D_1'$ and $p_2^*D_2$ are all proportional, which concludes the proof. $\square$

Now we construct fiber products showing that Theorem 17.3 fails in general when $\dim B \geq 2$. First we construct such examples of fiber products over a surface.

Example 19.5. Take $S := \mathbb{P}^2$, and take four points P_1, P_2, P_3, P_4 in S so that no three of them lie on a line. Let ℓ_1 be the line through P_1, P_2, and let ℓ_2 be the line through P_3, P_4. Take

$$W_1 := \mathrm{Bl}_{P_1,P_2}(S) \quad \text{and} \quad W_2 := \mathrm{Bl}_{P_3,P_4}(S).$$

As the blown-up points are distinct, $W := W_1 \times_S W_2$ is isomorphic to $\mathrm{Bl}_{P_1,P_2,P_3,P_4}(S)$, which is smooth. Moreover, the decomposition of the Picard group

$$\mathrm{Pic}(W) = p_1^*\mathrm{Pic}(W_1) + p_2^*\mathrm{Pic}(W_2)$$

clearly holds.

Denote by ℓ_1' and ℓ_2' the strict transforms of ℓ_1 and ℓ_2 in W_1 and W_2 respectively. Then ℓ_i' is an effective non-nef divisor on W_i as $(\ell_i')^2 = -1$. Let

$$D := p_1^*\ell_1' + p_2^*\ell_2'.$$

We show that D is nef; this also shows that Lemma 19.2 fails when $\dim B \geq 2$. As D is effective, it is enough to check that its intersections with its components are all non-negative. By symmetry, it is enough to compute

$$D \cdot p_1^*\ell_1' = (\ell_1')^2 + \ell_2' \cdot \phi_2^*\ell_1 = -1 + 1 = 0.$$

So D is nef, and has vanishing intersection with the curves $p_1^*\ell_1'$ and $p_2^*\ell_2'$.

Now assume by contradiction that D has another decomposition $D = p_1^*D_1 + p_2^*D_2$ with $D_i \in \mathrm{Nef}(W_i)$. Then we have

$$p_1^*(\ell_1' - D_1) = p_2^*(D_2 - \ell_2').$$

As $p_1^*N^1(W_1)_{\mathbb{R}} \cap p_2^*N^1(W_2)_{\mathbb{R}}$ clearly has dimension one, it equals $\mathbb{R}[p^*\ell]$, where $p : W \to S$ is the blow up, and ℓ is a line passing through none of P_1, P_2, P_3, P_4 in S. It follows that

$$p_1^*(\ell_1' - D_1) = p_2^*(D_2 - \ell_2') = cp^*\ell$$

for some $c \in \mathbb{R}$.

Since

$$p_1^*D_1 \cdot p_i^*\ell_i' + p_2^*D_2 \cdot p_i^*\ell_i' = D \cdot p_i^*\ell_i' = 0,$$

and both $p_1^*D_1$ and $p_2^*D_2$ are nef, we have $p_i^*D_i \cdot p_i^*\ell_i' = 0$. Thus

$$-1 = p_1^*\ell_1' \cdot p_1^*(\ell_1' - D_1) = cp_1^*\ell_1' \cdot p^*\ell = c$$

and similarly,

$$1 = p_2^*\ell_2' \cdot p_2^*(D_2 - \ell_2') = cp_2^*\ell_2' \cdot p^*\ell = c,$$

which is a contradiction.

Example 19.6. As for examples of fiber products over a base of higher dimension, we continue with the notations of Example 19.5, and introduce

$$W \times T = (W_1 \times T) \times_{(S \times T)} (W_2 \times T)$$

where T is an arbitrary smooth projective variety. As in Example 19.5, W, W_1 and W_2 are rationally connected, hence have trivial irregularity, so that

$$N^1(Z \times T)_{\mathbb{R}} = p_Z^*N^1(Z)_{\mathbb{R}} \oplus p_T^*N^1(T)_{\mathbb{R}},$$

for $Z = W, W_1$ or W_2. This implies that

$$N^1(W \times T)_{\mathbb{R}} = (p_1 \times \mathrm{id}_T)^*N^1(W_1 \times T)_{\mathbb{R}} + (p_2 \times \mathrm{id}_T)^*N^1(W_2 \times T)_{\mathbb{R}}.$$

Note that by the projection formula,

$$\mathrm{Nef}(Z \times T) = p_Z^*\mathrm{Nef}(Z) \oplus p_T^*\mathrm{Nef}(T),$$

for $Z = W, W_1$ or W_2. So, if we assume by contradiction that

$$\mathrm{Nef}(W \times T) = (p_1 \times \mathrm{id}_T)^*\mathrm{Nef}(W_1 \times T) + (p_2 \times \mathrm{id}_T)^*\mathrm{Nef}(W_2 \times T),$$

we get $\mathrm{Nef}(W) = p_1^*\mathrm{Nef}(W_1) + p_2^*\mathrm{Nef}(W_2)$, which contradicts Example 19.5.

For a morphism $\pi : X \to Y$, we define

$$\mathrm{Aut}(X/Y) = \{g \in \mathrm{Aut}(X) \mid \pi \circ g = \pi\}.$$

We have the following corollary of Theorem 17.3.

Corollary 19.7. *For $i = 1, 2$, let $\phi_i : W_i \to B$ be a surjective morphism with connected fibers from a smooth projective variety to a smooth projective curve B. Assume that*

1 the variety $W = W_1 \times_B W_2$ is smooth;

2 it holds
$$p_1^* N^1(W_1)_{\mathbb{R}} + p_2^* N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}},$$
where p_i denotes the projection from W onto W_i.

For $i = 1, 2$, let $H_i \leq \mathrm{Aut}(W_i/B)$ be a subgroup. Let $H \leq \mathrm{Aut}(W)$ be a subgroup containing $H_1 \times H_2$. Assume that there exists a rational polyhedral cone $\Pi_i \subset \mathrm{Nef}^+(W_i)$ such that $H_i \cdot \Pi_i \supset \mathrm{Amp}(W_i)$. Then $\mathrm{Nef}^+(W)$ admits a rational polyhedral fundamental domain for the H-action.

Proof. Let Π be the convex hull of $p_1^*\Pi_1 + p_2^*\Pi_2$. Then Π is a rational polyhedral cone contained in $\mathrm{Nef}^+(W)$. Moreover,
$$\mathrm{Amp}(W) \subset (H_1 \times H_2) \cdot \Pi \subset H \cdot \Pi$$
as $p_1^*\mathrm{Amp}(W_1) + p_2^*\mathrm{Amp}(W_2) = \mathrm{Amp}(W)$ by Theorem 17.3. The existence of a rational polyhedral fundamental domain then follows from Proposition 18.3.(1). □

CHAPTER 20

CONSTRUCTION OF SCHOEN VARIETIES

Schoen varieties will be constructed as a fiber product of two fibrations over $\mathbb{P}^1$. Let us first construct these fibrations.

20.1 The factor W with a fibration over $\mathbb{P}^1$. The construction relies on a pencil of ample hypersurfaces in a Fano manifold.

Let Z be a Fano manifold of dimension at least 2, and let D be an ample divisor in Z such that both $\mathcal{O}_Z(D)$ and $\mathcal{O}_Z(-K_Z - D)$ are globally generated. Note that $\mathcal{O}_Z(-K_Z)$ is then globally generated as well.

Example 20.1. Take any toric Fano manifold Z. Since nef line bundles on a projective toric manifold are globally generated, any decomposition $-K_Z = D + D'$ as the sum of an ample divisor D and a nef divisor D' yields a pair (Z, D) satisfying the above condition.

Let $W \subset \mathbb{P}^1 \times Z$ be a general member of the ample and basepoint-free linear system $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Z(D)|$. We have a fibration $\phi : W \to \mathbb{P}^1$ via the first projection, and the second projection $\varepsilon : W \to Z$ is the blow-up of Z along the smooth subvariety Y of codimension two cut out by the members of the pencil in $|D|$ defined by W. Since Z is Fano, W is rationally connected. By construction, the rational curve $\varepsilon^{-1}(y) \simeq \mathbb{P}^1$ for any $y \in Y$ is a section of $\phi : W \to \mathbb{P}^1$.

Note that

$$\mathcal{O}_W(-K_W) = (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Z(-K_Z - D))\,|_W \tag{20.1}$$

by the adjunction formula. So $\mathcal{O}_W(-K_W)$ is globally generated, in particular, nef and effective.

The following lemma describes the possibilities for W in dimension 2. Recall that a smooth projective surface S is called *weak del Pezzo* if its anticanonical divisor $-K_S$ is nef and big.

Lemma 20.2. *If $\dim W = 2$, then either $D \in |-K_Z|$ and $W \xrightarrow{\phi} \mathbb{P}^1$ is a rational elliptic surface with $-K_W$ globally generated, or W is a weak del Pezzo surface.*

Proof. Since W is rationally connected and $\dim W = 2$, we know that W is rational.

If $D \in |-K_Z|$, then $\mathcal{O}_W(-K_W) = \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$. So $-K_W$ is globally generated and W is a rational elliptic surface.

Suppose that $D \notin |-K_Z|$. As $-K_Z - D$ is effective and $-K_Z$ and D are ample, we have $-K_Z(-K_Z - D) > 0$ and $D(-K_Z - D) > 0$, and thus,

$$K_Z^2 > -K_Z \cdot D > D^2.$$

As W is the blowup of Z at D^2 points, we have ${K_W}^2 = {K_Z}^2 - D^2 > 0$. Since $-K_W$ is nef, W is a weak del Pezzo surface. □

Let us say something about the nef cone of W, in either case of Lemma 20.2.

On one hand, let W be a rational elliptic surface. Clearly $\text{Nef}^e(W)$ and $\text{Nef}^+(W)$ are subcones of $\text{Nef}(W)$. Moreover, by [192, Lemma 4.2], $\text{Nef}^+(W) \subset \text{Nef}^e(W)$. Then in the papers [190], [192, p.256, 2nd paragraph], Totaro covers the nef cone $\text{Nef}(W)$ by a set of rational polyhedral subcones $\{\Pi_E : E \text{ is a (-1)-curve}\}$. Since they are rational polyhedral, the Π_E are subcones of $\text{Nef}^+(W)$. Hence $\text{Nef}(W) \subset \text{Nef}^+(W)$. This proves the equalities $\text{Nef}^e(W) = \text{Nef}^+(W) = \text{Nef}(W)$.

On the other hand, a weak del Pezzo surface is easily seen to be a log del Pezzo surface (see [137, Proposition 2.6]), hence by the Cone Theorem [112, Theorem 3.7], its nef cone is a rational polyhedral cone spanned by classes of semiample divisors.

When $\dim W \geq 3$, the cone $\text{Nef}(W)$ is rational polyhedral, spanned by classes of semiample divisors. This is an immediate corollary of Theorem 20.3 below.

Theorem 20.3 ([20, Appendix], [83, Theorem 4.3], [15, Proposition 3.5])**.** *Let Y be a smooth projective variety of dimension ≥ 4 and let $j : D \hookrightarrow Y$ be a smooth ample divisor. Suppose that $-(K_Y + D)$ is nef. Then Y is a Fano manifold, and*

$$j^*(\text{Nef}(Y)) = \text{Nef}(D).$$

In particular, $\text{Nef}(D)$ *is rational polyhedral, spanned by classes of semiample divisors.*

In summary, we established the following result.

Proposition 20.4. *We have*

$$\text{Nef}^e(W) = \text{Nef}^+(W) = \text{Nef}(W).$$

Moreover, if $\dim W \geq 3$*, or if* $\dim W = 2$ *and W is a weak del Pezzo surface, the cone* $\text{Nef}(W)$ *is rational polyhedral, spanned by classes of semiample divisors.*

Finally, note that if $D \in |-K_Z|$, then by (20.1), a general fiber of $\phi : W \to \mathbb{P}^1$ is a smooth K-trivial variety. If W has dimension 2, it must be an elliptic curve. In general, we can say the following.

Lemma 20.5. *If $D \in |-K_Z|$, then a general fiber F of $\phi : W \to \mathbb{P}^1$ is a Calabi–Yau manifold, that is, $\omega_F \simeq \mathcal{O}_F$ and $h^i(F, \mathcal{O}_F) = 0$ for $0 < i < \dim F$.*

Proof. Since $D \in |-K_Z|$, we have $\mathcal{O}_W(F) \simeq \mathcal{O}_W(-K_W)$ by (20.1). So by adjunction, $\omega_F \simeq \mathcal{O}_F$, and also we have the exact sequence

$$0 \to \omega_W \to \mathcal{O}_W \to \mathcal{O}_F \to 0.$$

Since W is rationally connected, we have

$$h^{\dim W - i}(W, \omega_W) = h^i(W, \mathcal{O}_W) = 0$$

for $i \geq 1$. Hence $h^i(F, \mathcal{O}_F) = 0$ whenever $1 \leq i \leq \dim W - 2 = \dim F - 1$. □

20.2 The fiber product $X = W_1 \times_{\mathbb{P}^1} W_2$. We are ready to generalize Schoen's construction and obtain Calabi–Yau pairs in arbitrary dimension. For $i = 1, 2$, let Z_i, D_i, W_i be as in 20.1. We denote by $\phi_i : W_i \to \mathbb{P}^1$ the associated fibration, and recall that it has a section.

Denoting by S_i the locus of singular fibers of ϕ_i in $\mathbb{P}^1$, we assume $S_1 \cap S_2 = \varnothing$. Moreover, if $\phi_1 : W_1 \to \mathbb{P}^1$ and $\phi_2 : W_2 \to \mathbb{P}^1$ are two rational elliptic surfaces with sections, we require that the elliptic curves $\phi_1^{-1}(t)$ and $\phi_2^{-1}(t)$ are non-isogenous for a general point $t \in \mathbb{P}^1$.

We consider the fiber product over $\mathbb{P}^1$

$$\begin{tikzcd} & X = W_1 \times_{\mathbb{P}^1} W_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\phi"] & \\ W_1 \arrow[dr, "\phi_1"'] & & W_2 \arrow[dl, "\phi_2"] \\ & \mathbb{P}^1 & \end{tikzcd} .$$

As $S_1 \cap S_2 = \varnothing$, the variety X is smooth.

One can also regard X as a complete intersection in $\mathbb{P}^1 \times Z_1 \times Z_2$ of two hypersurfaces in the linear systems

$$|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_1}(D_1) \boxtimes \mathcal{O}_{Z_2}| \quad \text{and} \quad |\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_1} \boxtimes \mathcal{O}_{Z_2}(D_2)|,$$

respectively. In particular,

$$\mathcal{O}_X(-K_X) = (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{Z_1}(-K_{Z_1} - D_1) \boxtimes \mathcal{O}_{Z_2}(-K_{Z_2} - D_2))\,|_X \tag{20.2}$$

is globally generated.

Let $m \geq 1$ be an integer. As in Example 18.1, we let

$$\Delta_{m,X} = \frac{1}{m}\Delta'_{m,X}, \quad \text{where } \Delta'_{m,X} \in |-mK_X|.$$

Notice that by construction,

$$\dim X = \dim Z_1 + \dim Z_2 - 1,$$

and

$$K_X + \Delta_{m,X} \sim_{\mathbb{Q}} 0.$$

Thus, the pair $(X, \Delta_{m,X})$ is Calabi–Yau, and is klt if $m \geq 2$ and $\Delta'_{m,X} \in |-mK_X|$ is general.

Definition 20.6. The pair $(X, \Delta_{m,X})$ constructed above is called a *Schoen pair*. We may also refer to X alone as a *Schoen variety*.

Lemma 20.7. *Any Schoen variety X is simply connected.*

Proof. The proof is similar to [178, Lemma 1] and [187, Lemma 2.1].

Let $U \subset \mathbb{P}^1$ be the open subset over which the morphism $\phi : X \to \mathbb{P}^1$ is smooth and set $V := \phi^{-1}(U)$. The natural map $\phi|_V : V \to U$ is topologically locally trivial.

Denote its fiber by F. Since both ϕ_1 and ϕ_2 have sections, $\phi : X \to \mathbb{P}^1$ also admits a section $\sigma : \mathbb{P}^1 \to X$. Consider the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(F) \arrow[r] & \pi_1(V) \arrow[r, two heads] \arrow[d] & \pi_1(U) \arrow[r] \arrow[d] \arrow[l, bend right, "\sigma_{U*}"'] & 1 \\
& & \pi_1(X) \arrow[r, two heads] & \pi_1(\mathbb{P}^1). \arrow[l, bend right, "\sigma_*"'] &
\end{tikzcd}$$

Here the first row is exact by the homotopy long exact sequence. By a diagram chase and the fact that $\pi_1(\mathbb{P}^1)$ is trivial, it is enough to check that the image of $\pi_1(F)$ in $\pi_1(X)$ is trivial. Write $F = F_1 \times F_2$, where F_i is a general fiber of $\phi_i : W_i \to \mathbb{P}^1$ for $i = 1, 2$. Since $\pi_1(F) = \pi_1(F_1) \times \pi_1(F_2)$, it is enough to show that the image of $\pi_1(F_i)$ in $\pi_1(X)$ is trivial, which we prove for $i = 1$.

A section of $\phi_2 : W_2 \to \mathbb{P}^1$ gives rise to a section s of $p_1 : X \to W_1$. By construction, the homomorphism $\pi_1(F_1) \to \pi_1(X)$ is induced by $F_1 \hookrightarrow W_1 \xrightarrow{s} X$, thus factors through $\pi_1(W_1)$. Since it is rationally connected, W_1 is simply-connected, and hence the image of $\pi_1(F_1)$ in $\pi_1(X)$ is trivial. □

Proposition 20.8. *Suppose that $D_i \in |-K_{Z_i}|$ for both $i = 1, 2$. Then the Schoen variety X is a strict Calabi–Yau manifold; namely, X is simply connected with $\omega_X \simeq \mathcal{O}_X$, and $h^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$.*

Proof. First of all, (20.2) shows that K_X is trivial. Since X is simply-connected by Lemma 20.7, it is enough to show that $h^p(X, \mathcal{O}_X) = 0$ for $1 < p < \dim X$. A general fiber of p_2, i.e., of ϕ_1 is a Calabi–Yau manifold by Lemma 20.5.

Lemma 20.9. *Let $g : \mathfrak{X} \to \mathfrak{Y}$ be a surjective morphism between smooth projective varieties. Assume that a general fiber F of g is a Calabi–Yau manifold and that $\omega_{\mathfrak{X}} = \mathcal{O}_{\mathfrak{X}}$. Then for every integer $i > 0$, we have*

$$R^i g_* \mathcal{O}_{\mathfrak{X}} = \begin{cases} \omega_{\mathfrak{Y}}, & \text{if } i = \dim \mathfrak{X} - \dim \mathfrak{Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Set $r := \dim \mathfrak{X} - \dim \mathfrak{Y}$.

Since $R^q g_* \omega_{\mathfrak{X}} = R^q g_* \mathcal{O}_{\mathfrak{X}}$ is reflexive by [104, Theorem 2.1.(i)] and [105, Corollary 3.9], and since $H^q(F, \mathcal{O}_F) = 0$ for all $0 < q < r$ and $\dim H^q(F, \mathcal{O}_F) = 1$ for $q = 0$ or r, we have

$$R^q g_* \mathcal{O}_{\mathfrak{X}} = \begin{cases} \text{an invertible sheaf, if } q = 0 \text{ or } r, \\ 0, \text{ otherwise.} \end{cases}$$

By Grothendieck–Verdier duality [88, Theorem 3.34], we have

$$Rg_* \omega_{\mathfrak{X}} \simeq R\mathcal{H}om(Rg_* \mathcal{O}_{\mathfrak{X}}, \omega_{\mathfrak{Y}}[-r]).$$

The Grothendieck spectral sequence then gives

$$E_2^{p,-q} := \mathcal{E}xt^p(R^q g_* \mathcal{O}_{\mathfrak{X}}, \omega_{\mathfrak{Y}}) \Rightarrow R^{p-q+r} g_* \omega_{\mathfrak{X}}.$$

Note that since $\mathcal{H}om(\bullet, \omega_{\mathfrak{Y}})$ is contravariant, we have $-q$ instead of q in $E_2^{p,-q}$. (see also [88, Example 2.70.ii)]).

So $E_2^{p,-q} \neq 0$ only if $(p, q) = (0, 0)$ or $(0, r)$, and Lemma 20.9 follows. □

Let $w_i := \dim W_i$. Applying Lemma 20.9 to $p_2 : X \to W_2$ shows that

$$R^j p_{2*}\omega_X = \begin{cases} \mathcal{O}_{W_2}, & \text{if } j = 0, \\ \omega_{W_2}, & \text{if } j = \dim w_1 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from [105, Corollary 3.2] that

$$h^p(X, \omega_X) = h^p(W_2, \mathcal{O}_{W_2}) + h^{p-w_1+1}(W_2, \omega_{W_2})$$

for all $0 \leq p \leq \dim X$. Since W_2 is rationally connected, this is zero unless $p = 0$ or $w_1 + w_2 - 1$. $\square$

CHAPTER 21

APPLICATION TO THE CONE CONJECTURE

In this section, we prove Theorem 17.5.

We have defined Schoen pairs $(X, \Delta_{m,X})$ in Section 20, arising from fiber products

$$\begin{tikzcd} & X = W_1 \times_{\mathbb{P}^1} W_2 \arrow[dl, "p_1"'] \arrow[d, "\phi"] \arrow[dr, "p_2"] & \\ W_1 \arrow[r, "\phi_1"'] & \mathbb{P}^1 & W_2 \arrow[l, "\phi_2"] \end{tikzcd} .$$

Lemma 21.1. *Any line bundle L on a Schoen variety X can be written $L = p_1^* L_1 \otimes p_2^* L_2$, where L_i is a line bundle on W_i.*

Proof. Let $p \in \mathbb{P}^1$ be a general point and let $F_i := \phi_i^{-1}(p) \subset W_i$.

Claim 21.2. *The map*

$$\Psi : \mathrm{Pic}(F_1) \times \mathrm{Pic}(F_2) \to \mathrm{Pic}(F_1 \times F_2)$$

defined by $\Psi(L, M) = L \boxtimes M$ is an isomorphism.

Proof. First suppose that either W_1 or W_2 is not a rational elliptic surface. Since $H^1(F_i, \mathcal{O}_{F_i}) = 0$ for at least one $i \in \{1, 2\}$, Claim 21.2 follows from [81, Exercise III.12.6].

Assume now that W_1 and W_2 are rational elliptic surfaces. Then F_1 and F_2 are elliptic curves, and we have a short exact sequence of abelian groups [17, Theorem 11.5.1]

$$0 \to \mathrm{Pic}(F_1) \times \mathrm{Pic}(F_2) \xrightarrow{\Psi} \mathrm{Pic}(F_1 \times F_2) \to \mathrm{Hom}(F_1, F_2) \to 0$$

where $\mathrm{Hom}(F_1, F_2)$ is the group of homomorphisms of group varieties $F_1 \to F_2$. Since $p \in \mathbb{P}^1$ is general, the elliptic curves F_1 and F_2 are non-isogenous by our definition of Schoen varieties. Thus $\mathrm{Hom}(F_1, F_2) = 0$, which proves Claim 21.2. $\square$

Let L be a line bundle on X. Claim 21.2 implies that

$$L_{|\phi^{-1}(p)} \simeq L_{|F_1 \times \{u\}} \boxtimes L_{|\{v\} \times F_2},$$

for any points $u \in F_2$ and $v \in F_1$.

For each $i = 1, 2$, we choose a section $s_i : \mathbb{P}^1 \to W_i$ and let $\sigma_i : W_i \to X$ be the induced section:

$$\sigma_1(w_1) := (w_1, s_2(\phi_1(w_1))) \in W_1 \times_{\mathbb{P}^1} W_2,$$

and similarly for σ_2. We have

$$\begin{aligned}L_{|\phi^{-1}(p)} &\simeq L_{|F_1\times\{s_1(p)\}} \boxtimes L_{|\{s_2(p)\}\times F_2}\\ &\simeq (\sigma_1^*L)_{|F_1} \boxtimes (\sigma_2^*L)_{|F_2}\\ &\simeq (p_1^*\sigma_1^*L\otimes p_2^*\sigma_2^*L)_{|\phi^{-1}(p)}.\end{aligned}$$

Since $p\in\mathbb{P}^1$ is general, by [81, Exercise III.12.4]

$$L\simeq p_1^*\sigma_1^*L\otimes p_2^*\sigma_2^*L\otimes\mathcal{O}_X(D)$$

for some divisor D whose support is contained in a finite union of fibers of $\phi: X\to\mathbb{P}^1$. Since the subsets S_1, S_2 parametrizing singular fibers of ϕ_1 and ϕ_2 respectively are disjoint, the subsets paramatrizing reducible fibers are disjoint as well. Hence, an irreducible component R of a fiber of ϕ is of the form p_i^*R' where R' is a multiple of an irreducible component of a fiber of $\phi_i: W_i\to\mathbb{P}^1$. Applied to the irreducible components of D, that yields that

$$\operatorname{Pic}(W_1)\times\operatorname{Pic}(W_2)\xrightarrow{p_1^*\otimes p_2^*}\operatorname{Pic}(X)$$

is surjective. □

Lemma 21.3. *For every $D\in\operatorname{Nef}(X)$, one can write $D=p_1^*D_1+p_2^*D_2$, where $D_i\in\operatorname{Nef}(W_i)$.*

Proof. Lemma 21.3 follows from Lemma 21.1, which by $\mathbb{R}$-linearity, yields the decomposition at the level of $N^1(W)_{\mathbb{R}}$, and Theorem 17.3. □

Theorem 21.4 (= Theorem 17.5)**.** *Let $(X,\Delta_{m,X})$ be a Schoen pair. Then*

$$\operatorname{Nef}(X)=\operatorname{Nef}^+(X)=\operatorname{Nef}^e(X),$$

and moreover, there exists a rational polyhedral fundamental domain for the action of $\operatorname{Aut}(X,\Delta_{m,X})$ on $\operatorname{Nef}^e(X)$.

Proof. Since $\operatorname{Nef}(W_i)=\operatorname{Nef}^+(W_i)=\operatorname{Nef}^e(W_i)$ by Proposition 20.4, we have, by Theorem 17.3 and Lemma 21.3, $\operatorname{Nef}(X)=p_1^*\operatorname{Nef}^+(W_1)+p_2^*\operatorname{Nef}^+(W_2)\subset\operatorname{Nef}^+(X)$, so $\operatorname{Nef}(X)=\operatorname{Nef}^+(X)$. Similarly, we have $\operatorname{Nef}(X)=\operatorname{Nef}^e(X)$. Therefore,

$$\operatorname{Nef}(X)=\operatorname{Nef}^+(X)=\operatorname{Nef}^e(X).$$

Define the subgroups $H_i\le\operatorname{Aut}(W_i)$ by

$$H_i=\begin{cases}\operatorname{Aut}(W_i/\mathbb{P}^1), & \text{if } W_i \text{ is a rational elliptic surface},\\ \{\mathrm{id}_{W_i}\}, & \text{otherwise.}\end{cases}$$

Then there exists a rational polyhedral cone $\Pi_i\subset\operatorname{Nef}^+(W_i)$ such that $H_i\cdot\Pi_i$ contains $\operatorname{Amp}(W_i)$. Indeed, the case where W_i is a rational elliptic surface with $-K_{W_i}$ semiample follows from [190, Theorem 8.2], and the other cases follow from Proposition 20.4.

We claim that $H_1\times H_2\le\operatorname{Aut}(X,\Delta_{m,X})$. Indeed, if neither W_1 nor W_2 is a rational elliptic surface, then $H_1\times H_2$ is trivial by definition. If both W_1 and W_2 are rational elliptic surfaces, then $\Delta_{m,X}=0$ and clearly, $H_1\times H_2\le\operatorname{Aut}(X)$. Finally, if one of the W_i, say W_1, is a rational elliptic surface, and the other, say W_2, is not, then $\mathcal{O}_X(-K_X) \simeq p_2^* \mathcal{O}_{W_2}(-K_{Z_2} - D_2)$. Since p_2 is proper surjective with connected fibers, the pullback p_2^* induces an isomorphism

$$H^0(X, p_2^*\mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))) \simeq H^0(W_2, \mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))).$$

So $\Delta_{m,X} = \frac{1}{m} p_2^* \Delta_{m,W_2}$, for some divisor $\Delta_{m,W_2} \in |\mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))|$. Since $H_2 = \{\mathrm{id}_{W_2}\}$ in this case, it follows that $\Delta_{m,X}$ is invariant under $H_1 \times H_2$. This proves the claim.

It then follows from Corollary 19.7 that $\mathrm{Nef}^e(X) = \mathrm{Nef}^+(X)$ has a rational polyhedral fundamental domain Π for the $\mathrm{Aut}(X, \Delta_{m,X})$-action. □

Remark 21.5. In [68], the authors verified the Cone Conjecture for a strict Calabi–Yau threefold $X = W_1 \times_{\mathbb{P}^1} W_2$, where both W_i are rational elliptic surfaces with section, each of whose singular fibers is an irreducible rational curve with a node, and two generic fibers are non-isogenous.

Our proof bypasses the identification shown by Namikawa [151, Proposition 2.2 and Corollary 2.3]

$$\mathrm{Aut}(X) \cong \mathrm{Aut}(W_1) \times \mathrm{Aut}(W_2),$$

an identification that is crucial in [68] due to the lack of Looijenga's result (Lemma 18.4) at that time.

Example 21.6. Assume that $\dim Z_1 = 2$ and W_1 is a general rational elliptic surface obtained by a pencil of cubic curves in $\mathbb{P}^2$. Then $\mathrm{Nef}(W_1)$ admits infinitely many faces, and so does $\mathrm{Nef}(X)$ by Lemma 21.1 and Corollary 17.4. If in addition $D_2 \in |-K_{Z_2}|$, then the Schoen variety X is a strict Calabi–Yau manifold by Proposition 20.8.

Corollary 21.7. *Let X be a Schoen variety. Then $\pi_0\mathrm{Aut}(X)$ is finitely presented and there are only finitely many real structures on X up to equivalence.*

Proof. Recall the linear action $\rho : \mathrm{Aut}(X) \to \mathrm{GL}(N^1(X))$ and the induced action $\overline{\rho} : \pi_0\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X))$ defined in 18.1. We let $\mathrm{Aut}^*(X) = \rho(\mathrm{Aut}(X)) = \overline{\rho}(\pi_0\mathrm{Aut}(X))$. By Theorem 17.5, there exists a rational polyhedral cone $\Pi \subset \mathrm{Nef}^+(X)$ such that

$$\mathrm{Amp}(X) \subset \mathrm{Aut}(X, \Delta_{1,X}) \cdot \Pi \subset \mathrm{Aut}^*(X) \cdot \Pi.$$

Then, from Proposition 18.3, it follows that there is a rational polyhedral fundamental domain for the $\mathrm{Aut}^*(X)$-action on $\mathrm{Nef}^+(X)$ and the group $\mathrm{Aut}^*(X)$ is finitely presented.

Since $\mathrm{Ker}(\overline{\rho})$ is finite by [23, Corollary 2.11], the first claim follows from [95, Corollary 10.2]. The second statement follows from Theorem 21.8 below. □

Theorem 21.8 ([52, Theorem 1.4])**.** *Let V be a smooth complex projective variety. Assume that there exists a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(V)$ on $\mathrm{Nef}^+(V)$. Then the set of real structures of V is at most finite.*

We end this paper with a short discussion on the minimal models of X. In [151], Namikawa proved that the number of minimal models of X modulo isomorphisms (as abstract varieties) is finite when X is a strict Calabi–Yau threefold obtained from a certain fiber product of rational elliptic surfaces. It would be interesting to investigate the general case. Such finiteness is predicted by a birational version of the Cone Conjecture concerning the structure of movable cones of Calabi–Yau varieties. See [144, 101, 192, 125] for more details.

PART IV
POSITIVITY OF HIGHER EXTERIOR POWERS OF THE TANGENT BUNDLE

CHAPTER 22

INTRODUCTION

Positivity notions are numerous in algebraic geometry: a line bundle can be considered positive, e.g., if it is very ample, ample, strictly nef, nef, big, semiample, effective, pseudoeffective... Some of these notions relate: a very ample line bundle is ample, an ample line bundle is strictly nef and big, a strictly nef line bundle (i.e., a line bundle that has positive intersection with any curve) is nef, a nef line bundle and an effective line bundle are pseudoeffective. These positivity notions, as they tremendously matter in algebraic geometry, have been the subject of a lot of work, to which the books by Lazarsfeld [121, 122] are a great introduction. Proving new relationships between these various positivity notions is however a rather naive ambition, if not under strong additional assumptions.

From this perspective, the conjecture by Campana and Peternell [28] is surprising: they predict that, if X is a smooth projective variety, and the anticanonical bundle $-K_X$ is strictly nef, then $-K_X$ is ample, i.e., X is a Fano manifold. Their conjecture was in fact proven in dimension 2 and 3, by Maeda and Serrano [134, 181]. As all Fano manifolds are rationally connected [27, 110], an interesting update on the conjecture is the recent proof by Li, Ou and Yang [128, Theorem 1.2] that if X is a smooth projective variety, and the anticanonical bundle $-K_X$ is strictly nef, then X is rationally connected. Their proof uses important results on the Albanese map of varieties with nef anticanonical bundle. Such varieties have been extensively studied too [47, 212, 164, 45, 31, 30, 32].

Positivity notions extend to vector bundles [122, Definition 6.1.1] in the following fashion: a vector bundle E is stricly nef if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is strictly nef on $\mathbb{P}(E)$. Instead of asking about the positivity of the top exterior power of the tangent bundle, $-K_X = \bigwedge^{\dim(X)} T_X$, it makes sense to ask about the positivity of intermediate exterior powers $\bigwedge^r T_X$, for $1 \leq r \leq \dim(X) - 1$.

For $r = 1$, it is known since Mori [142] that projective spaces are the only smooth projective varieties with ample tangent bundle. They are also the only smooth projective varieties with strictly nef tangent bundle, by [128, Theorem 1.4]. Varieties with nef tangent bundle are, on the other hand, governed by another conjecture of Campana and Peternell [28] which has received a lot of attention: see the survey [146], and *inter alia* [28, 47, 200, 98, 97, 145, 209, 129, 46, 201, 99].

For $r = 2$, it has been proven that varieties with ample second exterior power of the tangent bundle are projective spaces and quadric hypersurfaces [35], varieties with strictly nef second exterior power of the tangent bundle alike.

Theorem 22.1. [128, Theorem 1.5] *Let X be a smooth projective variety of dimension $n \geq 2$, such that $\bigwedge^2 T_X$ is strictly nef. Then X is isomorphic to the projective space $\mathbb{P}^n$, or to a smooth quadric hypersurface Q^n.*

Partial results were obtained under the nef assumption [202, 177].

These results lead us to the following questions.

Question 1. Let X be a smooth projective variety of dimension n. Suppose that $\bigwedge^r T_X$ is strictly nef for some integer $1 \leq r \leq n$. Is X a Fano variety?

Question 2. Let X be a smooth projective variety of dimension n. Suppose that $\bigwedge^r T_X$ is nef for some integer $1 \leq r < n$, and that X is rationally connected. Is X a Fano variety?

Note that an affirmative answer to the second question would imply an affirmative answer to the first question, by [128, Theorem 1.2]. Also note that the second question is answered negatively for $r = n$, as there are smooth rationally connected threefolds with $-K_X$ nef but not semiample [208]. The first question is answered affirmatively for smooth toric varieties by [177]. In this paper, we answer the second question for $r = n - 1$.

Theorem 22.2. *Let X be a smooth projective variety of dimension $n \geq 2$ such that the vector bundle $\bigwedge^{n-1} T_X$ is nef and X is rationally connected. Then X is a Fano variety.*

This theorem is reminiscent of [47, Proposition 3.10], which states a dichotomy for varieties X with nef tangent bundle: either X is a Fano manifold, or $\chi(X, \mathcal{O}_X) = 0$. The proof similarly involves Chern classes inequalities and the Hirzebruch-Riemann-Roch formula. Note that, building on this theorem, [198, Proposition 1.4] very recently gave an affirmative answer to Question 2 in general.

Theorem 22.1 is based on the results of [34] and [44], which instead of the assumption on $\bigwedge^2 T_X$, feature a much weaker assumption on the length of rational curves. In a similar spirit, we provide the following partial characterizations and their corollaries.

Theorem 22.3. *Let X be a smooth projective rationally connected variety of dimension $n \geq 4$ such that for each rational curve C in X, we have $-K_X \cdot C \geq n - 1$. Then either $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$, or X is a Fano variety of Picard rank $\rho(X) = 1$.*

Corollary 22.4. *Let X be a smooth projective variety of dimension at least 4 such that the vector bundle $\bigwedge^3 T_X$ is strictly nef. Then either $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$, or X is a Fano variety of Picard rank $\rho(X) = 1$.*

This theorem and corollary are inspired by the author's Master thesis.

Let us briefly discuss the case when $\rho(X) = 1$. We know that, if X is a cubic or a complete intersection of two quadrics in $\mathbb{P}^n$, the vector bundle $\bigwedge^3 T_X$ is ample. These are two examples of del Pezzo manifolds, i.e. Fano n-folds of Picard rank 1 and of index $n - 1$. However, we do not know whether other del Pezzo manifolds have strictly nef $\bigwedge^3 T_X$, or whether varieties with strictly nef $\bigwedge^3 T_X$ are in general del Pezzo manifolds. We can hardly hope for a characterization of Fano manifolds of Picard rank one on which $-K_X \cdot C \geq n - 1$ for every rational curve C, and it is moreover not clear how to use the positivity of $\bigwedge^3 T_X$ beyond that length inequality, *cf.* Lemma 23.1.

Theorem 22.5. *Let X be a smooth projective rationally connected variety of dimension $n \geq 6$ such that for each rational curve C in X, we have $-K_X \cdot C \geq n - 2$. Then either X is isomorphic to $\mathbb{P}^3 \times \mathbb{P}^3$ or X is a Fano variety of Picard rank $\rho(X) = 1$.*

Studying the possibilities in dimension 5 by hand yields the following result.

Corollary 22.6. *Let X be a smooth projective variety of dimension at least 5 such that the vector bundle $\bigwedge^4 T_X$ is strictly nef. Then either X is isomorphic to one of the following Fano varieties*

$$\mathbb{P}^2 \times Q^3;\ \mathbb{P}^2 \times \mathbb{P}^3;\ \mathbb{P}(T_{\mathbb{P}^3});\ \mathrm{Bl}_\ell(\mathbb{P}^5) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1));\ \mathbb{P}^3 \times \mathbb{P}^3$$

or X is a Fano variety of Picard rank $\rho(X) = 1$.

These two corollaries were to our knowledge unknown even under the stronger, more classical assumption that $\bigwedge^3 T_X$ or $\bigwedge^4 T_X$ be ample. The proof of both theorems goes by classifying possible Mori contractions for X. A delicate point is that, while we know that our varieties X with $\rho(X) \geq 2$ admit one Mori contraction by the Cone Theorem, we need to construct by hand a second Mori contraction, e.g., to control higher-dimensional fibres in case of a first fibred Mori contraction. Depending on circumstances, we use unsplit covering families of deformations of rational curves, and a result by Bonavero, Casagrande and Druel [18], or, if X has the right dimension, Theorem 22.2, to produce this second Mori contraction.

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CHAPTER 23

A FIRST LEMMA

We start with a simple lemma.

Lemma 23.1. *Let X be a smooth projective variety of dimension n, and let $1 \leq r \leq n-1$. The following results hold:*

(i) *If $\bigwedge^r T_X$ is strictly nef, then any rational curve C in X satisfies*

$$-K_X \cdot C \geq n + 2 - r.$$

(ii) *If $\bigwedge^r T_X$ is nef, then any rational curve C in X satisfies $-K_X \cdot C \geq 2$.*

Proof. The proof goes as [128, Proof of Theorem 1.5]. Let $f : \mathbb{P}^1 \to C$ be the normalization of the curve. Write

$$f^* T_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),$$

with $(a_i)_{1 \leq i \leq n}$ ordered increasingly. It holds $a_n \geq 2$, as $T_{\mathbb{P}^1}$ maps non-trivially to $f^* T_X$, and we have $a_1 + \ldots + a_r \geq 0$ because $\mathcal{O}_{\mathbb{P}^1}(a_1 + \ldots + a_r)$ is a direct summand of the nef vector bundle $\bigwedge^r f^* T_X$. Moreover, if $\bigwedge^r T_X$ is strictly nef, the inequality is strict, and in particular, $a_{r+1} \geq a_r \geq 1$. Hence, if $\bigwedge^r T_X$ is strictly nef, we obtain

$$-K_X \cdot C = \deg f^*(-K_X) = a_1 + \ldots + a_n \geq 1 + n - r - 1 + 2 = n + 2 - r,$$

whereas if it is merely nef, we similarly have $-K_X \cdot C \geq 2$. □

This result is all the more valuable as, by [128, Theorem 1.2], if X is a smooth projective variety of dimension n such that $\bigwedge^r T_X$ is strictly nef, then it is rationally connected, in particular, it contains numerous rational curves.

CHAPTER 24

RESULTS ON $\bigwedge^{n-1} T_X$

The following lemma is the main step in the proof of Theorem 22.2.

Lemma 24.1. *Let X be a projective n-dimensional manifold such that $\bigwedge^{n-1} T_X$ is nef and X is rationally connected. Then $-K_X$ is nef and big.*

Proof. By [122, Theorem 6.2.12(iv)], the anticanonical bundle $-K_X$ is nef. By the Hirzebruch-Riemann-Roch formula, there is a homogeneous polynomial P of degree n in $\mathbb{Q}[X_1, \ldots, X_n]$ with grading $\deg X_i = i$ such that

$$\chi(X, \mathcal{O}_X) = P(c_1(X), \ldots, c_n(X)).$$

Note that, as $\bigwedge^{n-1} T_X = \Omega_X^1 \otimes \mathcal{O}_X(-K_X)$, and by [63, Remark 3.2.3(b)], we have

$$c_i\left(\bigwedge^{n-1} T_X\right) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{i-j} c_j(X) c_1(-K_X)^{i-j}. \tag{$*$}$$

Let us show by induction that $c_i(X)$ is a rational polynomial in the $c_j(\bigwedge^{n-1} T_X)$, for $0 \leq j \leq i$. Indeed, $c_1(X) = \frac{1}{n} c_1(\bigwedge^{n-1} T_X)$. Assume now that for some i, for all $0 \leq j \leq i$, there is a polynomial $P_j \in \mathbb{Q}[X_1, \ldots, X_j]$ such that $c_j(X) = P_j(c_1(\bigwedge^{n-1} T_X), \ldots, c_j(\bigwedge^{n-1} T_X))$. Then, setting

$$P_{i+1}(X_1, \ldots, X_{i+1}) = (-1)^{i+1} X_{i+1} - \sum_{j=0}^{i} (-1)^{i+j+1} \binom{n-j}{i+1-j} P_j(X_1, \ldots, X_j)(P_1(X_1))^{i+1-j},$$

we have $c_{i+1}(X) = P_{i+1}(c_1(\bigwedge^{n-1} T_X), \ldots, c_{i+1}(\bigwedge^{n-1} T_X))$ by ($*$). This perpetuates the induction.

In particular, we have

$$\chi(X, \mathcal{O}_X) = P\left(P_1\left(c_1\left(\bigwedge^{n-1} T_X\right)\right), \ldots, P_n\left(c_1\left(\bigwedge^{n-1} T_X\right), \ldots, c_n\left(\bigwedge^{n-1} T_X\right)\right)\right),$$

which is a homogeneous polynomial of degree n in $c_1(\bigwedge^{n-1} T_X), \ldots, c_n(\bigwedge^{n-1} T_X)$.

Now, if we suppose that $-K_X$ is not big, then $c_1(\bigwedge^{n-1} T_X)$ is not big. Thus, [47, Corollary 2.7] implies $\chi(X, \mathcal{O}_X) = 0$. But on the other hand, X is rationally connected, so $\chi(X, \mathcal{O}_X) = 1$, contradiction. $\square$

Remark 24.2. If $n = 4$, we cannot write $c_3(X)$ as a polynomial in

$$c_1\left(\bigwedge^{n-2} T_X\right) = 3c_1(X),$$
$$c_2\left(\bigwedge^{n-2} T_X\right) = 3c_1(X)^2 + 2c_2(X),$$
$$c_3\left(\bigwedge^{n-2} T_X\right) = c_1(X)^3 + 4c_1(X)c_2(X),$$

these formulas coming from [90, 4.5.2].

Lemma 24.3. *Let X be a projective n-dimensional manifold such that $\bigwedge^{n-1} T_X$ is nef and X is rationally connected. Then $-K_X$ is ample.*

Proof of Theorem 22.2. By Lemma 24.1, $-K_X$ is nef and big. By the base-point-free theorem [42, Theorem 7.32], we can find an integer m such that $-mK_X$ is globally generated. Let $\varepsilon : X \to Z$ be the $|-mK_X|$-morphism.

Suppose that it is not finite. By [100, Theorem 2], any irreducible component E of the exceptional locus is covered by rational curves that are contracted by ε. Let C be one of them: we have $0 = -K_X \cdot C \geq 2$ by Lemma 23.1, contradiction. So $-K_X$ is ample. $\square$

CHAPTER 25

STUDYING MORI CONTRACTIONS

The strategy for proving Theorems 22.3 and 22.5 is to show that there are only few possible birational contractions for X. In the following, if R is an extremal ray of the Mori cone $\overline{NE}(X)$, its *length* denoted by $\ell(R)$ is defined to be the minimal value of $-K_X \cdot C$, for a rational curve C with class in R. A Mori contraction is said to be *of length* ℓ if it is a contraction of a ray R with $\ell(R) = \ell$.

25.1 Small contractions.

Lemma 25.1. *Let $r \in [\![1, 4]\!]$. Let X be a smooth projective variety of dimension at least $r+1$ such that $\bigwedge^r T_X$ is strictly nef. Then X has no small contraction.*

Proof. Let n be the dimension of X. Let $\varphi : X \to Y$ be a birational contraction, E be an irreducible component of the exceptional locus, F an irreducible component of the general fiber of $\varphi|_E$, and R the corresponding extremal ray. Applying Ionescu-Wiśnewski inequality [92, Theorem 0.4], [206, Theorem 1.1] together with Lemma 23.1 yields

$$\dim E + \dim F \geq n + \ell(R) - 1 \geq 2n + 1 - r.$$

Since $r \leq 4$, we have $\dim E \geq n - 1$, and thus φ is a divisorial contraction. □

25.2 Fibred Mori contractions.

We move on to studying fibred Mori contractions.

25.2.1 Generalities about fibred Mori contractions

We use families of deformations of rational curves (see Section 2.12) to prove the following proposition.

Proposition 25.2. *Let X be a smooth projective rationally connected variety of dimension n. Let $r \in [\![1, n-1]\!]$. Suppose that $-K_X \cdot C \geq n+2-r$ for any rational curve C in X. Suppose that there is a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$. Then the general fiber of π has dimension at most $r-1$.*

If equality holds, then there is a rational curve C in X, not contracted by π, whose family of deformations $\mathcal{V}$ is unsplit covering and satisfies $\dim \mathrm{Locus}(\mathcal{V}_x) = n + 1 - r$ for $x \in \mathrm{Locus}(\mathcal{V})$ general.

Proof of Proposition 25.2. Since X is rationally connected and $-K_X$ is Cartier, we dispose of a rational curve C such that $\pi(C) \neq \{\mathrm{pt}\}$ and $-K_X \cdot C \geq n + 2 - r \geq 3$ is minimal with this condition. Let $\mathcal{V}$ be the corresponding family of deformations. By Lemma 2.123, it is unsplit.

Fix $x \in \operatorname{Locus}(\mathcal{V})$ general. By [108, Proposition IV.2.6] and Lemma 23.1, we derive

$$\dim \operatorname{Locus}(\mathcal{V}) + \dim \operatorname{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + n - 1 \geq 2n + 1 - r.$$

So $\dim \operatorname{Locus}(\mathcal{V}_x) \geq n + 1 - r$.

Let d denote the dimension of the general fiber of π. Then, by Lemma 2.124,

$$d \leq n - \dim \operatorname{Locus}(\mathcal{V}_x) \leq r - 1.$$

As for the equality case, if $d = r - 1$, then $\dim \operatorname{Locus}(\mathcal{V}_x) = n - r + 1$, and so C is such a rational curve as we claimed existed in the equality case of the proposition. □

Proposition 25.2 has an important consequence.

Corollary 25.3. *Let X be a smooth projective rationally connected variety of dimension n such that, for some $r \in [\![1, n-1]\!]$, one has $-K_X \cdot C \geq n+2-r$ for any rational curve $C \subset X$. Suppose that there is a fibred contraction $\pi : X \to Y$ with $\dim Y > 0$. Then $n \leq 2r - 2$.*

If equality holds, then a general fiber of π has dimension $r - 1$, and there is a rational curve C in X, not contracted by π, whose family of deformations $\mathcal{V}$ is unsplit covering and satisfies $\dim \operatorname{Locus}(\mathcal{V}_x) = n + 1 - r$ for $x \in \operatorname{Locus}(\mathcal{V})$ general.

Proof. Let F be a general fiber of π. By Proposition 25.2, we have $r - 1 \geq \dim F$. Adding n to both sides and applying Ionescu-Wiśnewski inequality (with the exceptional locus $E = X$ of dimension n), it holds

$$n + r - 1 \geq n + \dim F \geq n + \ell(R) - 1 \geq 2n + 1 - r.$$

If there is an equality, then $\dim F = r - 1$, and so we are in the equality case of Proposition 25.2. □

25.2.2 Fibred Mori contractions for certain varieties of even dimension

The set-up for this paragraph is the following. Let r be 3 or 4. Let X be a smooth projective rationally connected variety of dimension $2r - 2$ such that $-K_X \cdot C \geq r$ for any rational curve $C \subset X$. Suppose that there is a fibred contraction $\pi : X \to Y$ with $\dim Y > 0$. Let us classify what happens.

Lemma 25.4. *Let r be 3 or 4. Let X be a smooth projective rationally connected variety of dimension $2r - 2$ such that $-K_X \cdot C \geq r$ for any rational curve $C \subset X$. Suppose that there is a fibred contraction $\pi : X \to Y$ with $\dim Y > 0$. Then there is another equidimensional fibred Mori contraction $\varphi : X \to Z$ with $\dim Z = r - 1$.*

Proof. We are in the case of equality of Corollary 25.3. In particular, the general fiber F of π has dimension $r - 1$, and there is a rational curve C in X that is not contracted by π whose family of deformations $\mathcal{V}$ is unsplit covering and satisfies $\dim \operatorname{Locus}(\mathcal{V}_x) = r - 1 \geq (2r - 2) - 3 = \dim X - 3$.

By [18, Theorem 2, Proposition 1(i)], there is a fibred Mori contraction $\varphi : X \to Z$ whose fibers exactly are the $\mathcal{V}$-equivalence classes, and its general fiber has dimension $\dim \mathrm{Locus}(\mathcal{V}_x) = r-1$.

Let G be a fiber of φ. We claim that $\pi|_G$ is finite. Indeed, if it is not, then there is a curve $B \subset G$ that is contracted by π. The curve B lies in a $\mathcal{V}$-equivalence class, so by [18, Remark 1], as $\mathcal{V}$ is unsplit, B is numerically equivalent to a multiple of C, so it cannot be contracted by π, contradiction! So $\pi|_G$ is finite onto its image, which is contained in Y, so $\dim G \leq \dim Y = r-1$.

So φ is indeed equidimensional. □

Proposition 25.5. *Let $r \geq 3$ be an integer. Let X be a smooth projective rationally connected variety of dimension $2r-2$ such that $-K_X \cdot C \geq r$ for any rational curve $C \subset X$. Suppose that there is an equidimensional fibred Mori contraction $\pi : X \to Y$ with $\dim Y = r-1$. Then $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.*

This proposition relies on the following lemma.

Definition 25.6. Let $\pi : X \to Y$ be a fibration whose general fiber is a projective space. Let $f : \mathbb{P}^1 \to C \subset Y$ be a rational curve whose image lies in the smooth locus of π. The fiber product π_C of π by f is the projectivization of a bundle $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_k)$, with the (a_i) ordered increasingly. A *minimal section* over C is the section $s : \mathbb{P}^1 \to X$ of π_C corresponding to a quotient $\mathcal{O}_{\mathbb{P}^1}(a_1)$.

Remark 25.7. There may be several minimal sections as soon as $a_1 = a_2$.

Lemma 25.8. *Let X be a smooth projective variety with a fibration $\pi : X \to Y$ whose general fiber is a projective space. Then for any rational curve $f : \mathbb{P}^1 \to C \subset Y^0 \subset Y$ in the smooth locus of π, for any minimal section s of it, it holds $-K_Y \cdot C \geq -K_X \cdot s(\mathbb{P}^1)$. In particular,*

$$-K_Y \cdot C \geq \min\{-K_X \cdot C' \mid C' \textit{ is a rational curve in } X\}. \qquad (**)$$

If there is an equality in (**), then the base change of π by f is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}{}^{\oplus k}) \to \mathbb{P}^1$.

If there is almost an equality, i.e.,

$$-K_Y \cdot C = \min\{-K_X \cdot C' \mid C' \textit{ is a rational curve in } X\} + 1,$$

then the base change of π by f is isomorphic to to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}{}^{\oplus k}) \to \mathbb{P}^1$ or to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}{}^{\oplus k-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$.

Proof. By Tsen's theorem, the base change π_C of π by f is the natural projection morphism of a projectivized vector bundle V on $\mathbb{P}^1$. We write $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_k)$, with (a_i) ordered increasingly, and consider s the section of π_C satisfying $s^*\mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{O}_{\mathbb{P}^1}(a_1)$. The degree of $\det(s^*\mathcal{O}_{\mathbb{P}(V)}(1)) \otimes V^*$ is non-positive, equals zero if and only if $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k}$, and equals one if and only if $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k-1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1+1)$.

Pulling-back the Euler exact sequence of π_C by s, we get

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to s^*\mathcal{O}_{\mathbb{P}(V)}(1) \otimes V^* \to s^*T_{X/Y} \to 0.$$

Thus, $s^*T_{X/Y}$ has non-positive degree. We also have the tangent bundle exact sequence:

$$0 \to s^*T_{X/Y} \to s^*T_X \to f^*T_Y \to 0,$$

Since $s^*T_{X/Y}$ has non-positive degree, we obtain

$$-K_Y \cdot C \geq -K_X \cdot s(C) \geq \min\{-K_X \cdot C' \mid C' \text{ is a rational curve in } X\}.$$

Moreover, if there is an equality, then we have $-K_Y \cdot C = -K_X \cdot s(C)$, and so $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k}$.

If there is almost an equality, then $-K_Y \cdot C = -K_X \cdot s(C)$ or $-K_Y \cdot C = -K_X \cdot s(C) + 1$, so $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k}$ or $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k-1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1+1)$. □

Proof of Proposition 25.5. By [84, Theorem 1.3], as $\pi : X \to Y$ is an equidimensional fibration with fibres of dimension $r-1$, and as it is a Mori contraction of length at least r as well, it is a $\mathbb{P}^{r-1}$-bundle. Let us show that Y is isomorphic to $\mathbb{P}^{r-1}$. Since X is smooth and a projective bundle over Y, the variety Y is smooth. By Lemma 25.8, any rational curve C in Y satisfies $-K_Y \cdot C \geq r$. Moreover, X is rationally connected, so Y is too. By [34, Cor.0.4, $1 \Leftrightarrow 10$], we get $Y \simeq \mathbb{P}^{r-1}$.

As $\mathbb{P}^{r-1}$ has trivial Brauer group, there is a vector bundle V of rank r on Y such that π identifies with the natural projection $\mathbb{P}(V) \to \mathbb{P}^{r-1}$. Without loss of generality, we can twist V by a line bundle so that $\deg_\Delta V|_\Delta \in [\![0, r-1]\!]$ for any line Δ in $\mathbb{P}^{r-1}$. Let Δ be a line in $\mathbb{P}^{r-1}$. Then $-K_{\mathbb{P}^{r-1}} \cdot \Delta = r$. By the equality case in Lemma 25.8, the restriction $V|_\Delta$ is isomorphic to $L^{\oplus r}$ for some line bundle L on Δ. Hence $\deg L = 0$, so $L = \mathcal{O}_\Delta$. By [161, Theorem 3.2.1], the vector bundle V is globally trivial. Hence, $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$. □

25.2.3 Fibred Mori contractions for certain fivefolds

The goal in this section is prove the following result.

Proposition 25.9. *Let X be a smooth projective fivefold such that $\bigwedge^4 T_X$ is strictly nef. Suppose that X admits a fibred Mori contraction. Then X is isomorphic to one of the following projective manifolds*

$$\mathbb{P}^2 \times Q^3;\ \mathbb{P}^2 \times \mathbb{P}^3;\ \mathbb{P}(T_{\mathbb{P}^3});\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)).$$

We first establish this classification under the simplifying assumption that X has a $\mathbb{P}^2$-bundle structure, instead of a fibred Mori contraction.

Lemma 25.10. *Let X be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that $p : X \to Y$ is a $\mathbb{P}^2$-bundle. Then Y is a smooth projective variety, and X is isomorphic to one of the following projective manifolds*

$$\mathbb{P}^2 \times Q^3;\ \mathbb{P}^2 \times \mathbb{P}^3;\ \mathbb{P}(T_{\mathbb{P}^3});\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)).$$

Among other things, the proof uses the following lemma.

Lemma 25.11. *Let V be a vector bundle on a smooth quadric hypersurface Q^n. If V is trivial on all lines in Q^n, then V is trivial.*

Proof. Note that by [59, Theorem 7], it is enough to show that for any $x, z \in Q^n$, there exists a point $y \in Q^n$ such that the lines (xy) and (yz) belong to Q^n. Intersecting with $n-2$ hyperplanes, we can reduce to $n = 2$, in which case $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is covered by two family of lines corresponding to the two rulings. Hence, the point $y = (pr_1(x), pr_2(z))$ satisfies our requirement. □

Proof of Lemma 25.10. Since X is smooth and $X \to Y$ is a projective bundle, Y is smooth as well. Since X is rationally connected, Y is rationally connected and by Lemma 25.8, one has $-K_Y \cdot C \geq 3$ for any rational curve C in Y. By [44, Cor.1.4], Y is a quadric hypersurface Q^3 or the projective space $\mathbb{P}^3$. In either case, Y is rational and so it has trivial Brauer group. Hence, $X = \mathbb{P}(V)$ for some vector bundle V on Y.

If Y is a quadric, then all lines $\Delta \subset Y$ satisfy $-K_Y \cdot \Delta = 3$, and thus by the equality case in Lemma 25.8, $V|_\Delta \simeq L_\Delta{}^{\oplus 3}$ for some line bundle L on Δ. Fixing a line Δ_0, we have, as $\rho(Y) = 1$,

$$\deg L_\Delta \otimes L_{\Delta_0}{}^{-1} = \frac{1}{3}(\deg V|_\Delta \otimes V^*|_{\Delta_0}) = \frac{1}{3}(\det V \cdot \Delta - \det V \cdot \Delta_0) = 0,$$

so $(V \otimes L_{\Delta_0}{}^{-1})|_\Delta = \mathcal{O}_\Delta{}^{\oplus 3}$ for any line Δ in Y. By Lemma 25.11, this twist of V is globally trivial and thus $X \simeq \mathbb{P}^2 \times Q^3$.

Suppose now that Y is a projective space. By the almost-equality case in Lemma 25.8, for every line Δ in Y,

$$V|_\Delta \simeq \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(a_{i,\Delta}),$$

with either $a_{1,\Delta} = a_{2,\Delta} = a_{3,\Delta}$ or $a_{1,\Delta} = a_{2,\Delta} = a_{3,\Delta} - 1$. Note that the sum $a_{1,\Delta} + a_{2,\Delta} + a_{3,\Delta} = \det V \cdot \Delta$ is independent of the chosen line Δ. If it is divisible by 3, then we are in the first case, else it is congruent to 1 modulo 3 and we are in the second case. In both cases, the $a_{i,\Delta}$ are thus independent of the line Δ. Fixing a line Δ_0, the restricted twisted bundle $(V \otimes \mathcal{O}_{\mathbb{P}^1}(-a_{1,\Delta_0}))|_\Delta$ therefore is a uniform bundle of type $(0,0,0)$ or $(0,0,1)$. In the first case, this twist of V is globally trivial by [161], and so $X \simeq \mathbb{P}^2 \times \mathbb{P}^3$. In the second case, by [175], this twist of V is either $\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ or $T_{\mathbb{P}^3}(-1)$, which concludes the classification. □

Let us now study a more general fibred Mori contraction of X.

Lemma 25.12. *Let X be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$. Then $\dim Y \leq 3$.*

Proof. If $\dim(Y) = 4$, the general fiber of π is a smooth curve C with trivial normal bundle. By assumption,

$$2 = -K_X \cdot C = \deg_C(-K_C) \geq 3,$$

absurd. □

Let us cover the case when X has a fibred Mori contraction $\pi : X \to Y$ with $1 \leq \dim(Y) \leq 2$.

Lemma 25.13. *Let X be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $1 \leq \dim Y \leq 2$. Then there is a fibred Mori contraction $p : X \to Z$ that is a $\mathbb{P}^2$-bundle.*

Proof. We dispose of a rational curve C such that $\pi(C) \neq \{\mathrm{pt}\}$ and $-K_X \cdot C \geq 3$ is minimal with this condition. Let $\mathcal{V}$ be the corresponding family of deformations. By

Lemma 2.123, $\mathcal{V}$ is unsplit. Fix $x \in \operatorname{Locus}(\mathcal{V})$ general. By [108, Proposition IV.2.6] and by assumption, we derive

$$\dim \operatorname{Locus}(\mathcal{V}) + \dim \operatorname{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + 5 - 1 \geq 7.$$

So $\dim \operatorname{Locus}(\mathcal{V}_x) \geq 2$. By Lemma 2.124, $\dim \operatorname{Locus}(\mathcal{V}_x) \leq \dim Y \leq 2$.

As equality holds, $\mathcal{V}$ is a covering family of rational 1-cycles with $\dim \operatorname{Locus}(\mathcal{V}_x) = 2 \geq 5 - 3$, so by [18, Theorem 2, Proposition 1(i)], it admits a geometric quotient $p : X \to Z$, that is a fibred Mori contraction, with a general fiber of dimension 2. If a fiber F of p has dimension 3 or more, then since $\dim Y \leq 2$, $\pi|_F$ cannot be finite. So π contracts at least a curve B contained in F, which is numerically equivalent to a multiple of C as it lies in a $\mathcal{V}$-equivalence class [18, Remark 1], contradiction.

So p is an equidimensional fibred Mori contraction with fibres of dimension 2, of length $-K_X \cdot C \geq 3$. By [84, Theorem 1.3], the morphism p is a $\mathbb{P}^2$-bundle. □

We are left supposing that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim(Y) = 3$ that is not a $\mathbb{P}^2$-bundle. Let us first prove a few generalities about its fibres.

Lemma 25.14. *Let X be a smooth projective n-dimensional variety with a fibred Mori contraction π of length $n-k+1$ onto a variety Y of dimension k. Then the general fiber is isomorphic to $\mathbb{P}^{n-k}$.*

Proof. The general fiber is a smooth variety F of dimension $n-k$ such that $-K_F \cdot C \geq n-k+1$ for any rational curve C in F, and $-K_F$ is ample. By [34, 102], [84, Theorem 2.1], we obtain $F \simeq \mathbb{P}^{n-k}$. □

We recall and prove a fact mentioned in [84, 1.C].

Lemma 25.15. *Let X be a smooth projective variety of dimension $n \geq 4$ with a fibred Mori contraction π of length $n-2$ onto a threefold Y. Suppose that π is not equidimensional. Then for any irreducible component F of a fiber of π of dimension $n-2$, the normalization $\tilde{F}$ of F is isomorphic to $\mathbb{P}^{n-2}$.*

Proof. By [84, Theorem 1.3], and as $\operatorname{Univ}_{n-3}(X/Y) \to \operatorname{Chow}_{n-3}(X/Y)$ is a universal family for the $(n-3)$-cycles of X over Y, there is a commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "\eta'"'] \arrow[d, "\pi'"'] \arrow[rr, bend left, "\mu'"] & \overline{X} \arrow[r, "\varepsilon'"'] \arrow[d, "\overline{\pi}"'] & X \arrow[d, "\pi"] \\
Y' \arrow[r, "\eta"] \arrow[rr, bend right, "\mu"'] & \overline{Y} \arrow[r, "\varepsilon"] & Y
\end{tikzcd}$$

where $\overline{Y}$ is the normalization of the closure of the π-equidimensional locus of Y in $\operatorname{Chow}_{n-3}(X/Y)$, $\overline{X}$ is the normalization of the universal family over it, ε' is the evaluation map, Y' is a resolution of $\overline{Y}$, X' is the corresponding normalized fibred product, π' is a $\mathbb{P}^{n-3}$ bundle. Note that since Y is $\mathbb{Q}$-factorial, the exceptional loci of μ and of ε are unions of surfaces, hence the exceptional locus of μ' is a union of $\mathbb{P}^{n-3}$-bundles on surfaces.

Let F be an irreducible component of dimension $n-2$ of a fiber of π, let $\nu : \tilde{F} \to F$ be its normalization. Let $\Sigma \subset \overline{Y}$ be one of the surfaces that ε contracts onto $\pi(F)$, chosen such that $\Gamma := \overline{\pi}^{-1}(\Sigma)$ dominates F. Let S be the strict transform of Σ by η, and let $P := \pi'^{-1}(S)$: it is a $\mathbb{P}^{n-3}$-bundle over S and it dominates Γ. By the universal property of the normalization, we have a map $f : P \to \tilde{F}$, that fits into the following commutative diagram.

$$\begin{tikzcd}
 & & \tilde{F} \arrow[d, "\nu"] \\
P \arrow[urr, bend left, "f"] \arrow[rr, bend left, "\mu'"] \arrow[r, "\eta'"'] \arrow[d, "\pi'"'] & \Gamma \arrow[r, "\varepsilon'"'] \arrow[d, "\overline{\pi}"'] & F \arrow[d, "\pi"] \\
S \arrow[r, "\eta"] \arrow[rr, bend right, "\mu"'] & \Sigma \arrow[r, "\varepsilon"] & \{\mathrm{pt}\}
\end{tikzcd}$$

Let ℓ be a line contained in a fiber of $\pi'|_P$. Let $\mathcal{V}$ be the family of deformation of $f_*\ell$ in $\tilde{F}$.

Let us show that this family satisfies the hypotheses of [84, Theorem 2.1]. First, note that $\nu^*(-K_X|_F)$ is ample. Since there is a line in X' numerically equivalent to ℓ that is disjoint from all exceptional divisors of μ', and since ℓ is contracted by π',

$$\nu^*(-K_X|_F) \cdot f_*\ell = -K_X \cdot \mu'_*\ell = -K_{X'} \cdot \ell = -K_{X'/Y'} \cdot \ell = -K_{\mathbb{P}^{n-3}} \cdot \ell = n-2.$$

Since for any rational curve C in $\tilde{F}$, it holds $\nu^*(-K_X|_F) \cdot C \geq n-2$ by assumption, the family $\mathcal{V}$ is unsplit. Moreover, it is a covering family, as ν is birational, μ' is surjective and the family of deformations of ℓ is covering. Hence, by [108, Proposition IV.2.5], for a general point $x \in \tilde{F}$,

$$\dim \mathcal{V} = n - 2 + \dim \operatorname{Locus}(\mathcal{V}_x) + 1 - 3,$$

so we are left to show that $\dim \operatorname{Locus}(\mathcal{V}_x) = n-2$ to conclude.

Let us take x and y general in F. It suffices to show that the image by $\mu'|_P$ of a certain fiber $\mathbb{P}^{n-3}$ of $\pi'|_P$ contains both x and y, since then there is a line through any two points in $\mathbb{P}^{n-3}$.

Since x is general and Γ dominates F, it holds $\dim \varepsilon'^{-1}(x) = \dim \Gamma - \dim F = n-3+2-(n-2) = 1$, so there is a one-dimensional family of cycles passing through x, parametrized by a curve in Σ. As there is a finite map $\Sigma \to \mathrm{Chow}_{n-3}(F)$ (a composition of inclusions and a normalization), this is a non-trivial family of divisors. Hence, it must cover F, in particular there is one divisor passing through y and x. This divisor is dominated by a fiber of $\pi'|_P$, which concludes. $\square$

We now use the fact that π is not a $\mathbb{P}^2$-bundle (in fact, that π is not equidimensional) to construct covering families of rational curves on X. Before that, we prove a simple lemma.

Definition 25.16. Let $f : X \dashrightarrow Y$ be a rational map. We say that f is *almost holomorphic* if there is are Zariski open subsets $U \subset X$ and $V \subset Y$ such that $f|_U : U \to V$ is a proper holomorphic map.

Lemma 25.17. *Let $f : X \dashrightarrow Y$ be an almost holomorphic map. If Y is a curve, then f is holomorphic.*

Proof. Let $\varepsilon : X' \to X$ be a resolution of indeterminacies for f, let $f' : X' \to Y$ be the induced holomorphic map. As f is almost holomorphic, no component of the exceptional locus of ε is dominant onto Y. As Y is curve, this means that the exceptional locus of ε is sent onto finitely many points in Y. So f' factors through ε, i.e., f is holomorphic. $\square$

Lemma 25.18. *Let X be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y = 3$. If π is not a $\mathbb{P}^2$-bundle, then any rational curve $C \subset X$ such that $\pi(C) \neq \{\mathrm{pt}\}$, and which deforms in an unsplit family, deforms in a family covering X.*

Proof. Note that if π is equidimensional, by [84, Theorem 1.3] it is a $\mathbb{P}^2$-bundle. Hence, we assume that a variety F of dimension 3 is contained in a fiber of π. By contradiction, we consider a rational curve $C \subset X$ such that $\pi(C) \neq \{\mathrm{pt}\}$, and the family $\mathcal{V}$ of deformations of C is unsplit and not covering X.

Fix $x \in \mathrm{Locus}(\mathcal{V})$ general. By Lemma 2.124, $\dim \mathrm{Locus}(\mathcal{V}_x) \leq \dim Y \leq 3$. Since the family $\mathcal{V}$ is unsplit,

$$\dim \mathrm{Locus}(\mathcal{V}) + \dim \mathrm{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + 5 - 1 \geq 7,$$

in particular as $\mathcal{V}$ is not covering, $\dim \mathrm{Locus}(\mathcal{V}) = 4$ and $\dim \mathrm{Locus}(\mathcal{V}_x) = 3$.

Let $n : \tilde{D} \to D$ denote the normalization of $D = \mathrm{Locus}(\mathcal{V})$, and let $\tilde{\mathcal{V}}$ be the covering family on $\tilde{D}$. Note that π induces a fibration of $\tilde{D}$ onto a variety of smaller dimension that is not a point, in particular $\rho(\tilde{D}) \geq 2$. Thus, by [3, Corollary 4.4], $\tilde{D}$ cannot be $\tilde{\mathcal{V}}$-chain-connected.

Considering the dominant almost holomorphic map $r : \tilde{D} \dashrightarrow Z$ whose general fiber is a $\tilde{\mathcal{V}}$-equivalence class [18, Section 2], the variety Z is thus not a point. Since $\dim \mathrm{Locus}(\tilde{\mathcal{V}}_x) = 3$ for a general $x \in \mathrm{Locus}(\tilde{\mathcal{V}})$, the variety Z must be a curve, in particular, by Lemma 25.17, the map r is holomorphic.

Note that, as D is a relatively ample Cartier divisor with respect to π, it intersects the three-dimensional variety F along a surface S. Since $\dim n^{-1}(S) = 2 > \dim Z = 1$, the restriction $r|_{n^{-1}(S)} : n^{-1}(S) \to Z$ cannot be finite. So it contracts a curve B. Its image $n(B)$ is in a $\mathcal{V}$-equivalence class, so as $\mathcal{V}$ is unsplit, it is numerically equivalent to a multiple of C. But $n(B) \subset F$, so this curve is contracted by π, contradiction. $\square$

Definition 25.19. Let $f : X \to Y$ be a finite surjective map. We say that f is quasiétale if it is étale in codimension 1.

Remark 25.20. Note that if $f : X \to Y$ is quasiétale and Y is smooth, then by Zariski purity of the branch locus, f is étale.

Lemma 25.21. *Let X be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$. If X is not a $\mathbb{P}^2$-bundle over any smooth projective base, then $Y \simeq \mathbb{P}^3$. Moreover, $\rho(X) = 2$, and if C is a line in the smooth locus $Y^0 \subset Y$ of π and s a minimal section over C in X, the class of $s(\mathbb{P}^1)$ generates the other extremal ray in $\overline{NE}(X)$, induces a fibred Mori contraction to a positive dimensional variety too, and satisfies $-K_X \cdot s(\mathbb{P}^1) = 3$.*

Proof. Note that $\dim(Y) = 3$, by Lemmas 25.12, 25.13. By [56], let C be a minimal free rational curve in the smooth locus $Y^0 \subset Y$ of π. Let s be a minimal section over C. Lemma 25.8 yields

$$4 \geq -K_Y \cdot C \geq -K_X \cdot s(\mathbb{P}^1).$$

The family $\mathcal{V}$ of deformations of $s(\mathbb{P}^1)$ is unsplit. Indeed, suppose by contradiction that it is splitting, i.e. that there is a cycle

$$\sum_i a_i C_i \underset{\text{num}}{\equiv} s(\mathbb{P}^1),$$

with C_i rational curves, $a_i \geq 1$ integers, and $\sum_i a_i \geq 2$. Then, intersecting with $-K_X$ yields $4 \geq -K_X \cdot s(\mathbb{P}^1) \geq 6$, contradiction.

By Lemma 25.18, $\mathcal{V}$ therefore is a covering family. By [108, Proposition IV.2.6], it moreover holds

$$\dim \operatorname{Locus}(\mathcal{V}_x) \geq -K_X \cdot s(\mathbb{P}^1) - 1 \geq 2 = 5 - 3,$$

so by [18, Theorem 2, Proposition 1(i)], there is a geometric quotient $p : X \to Z$, that is a fibred Mori contraction, with general fiber of dimension at least $-K_X \cdot s(\mathbb{P}^1) - 1$. By Lemma 25.12, we have $\dim Z \leq 3$ and by Lemma 25.13, we have $\dim(Z) = 3$, or X is a $\mathbb{P}^2$-bundle over some three-dimensional base. So $\dim Z = 3$, hence $-K_X \cdot s(\mathbb{P}^1) = 3$. It also follows that $s(\mathbb{P}^1)$ is an extremal class in the Mori cone, as wished.

Again, X not being a $\mathbb{P}^2$-bundle over any smooth base, p is not equidimensional by [84, Theorem 1.3], so a variety F of dimension 3 is contained in a fiber of p. By Lemma 25.15, the normalization $n : \tilde{F} \to F$ satisfies $\tilde{F} \simeq \mathbb{P}^3$.

Since π and p are distinct Mori contractions, they contract no common numerical class of curve, in particular $\pi|_F : F \to Y$ is finite onto its image, hence finite surjective for dimensional reasons. There is an effective ramification divisor $R \in \operatorname{Pic}(\mathbb{P}^3)$ such that $-K_{\mathbb{P}^3} = n^*\pi|_F{}^*(-K_Y) - R$. As F is an irreducible component of a $\mathcal{V}$-equivalence class, and as $\mathcal{V}$ is unsplit, F contains a deformation of $s(\mathbb{P}^1)$. Let $\tilde{C}$ be the lift to $\tilde{F}$ of a deformation of $s(\mathbb{P}^1)$ that is contained in F. Then $-K_{\mathbb{P}^3} \cdot \tilde{C} \geq 4$, and $n^*\pi|_F{}^*(-K_Y) \cdot \tilde{C} = -K_Y \cdot C \leq 4$. So $R \cdot \tilde{C} \leq 0$, but $R \in \operatorname{Pic}(\mathbb{P}^3)$ is effective, thus ample or trivial, so R is trivial. The finite map $\pi|_F \circ n : \mathbb{P}^3 \to Y$ is thus quasiétale. So, its base change $\mathbb{P}^3 \underset{Y}{\times} X \to X$ is also quasiétale, as $\pi : X \to Y$ contracts no divisor. But X is rationally connected, hence simply-connected, and smooth, so $\mathbb{P}^3 \underset{Y}{\times} X \to X$ is an isomorphism. Hence $\pi|_F \circ n : \mathbb{P}^3 \to Y$ is an isomorphism too.

Since $\rho(Y) = 1$, we have $\rho(X) = 2$. Since $Y \simeq \mathbb{P}^3$ and $4 \geq -K_Y \cdot C$, the curve C is a line. □

Lemma 25.22. *Let X be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim(Y) > 0$. If X is not a $\mathbb{P}^2$-bundle over any smooth projective base, then $\rho(X) = 2$ and X has two distinct fibred Mori contractions onto $\mathbb{P}^3$, with corresponding extremal rays generated by the minimal sections $s(\mathbb{P}^1), \sigma(\mathbb{P}^1)$ above lines that lie in each $\mathbb{P}^3$ in the smooth locus of the fibration. Moreover,*

$$-K_X \cdot s(\mathbb{P}^1) = -K_X \cdot \sigma(\mathbb{P}^1) = 3.$$

Proof. Apply Lemma 25.21 twice. □

Proof of Proposition 25.9. If X has a $\mathbb{P}^2$-bundle structure, then Lemma 25.10 concludes. Suppose that X is not a $\mathbb{P}^2$-bundle. By Lemma 25.22, X admits exactly two fibred Mori contractions π and p, both onto $\mathbb{P}^3$. Given the intersection number of $-K_X$ with both extremal rays, and as $\pi_* s(\mathbb{P}^1)$ is a line in $\mathbb{P}^3$ and as $p_* s(\mathbb{P}^1) = 0$, we have

$$-K_X \cdot s(\mathbb{P}^1) = 3 = \pi^* \mathcal{O}_{\mathbb{P}^3}(3) \cdot s(\mathbb{P}^1) = (\pi^* \mathcal{O}_{\mathbb{P}^3}(3) \otimes p^* \mathcal{O}_{\mathbb{P}^3}(3)) \cdot s(\mathbb{P}^1),$$

and similarly

$$-K_X \cdot \sigma(\mathbb{P}^1) = (\pi^* \mathcal{O}_{\mathbb{P}^3}(3) \otimes p^* \mathcal{O}_{\mathbb{P}^3}(3)) \cdot \sigma(\mathbb{P}^1).$$

Hence, as $\rho(X) = 2$, and $s(\mathbb{P}^1)$ and $\sigma(\mathbb{P}^1)$ are independent,

$$\omega_X{}^* = \pi^* \mathcal{O}_{\mathbb{P}^3}(3) \otimes p^* \mathcal{O}_{\mathbb{P}^3}(3).$$

By Theorem 22.2, $-K_X$ is ample. So X is a Fano fivefold, and we just showed that it has index 3. By the classification in [207], X must then be a $\mathbb{P}^2$-bundle, which is a contradiction. □

25.3 Divisorial contractions. Let us classify divisorial Mori contraction of large length.

Proposition 25.23. *Let X be a smooth projective rationally connected variety of dimension n such that $-K_X \cdot C \geq 3$ for every rational curve C. Then X admits no divisorial Mori contraction of length greater or equal to $n-1$.*

Remark 25.24. In particular, the assumptions are fulfilled if there is $1 \leq r \leq n-1$ such that $\bigwedge^r T_X$ is strictly nef, by [128, Theorem 1.2] and Lemma 23.1.

The proof uses the following lemma, that excludes some special contractions of length $n-1$.

Lemma 25.25. *Let X be a smooth projective rationally connected variety of dimension n such that $-K_X \cdot C \geq 3$ for every rational curve C. Then there is no morphism $X \to Y$ that is a blow-up of a smooth point in a smooth variety.*

Proof of Lemma 25.25. By contradiction, consider such a smooth blow-up:

$$f : E \subset X \to p \in Y$$

Note that since X is rationally connected, so Y is too. Let C be a rational curve through p.

Since $-f^* K_Y = -K_X + (n-1)E$ and since no curve is contained in the blown-up locus p, the anticanonical divisor $-K_Y$ is stricly nef. By bend-and-break [42, Proposition 3.2] on the smooth variety Y, one can thus assume $-K_Y \cdot C \leq n+1$. The strict transform $C' \subset X$ of C satisfies $E \cdot C' > 0$. Since $K_X = f^* K_Y + (n-1)E$, we have

$$3 \leq -K_X \cdot C' \leq -K_Y \cdot C - (n-1) \leq 2,$$

contradiction! □

Proof of Proposition 25.23. By Ionescu-Wiśnewski inequality, if X admits a divisorial Mori contraction of length $\ell \geq n-1$, the exceptional divisor E and the general fiber $F \subset E$ satisfy:

$$\dim E + \dim F \geq n + \ell - 1 \geq 2n - 2,$$

i.e., $\ell = n-1$ and $E = F$ is contracted onto a point. So [4, Theorem 5.2] applies and shows that this divisorial Mori contraction of X correponds to a blow-up of a smooth point in a smooth variety, which contradicts Lemma 25.25. $\square$

We now consider divisorial Mori contractions of length $n-2$.

Proposition 25.26. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $-K_X \cdot C \geq n-2$ for any rational curve $C \subset X$. Then X has no divisorial Mori contraction contracting the exceptional divisor to a point.*

Remark 25.27. These assumptions are fulfilled if $\bigwedge^4 T_X$ is strictly nef, by [128, Theorem 1.2] and Lemma 23.1.

Proof. Assume that $\varepsilon : X \to Y$ is a divisorial Mori contraction contracting the exceptional divisor E to a point. Note that as X is rationally connected, there exists a rational curve C that intersects E without being contained in E. In particular, $E \cdot C > 0$. Among all such curves, let actually C be one such that $-K_X \cdot C$ is minimal. Then we claim that the family $\mathcal{V}$ of deformations of C is unsplit. Indeed, suppose by contradiction that it is splitting, i.e.,

$$C \underset{\text{num}}{\equiv} \sum_i a_i C_i,$$

with rational curves C_i and coefficients $a_i \geq 1$ such that $\sum a_i \geq 2$. Then $E \cdot C > 0$, so without loss of generality, $E \cdot C_1 > 0$. In particular, C_1 intersects E and is not contracted by ε, hence not contained in E. Since $-K_X$ has positive degree on all rational curves in X, we have $-K_X \cdot C_1 < -K_X \cdot C$, which contradicts the minimality of $-K_X \cdot C$.

By [108, Proposition IV.2.6.1], for a general $x \in \operatorname{Locus}(\mathcal{V})$,

$$\dim \operatorname{Locus}(\mathcal{V}) + \dim \operatorname{Locus}(\mathcal{V}_x) \geq n + n - 2 - 1.$$

In particular, $\dim \operatorname{Locus}(\mathcal{V}_x) \geq n-3$, and as X is smooth, E is Cartier, hence intersects $\operatorname{Locus}(\mathcal{V}_x)$ along a subscheme of dimension at least $n-4 \geq 1$. Let B be a curve in this intersection. It is contained in E, hence contracted by ε, hence satisfies $E \cdot B < 0$. On the other hand, it is contained in $\operatorname{Locus}(\mathcal{V}_x)$, hence is numerically equivalent to a multiple of C by [3, Lemma 4.1]. It has to be a positive multiple, as one sees when intersecting with any ample divisor. But $E \cdot C > 0$, contradiction. $\square$

Corollary 25.28. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $-K_X \cdot C \geq n-2$ for any rational curve $C \subset X$. Suppose that $\varepsilon : X \to Y$ is a divisorial Mori contraction. Then Y is smooth and ε is the blow-up of a smooth curve in Y.*

Proof. Recall [42, Proposition 6.10(b)] that the divisorial Mori contraction ε has a unique exceptional divisor E as its exceptional locus. By [111, Lemma 2.62], a ray $\mathbb{R}_+[C]$ associated to ε satisfies $E \cdot C < 0$, so such C has negative intersection with at least one effective divisor. Moreover, ε is a Mori contraction of length $n-2$. So [4, Theorem 5.3] applies, showing that ε either contracts a divisor to a point, or is a blow-up of a smooth curve in a smooth variety Y. By Proposition 25.26, only the latter can occur. □

Let us finally describe more precisely what happens in the occurrence of Corollary 25.28.

Lemma 25.29. *Let X be a smooth projective variety of dimension $n \geq 3$, that is rationally connected and such that for some $1 \leq r \leq n-1$, for any rational curve $C \subset X$, it holds $-K_X \cdot C \geq n+2-r$. If there is a morphism $\varepsilon : X \to Y$ that is a blow-up of a smooth curve in the smooth variety Y, then $r = n-1$.*

Proof. Consider such a smooth blow-up:

$$f : E \subset X \to \ell \subset Y$$

As X is rationally connected, so is Y. Fix H an ample divisor on Y. Let $C \subset Y$ be a rational curve other than ℓ passing through a point $p \in \ell$, with $H \cdot C$ minimal among the degrees of all rational curves intersecting ℓ other than ℓ. Fix another point $q \in C \setminus C \cap \ell$. By bend-and-break [42, Proposition 7.3], as Y is smooth, if $-K_Y \cdot C \geq n+2$, then there is a connected non-integral 1-cycle that is a deformation of C passing through p and q. In particular,

$$\sum_{i=1}^{k} a_i C_i \underset{\text{num}}{\equiv} C,$$

with rational curves C_i such that $p \in C_1$, $q \in C_{i_0}$ for some i_0, coefficients $a_i \geq 1$, and $\sum_{i=1}^{k} a_i \geq 2$. As $q \notin \ell$, we have that $C_{i_0} \neq \ell$, so either $C_1 \neq \ell$, or $C_1 = \ell$ and $k \geq 2$. Intersecting with H, we see that $H \cdot C_i < H \cdot C$ for all i, in particular for C_1. If $C_1 \neq \ell$, then $H \cdot C_1$ contradicts the minimality of $H \cdot C$. If $C_1 = \ell$, then $k \geq 2$ and by connectedness of the rational cycle, there is a curve $C_{i_1} \neq \ell$ that intersects $C_1 = \ell$. So $C_{i_1} \neq \ell$ intersects ℓ and contradicts the minimality, as $H \cdot C_{i_1} < H \cdot C$ again. So $-K_Y \cdot C \leq n+1$.

The strict transform $C' \subset X$ of C satisfies $E \cdot C' > 0$. Since $K_X = f^*K_Y + (n-2)E$, and by assumption,

$$n+2-r \leq -K_X \cdot C' \leq -K_Y \cdot C - (n-2) \leq 3,$$

so $r = n-1$. □

Proposition 25.30. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $\bigwedge^4 T_X$ is strictly nef. If there is a morphism $\varepsilon : X \to Y$ that is a blow-up of a smooth curve in the smooth variety Y, then X is a fivefold and there is a fibred Mori contraction $\pi : X \to Z$ with $\dim(Z) > 0$.*

Proof. By Lemma 25.29, we have $n = 5$. So by Theorem 22.2, $-K_X$ is ample. The Mori cone $NE(X)$ is closed, generated by finitely many classes of rational curves. Let E be the exceptional divisor of ε. Note that there exists an extremal ray $R = \mathbb{R}_+[C]$ of $NE(X)$ on which $E \cdot C > 0$. Indeed, if there were not such a ray, then E would be non-positive on all curves in X, which is absurd for an effective divisor. So, let $R = \mathbb{R}_+[C]$ be an extremal ray on which $E \cdot C > 0$.

Denote the associated Mori contraction by $\pi : X \to Z$. Since X already had a non-trivial Mori contraction ε, we have $\dim(Z) > 0$. Let us prove that π is a fibred Mori contraction.

By Lemma 25.1, π cannot be a small contraction. Assume by contradiction that it is a divisorial contraction. By Corollary 25.28, the variety Z is smooth and π is a blow-up along a smooth curve of Z. Let E' be the π-exceptional divisor. Let ℓ, respectively ℓ', be the image of E, respectively E', in Y, respectively Z. Let F' be a general fiber of $\pi|_{E'}$. It has dimension $n-2$. Note that F' and E intersect, since $E \cdot C > 0$. Hence, $E \cap F'$ is a subscheme of X of dimension at least $n-3$. Since ε and π are distinct Mori contractions, the restriction $\varepsilon|_{E\cap F'}$ must be finite onto its image, which is contained in ℓ. So $n-3 \leq 1$, contradiction!

So π is a fibred Mori contraction. □

Proposition 25.31. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $\bigwedge^4 T_X$ is strictly nef. If there is a morphism $\varepsilon : X \to Y$ that is a blow-up of a smooth curve, then $Y \simeq \mathbb{P}^5$ and ε is the blow-up of a line.*

Proof. By Proposition 25.30, X is a fivefold and admits a fibred Mori contraction onto a positive dimensional base. So Proposition 25.9 applies, showing that X belongs to a list of certain varieties of Picard number two. Only one of them has a divisorial Mori contraction, namely $\mathrm{Bl}_\ell(\mathbb{P}^5) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$. □

CHAPTER 26

RESULTS ON $\Lambda^3 T_X$ AND $\Lambda^4 T_X$

26.1 Proof of Theorem 22.3 and of Corollary 22.4, and examples.

Proof of Theorem 22.3. Note that $-K_X$ is nef, and non-trivial (as it is positive on rational curves, and X is rationally connected). If $\rho(X) = 1$, $-K_X$ is ample and X is thus a Fano variety. If $\rho(X) \geq 2$, by the Cone Theorem, X admits a Mori contraction, which by Lemma 25.1 and Proposition 25.23 is a fibred Mori contraction. Corollary 25.3 implies that X is a fourfold. By Lemma 25.4, X has an equidimensional fibred Mori contraction to a surface, so by Proposition 25.5, we have $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$. $\square$

Proof of Corollary 22.4. It is straightforward from Lemma 23.1, [128, Theorem 1.2], and Theorem 22.3. $\square$

Remark 26.1. It is easy to check that $\bigwedge^3 T_{\mathbb{P}^2 \times \mathbb{P}^2}$ is ample.

Example 26.2. Let X be a cubic in $\mathbb{P}^n$ with $n \geq 5$. From the tangent exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to \mathcal{O}_X(3) \to 0,$$

we can use [81, II.Ex.5.16(d)] to derive the existence of a surjection

$$0 \to F_4 \to \bigwedge^4 T_{\mathbb{P}^n}|_X \to \bigwedge^3 T_X \otimes \mathcal{O}_X(3) \to 0.$$

As $T_{\mathbb{P}^n}|_X \otimes \mathcal{O}_X(-1)$ is nef, the quotient of its fourth exterior power $\bigwedge^3 T_X \otimes \mathcal{O}_X(-1)$ is also nef, and thus $\bigwedge^3 T_X$ is ample.

Example 26.3. Let X be the complete intersection of two quadrics in $\mathbb{P}^n$ with $n \geq 6$. From the tangent exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to \mathcal{O}_X(2) \oplus \mathcal{O}_X(2) \to 0,$$

we can use [81, II.Ex.5.16(d)] to derive the existence of a surjection

$$0 \to F_4 \to \bigwedge^5 T_{\mathbb{P}^n}|_X \to \bigwedge^3 T_X \otimes \mathcal{O}_X(4) \to 0.$$

As $T_{\mathbb{P}^n}|_X \otimes \mathcal{O}_X(-1)$ is nef, the quotient of its fifth exterior power $\bigwedge^3 T_X \otimes \mathcal{O}_X(-1)$ is also nef, and thus $\bigwedge^3 T_X$ is ample.

26.2 Examples for Corollary 22.6.

Lemma 26.4. *Let X be the fivefold $\mathbb{P}(T_{\mathbb{P}^3})$. Then $\bigwedge^4 T_X$ is ample.*

Proof. Denote the natural projection by $p : X \to \mathbb{P}^3$, the tautological line bundle on X by $\mathcal{O}_X(1)$. By [81, II.Ex.5.16(d)], there is an exact sequence

$$0 \to \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3} \to \bigwedge^4 T_X \to T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3}) \to 0.$$

Let us prove that $E_1 = T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3})$ is ample. We have the relative Euler sequence

$$0 \to \mathcal{O}_X \to p^*\Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_X(1) \to T_{X/\mathbb{P}^3} \to 0.$$

The bundle E_1 is a quotient of $p^*\Omega^1_{\mathbb{P}^3}(4) \otimes \mathcal{O}_X(1)$. But as $T_{\mathbb{P}^3}$ is ample, $\mathcal{O}_X(1)$ is ample. Moreover, $\Omega^1_{\mathbb{P}^3}(4) \simeq \bigwedge^2 T_{\mathbb{P}^3}$ is ample too, which concludes by [122, 6.1.16].

Let us prove that $E_2 = \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3}$ is ample. This would settle the ampleness of $\bigwedge^4 T_X$ by [122, 6.1.13(ii)]. From [81, II.Ex.5.16(d)] and the relative Euler sequence, we derive

$$0 \to T_{X/\mathbb{P}^3} \to p^*T_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_X(2) \to \bigwedge^2 T_{X/\mathbb{P}^3} \to 0.$$

Since E_2 is a quotient of $p^*(T_{\mathbb{P}^3}(-4) \otimes \bigwedge^2 T_{\mathbb{P}^3}) \otimes \mathcal{O}_X(2)$, we are left proving that the latter is ample. Notice that $T_{\mathbb{P}^3}(-1)$ is globally generated and thus nef. So the bundle $T_{\mathbb{P}^3}(-3) \otimes \bigwedge^2 T_{\mathbb{P}^3} = T_{\mathbb{P}^3}(-1) \otimes \bigwedge^2 T_{\mathbb{P}^3}(-1)$ is nef as well. Finally, $\mathcal{O}_X(1)$ is ample, and we see that $\mathcal{O}_X(1) \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-1)$ is a quotient of $p^*T_{\mathbb{P}^3}(-1)$ (dualizing the relative Euler exact sequence and twisting by $\mathcal{O}_X(1)$), hence it is nef. We conclude by [122, 6.2.12(iv)]. $\square$

Lemma 26.5. *Let X be the fivefold $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$. Then $\bigwedge^4 T_X$ is ample.*

Remark 26.6. Note that $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$ is isomorphic to the blow-up of line in $\mathbb{P}^5$ [58, Section 9.3.2].

Proof. Denote the natural projection by $p : X \to \mathbb{P}^3$, the tautological line bundle on X by $\mathcal{O}_X(1)$. By [81, II.Ex.5.16(d)], there is an exact sequence

$$0 \to \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3} \to \bigwedge^4 T_X \to T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3}) \to 0.$$

Let us prove that $E_1 = T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3})$ is ample. We have the relative Euler sequence

$$0 \to \mathcal{O}_X \to p^*(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)) \otimes \mathcal{O}_X(1) \to T_{X/\mathbb{P}^3} \to 0.$$

The bundle E_1 is a quotient of $p^*(\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(4) \oplus \mathcal{O}_{\mathbb{P}^3}(4)) \otimes \mathcal{O}_X(1)$. Since $\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(4) \oplus \mathcal{O}_{\mathbb{P}^3}(4)$ is ample and $\mathcal{O}_X(1)$ is nef and p-ample, the bundle E_1 is thus ample.

Let us prove that $E_2 = \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3}$ is ample. From [81, II.Ex.5.16(d)] and the relative Euler sequence, we derive

$$0 \to T_{X/\mathbb{P}^3} \to p^*(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}) \otimes \mathcal{O}_X(2) \to \bigwedge^2 T_{X/\mathbb{P}^3} \to 0.$$

It is thus enough to prove that $p^* \bigwedge^2 T_{\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_X(2)$ is ample, which is clear since $\bigwedge^2 T_{\mathbb{P}^3}(-1) = (\bigwedge^2 T_{\mathbb{P}^3})(-2)$ is globally generated and thus nef, and since $p^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_X(2)$ is ample. $\square$

Remark 26.7. It is easy check to that $\bigwedge^4 T_{\mathbb{P}^2 \times \mathbb{P}^3}$, $\bigwedge^4 T_{\mathbb{P}^2 \times Q^3}$, $\bigwedge^4 T_{\mathbb{P}^3 \times \mathbb{P}^3}$ are ample.

26.3 Proof of Theorem 22.5 and Corollary 22.6.

Proof of Theorem 22.5. Note that $-K_X$ is nef, and non-trivial (as it is positive on rational curves, and X is rationally connected). If $\rho(X) = 1$, $-K_X$ is ample and X is thus a Fano variety. If $\rho(X) \geq 2$, by the Cone Theorem, X admits a Mori contraction. By Lemma 25.1, it cannot be a small contraction.

Suppose that it is a divisorial contraction. By Corollary 25.28 and Lemma 25.29, it is a smooth blow-up of a smooth curve in a fivefold, but we are assuming that X has dimension at least six, contradiction!

So X has no divisorial contraction. Thus, it has a fibred Mori contraction onto a positive dimensional variety. Corollary 25.3 implies that X is a fivefold or a sixfold. By assumption, X is thus a sixfold. By Lemma 25.4, X has an equidimensional fibred Mori contraction to a threefold, so by Proposition 25.5, we have $X \simeq \mathbb{P}^3 \times \mathbb{P}^3$, which concludes. □

Proof of Corollary 22.6. By Theorem 22.5, is is enough to consider the case when X is a fivefold. In particular, by Theorem 22.2, X is a Fano variety. Again, if $\rho(X) = 1$, there is nothing to prove.

If $\rho(X) \geq 2$, by the Cone Theorem, X admits a Mori contraction. By Lemma 25.1, it cannot be a small contraction.

Suppose that it is a divisorial contraction. By Corollary 25.28, it is a smooth blow-up of a smooth curve, and by Proposition 25.31, $X \simeq \mathrm{Bl}_\ell \mathbb{P}^5$.

Otherwise, it is a fibred Mori contraction onto a positive dimensional variety. Since X is a fivefold such that $\bigwedge^4 T_X$ is strictly nef, Proposition 25.9 applies and concludes. □

BIBLIOGRAPHY

[1] Amerik, E., and Verbitsky, M. Rational curves and MBM classes on hyperkähler manifolds: a survey. Preprint arXiv:2011.08727.

[2] Amerik, E., and Verbitsky, M. Morrison-Kawamata cone conjecture for hyperkähler manifolds. *Ann. Sci. Éc. Norm. Supér. (4) 50*, 4 (2017), 973–993.

[3] Andreatta, M., Chierici, E., and Occhetta, G. Generalized Mukai conjecture for special Fano varieties. *Central Europen Journal of Mathematics 2*, 2 (2009), 272–293.

[4] Andreatta, M., and Occhetta, G. Special rays in the Mori cone of a projective variety. *Nagoya Mathematical Journal 168* (2002), 127–137.

[5] Andreatta, M., and Wiśniewski, J. On the Kummer construction. *Rev. Mat. Complut. 23*, 1 (2010), 191–215.

[6] Auffarth, R., and Arteche, G. L. Smooth quotients of complex tori by finite groups. *Math. Z. 300*, 2 (2022), 1071–1091.

[7] Auffarth, R., Arteche, G. L., and Quezada, P. Smooth quotients of abelian surfaces by finite groups that fix the origin. *Cubo 24*, 1 (2022), 37–51.

[8] Bakker, B., Guénancia, H., and Lehn, C. Algebraic approximation and the decomposition theorem for Kähler Calabi-Yau varieties. Preprint arXiv:2012.00441.

[9] Balaji, V., and Kollár, J. Holonomy groups of stable vector bundles. *Publ. Res. Inst. Math. Sci. 44* (2008), 183–211.

[10] Beauville, A. *Some remarks on Kähler manifolds with* $c_1 = 0$, vol. 39. 1983, pp. 1–26.

[11] Beauville, A. Variétés kählériennes dont la première classe de Chern est nulle. *J. Differ. Geom. 18* (1984), 755–782.

[12] Bellamy, G. Counting resolutions of symplectic quotient singularities. *Compos. Math. 152*, 1 (2016), 99–114.

[13] Bellamy, G., Craw, A., Rayan, S., Schedler, T., and Weiss, H. All 81 crepant resolutions of a finite quotient singularity are hyperpolygon spaces. Preprint arXiv:2112.09878.

[14] Beltrametti, M., and Robbiano, L. Introduction to the theory of weighted projective spaces. *Expo. Math. 4* (1986), 111–162.

[15] Beltrametti, M. C., and Ionescu, P. A view on extending morphisms from ample divisors. In *Interactions of classical and numerical algebraic geometry*, vol. 496 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 71–110.

[16] Birkar, C., Di Cerbo, G., and Svaldi, R. Boundedness of elliptic Calabi-Yau varieties with a rational section. Preprint arXiv:2010.09769.

[17] Birkenhake, C., and Lange, H. *Complex abelian varieties*, second ed., vol. 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2004.

[18] Bonavero, L., Casagrande, C., and Druel, S. On covering and quasi-covering families of curves. *Journal of the European Mathmatical Society 9* (2007), 45–57.

[19] Borcea, C. Homogeneous vector bundles and families of Calabi–Yau threefolds. II. In *Several Complex Variables and Complex Geometry, Part II*, vol. 52 of *Proc. Symp. Pure Math.* American Mathematical Society, Providence, 1991, pp. 83–90. With an Appendix by Kollár, J.

[20] Borcea, C. Homogeneous vector bundles and families of Calabi–Yau threefolds. II. In *Several Complex Variables and Complex Geometry, Part II*, vol. 52 of *Proc. Symp. Pure Math.* American Mathematical Society, Providence, 1991, pp. 83–90. With an Appendix by Kollár, J.

[21] Borcea, C. On desingularized Horrocks-Mumford quintics. *J. Reine Angew. Math. 421* (1991), 23–41.

[22] Bost, J.-B. Algebraic leaves of algebraic foliations over number fields. *Publ. Math. Inst. Hautes Études Sci.*, 93 (2001), 161–221.

[23] Brion, M. Notes on automorphism groups of projective varieties. http://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos_final.pdf.

[24] Brozellino, J., and Brunsden, V. Elementary orbifold differential topology. *Topology Appl. 159* (2012), 3583–3589.

[25] Burek, D. Higher dimensional Calabi-Yau manifolds of Kummer type. *Math. Nachr. 293*, 4 (2020), 638–650.

[26] Campana, F. The Bogomolov-Beauville-Yau decomposition for klt projective varieties with trivial first Chern class -without tears-. Preprint arXiv:2004.08261.

[27] Campana, F. Connexité rationnelle des variétés de Fano. *Ann. Sci. École Norm. Sup. (4) 25*, 5 (1992), 539–545.

[28] Campana, F., and Peternell, T. Projective manifolds whose tangent bundles are numerically effective. *Mathematische Annalen 289*, 1 (1991), 1–33.

[29] Campana, F., and Păun, M. Foliations with positive slopes and birational stability of orbifold cotangent bundles. *Publ. Math. Inst. Hautes Études Sci. 129* (2019), 1–49.

[30] Cao, J. Albanese maps of projective manifolds with nef anticanonical bundles. *Ann. Sci. Éc. Norm. Supér. (4) 52*, 5 (2019), 1137–1154.

[31] Cao, J., and Höring, A. Manifolds with nef anticanonical bundle. *J. Reine Angew. Math. 724* (2017), 203–244.

[32] Cao, J., and Höring, A. A decomposition theorem for projective manifolds with nef anticanonical bundle. *J. Algebraic Geom. 28*, 3 (2019), 567–597.

[33] Chen, W., Di Cerbo, G., Han, J., Jiang, C., and Svaldi, R. Birational boundedness of rationally connected Calabi-Yau 3-folds. *Adv. Math. 378* (2021), Paper No. 107541, 32.

[34] Cho, K., Miyaoka, Y., and Shepherd-Barron, N. Characterizations of projective space and applications to complex symplectic manifolds. In *Higher dimensional birational geometry*, vol. 35 of *Higher dimensional birational geometry.* Advanced Studies in Pure Mathematics, Tokyo, 2002, pp. 1–88.

[35] Cho, K., and Sato, E. Smooth projective varieties with the ample vector bundle $\bigwedge^2 T_X$ in any characteristic. *J. Math. Kyoto Univ. 35*, 1 (1995), 1–33.

[36] Claudon, B., Graf, P., Guénancia, H., and Naumann, P. Kähler spaces with zero first Chern class: Bochner principle, fundamental groups, and the Kodaira problem. Preprint arXiv:2008.13008.

[37] Coskun, I., and Prendergast-Smith, A. Fano manifolds of index $n - 1$ and the cone conjecture. *Int. Math. Res. Not. IMRN*, 9 (2014), 2401–2439.

[38] Coskun, I., and Prendergast-Smith, A. Fano manifolds of index $n-2$ and the cone conjecture. *Math. Proc. Cambridge Philos. Soc. 166*, 1 (2019), 1–31.

[39] Cynk, S., and Hulek, K. Higher-dimensional modular Calabi-Yau manifolds. *Canad. Math. Bull. 50*, 4 (2007), 486–503.

[40] Cynk, S., and Schütt, M. Generalised Kummer constructions and Weil restrictions. *J. Number Theory 129*, 8 (2009), 1965–1975.

[41] Debarre, O. Hyperkähler manifolds. Preprint arXiv:1810.02087v2.

[42] Debarre, O. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[43] Debarre, O., Ein, L., Lazarsfeld, R., and Voisin, C. Pseudoeffective and nef classes on abelian varieties. *Compos. Math. 147*, 6 (2011), 1793–1818.

[44] Dedieu, T., and Höring, A. Numerical characterisation of quadrics. *Algebraic Geometry 4*, 1 (2017), 120–135.

[45] Demailly, J.-P. Structure theorems for compact Kähler manifolds with nef anticanonical bundles. In *Complex analysis and geometry*, vol. 144 of *Springer Proc. Math. Stat.* Springer, Tokyo, 2015, pp. 119–133.

[46] Demailly, J.-P. Fano manifolds with nef tangent bundles are weakly almost Kähler-Einstein. *Asian J. Math. 22*, 2 (2018), 285–290.

[47] Demailly, J.-P., Peternell, T., and Schneider, M. Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom. 3*, 2 (1994), 295–345.

[48] Di Cerbo, G., and Svaldi, R. Birational boundedness of low-dimensional elliptic Calabi-Yau varieties with a section. *Compos. Math. 157*, 8 (2021), 1766–1806.

[49] Dieudonné, J.-A., and Grothendieck, A. *Éléments de géométrie algébrique II. Deuxième partie.* Publ. math. IHES, 1965.

[50] Dieudonné, J.-A., and Grothendieck, A. *Éléments de géométrie algébrique IV. Troisi ème partie.* Publ. math. IHES, 1965.

[51] Dimca, A. Singularities and coverings of weighted complete intersections. *J. für reine und angewandte Math. 366* (1986), 184–193.

[52] Dinh, T.-C., Gachet, C., Lin, H.-Y., Oguiso, K., Wang, L., and Yu, X. Smooth projective surfaces with infinitely many real forms. Work in progress.

[53] Dolgachev, I. Weighted projective varieties. In *Groups actions and vector fields*, vol. LNM 956. Springer, 1982, pp. 34–71.

[54] Donten-Bury, M., and Wiśniewski, J. a. A. On 81 symplectic resolutions of a 4-dimensional quotient by a group of order 32. *Kyoto J. Math. 57*, 2 (2017), 395–434.

[55] Druel, S. A decomposition theorem for singular spaces with trivial canonical class of dimension at most five. *Invent. math. 211* (2018), 245–296.

[56] Druel, S., and Paris, M. Erratum to "characterizations of projective spaces and hyperquadrics". https://druel.perso.math.cnrs.fr/textes/epq2_erratum.pdf.

[57] Ein, L., Lazarsfeld, R., Mustaţă, M., Nakamaye, M., and Popa, M. Asymptotic invariants of base loci. *Ann. Inst. Fourier (Grenoble) 56*, 6 (2006), 1701–1734.

[58] Eisenbud, D., and Harris, J. *3264 and all that—a second course in algebraic geometry.* Cambridge University Press, Cambridge, 2016.

[59] Ermakova, S. Vector bundles of finite rank on complete intersections of finite codimension in ind-Grassmannians. *Complex Manifolds 2*, 1 (2015), 78–88.

[60] Filipazzi, S., Hacon, C. D., and Svaldi, R. Boundedness of elliptic Calabi-Yau threefolds. arXiv:2112.01352.

[61] Fryers, M. J. The movable fan of the Horrocks–Mumford quintic. arXiv:math/0102055.

[62] Fulger, M., and Lehmann, B. Positive cones of dual cycle classes. *Algebr. Geom. 4*, 1 (2017), 1–28.

[63] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 1998.

[64] Gachet, C. Finite quotients of an abelian variety with a Calabi-Yau resolution. Preprint arXiv:2201.00619.

[65] Gachet, C. Positivity of higher exterior powers of the tangent bundle. Preprint arXiv:2207.10854.

[66] Gachet, C. Positivity of the cotangent sheaf of singular Calabi-Yau varieties. Preprint arXiv:2009.10044, to appear in Math. Res. Let.

[67] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021.

[68] Grassi, A., and Morrison, D. R. Automorphisms and the Kähler cone of certain Calabi–Yau manifolds. *Duke Math. J. 71*, 3 (1993), 831–838.

[69] Grayson, D. R., and Stillman, M. E. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[70] Greb, D., Guénancia, H., and Kebekus, S. Klt varieties with trivial canonical class. Holonomy, differential forms and fundamental groups. *Geometry & Topology 23* (2019), 2051–2124.

[71] Greb, D., Kebekus, S., Kovács, S., and Peternell, T. Differential forms on log canonical spaces. *Publ. Math. Inst. Hautes Études Sci. 114* (2011), 87–169.

[72] Greb, D., Kebekus, S., and Peternell, T. Etale fundamental groups of Kawamata-log-terminal spaces, falt sheaves and quotients of abelian varieties. *Duke Math. J. 165* (2016), 1965–2004.

[73] Greb, D., Kebekus, S., and Peternell, T. Singular spaces with trivial canonical class. In *Minimal Models and Extremal Rays (Kyoto 2011)*, vol. 70. Adv. Stud. Pure Math., 2016, pp. 67–113.

[74] Greb, D., Kebekus, S., Peternell, T., and Taji, B. Harmonic metrics on Higgs sheaves and uniformization of varieties of general type. *Math. Ann. 378*, 3-4 (2020), 1061–1094.

[75] Greb, D., Kebekus, S., Peternell, T., and Taji, B. The Miyaoka-Yau inequality and uniformisation of canonical models. *Ann. Sci. éc. Norm. Supér. 52* (2020), 1487–1535.

[76] Greb, D., Kebekus, S., and T., P. Projective flatness over klt spaces and uniformisation of varieties with nef anticanonical bundle. Preprint arXiv:2006.08769.

[77] Greb, D., Kebekus, S., and Taji, B. Uniformisation of higher-dimensional minimal varieties. In *Algebraic geometry: Salt Lake City 2015*, vol. 97 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 2018, pp. 277–308.

[78] Grothendieck, A. La théorie des classes de Chern. *Bull. Soc. Math. France 86* (1958), 137–154.

[79] Guénancia, H., and Taji, B. Orbifold stability and Miyaoka-Yau inequality for minimal pairs. Preprint arXiv:1611.05981.

[80] Hartshorne, R. Ample subvarietes of algebraic varieties.

[81] Hartshorne, R. *Algebraic Geometry*, vol. 52 of *Graduate Texts in Mathematics.* Springer, 1977.

[82] Hartshorne, R. Stable reflexive sheaves. *Math. Ann. 254* (1980), 121–176.

[83] Hassett, B., Lin, H.-W., and Wang, C.-L. The weak Lefschetz principle is false for ample cones. *Asian J. Math. 6*, 1 (2002), 95–99.

[84] Höring, A., and Novelli, C. Mori contractions of maximal length. *Publ. Res. Inst. Math. Sci. 49*, 1 (2013), 215–228.

[85] Höring, A., and Peternell, T. A non–vanishing conjecture for cotangent bundles. Preprint arXiv:2006.05225.

[86] Höring, A., and Peternell, T. Algebraic integrability of foliations with numerically trivial canonical bundle. *Invent. math. 216* (2019), 395–419.

[87] Huppert, B. *Character theory of finite groups*, vol. 25 of *De Gruyter Expositions in Mathematics.* Walter de Gruyter & Co., Berlin, 1998.

[88] Huybrechts, D. *Fourier-Mukai transforms in algebraic geometry.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[89] Iano-Fletcher, A. R. *Working with weighted complete intersections.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2000, pp. 101–174.

[90] Iena, O. On symbolic compoutations with Chern classes: Remarks on the library `chern.lib` for `singular`. https://orbilu.uni.lu/bitstream/10993/21949/2/ChernLib.pdf.

[91] Im, B.-H., and Larsen, M. Rational curves on quotients of abelian varieties by finite groups. *Math. Res. Lett. 22*, 4 (2015), 1145–1157.

[92] Ionescu, P. Generalized adjunction and applications. *Mathematical Proceedings of the Cambridge Philosophical Society 99*, 3 (1986), 457–472.

[93] Isaacs, I. *Finite group theory*, vol. 92 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[94] Ito, Y., and Reid, M. The McKay correspondence for finite subgroups of $SL(3, \mathbf{C})$. In *Higher-dimensional complex varieties (Trento, 1994)*. de Gruyter, Berlin, 1996, pp. 221–240.

[95] Johnson, D. L. *Presentations of groups*, second ed., vol. 15 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[96] Kamenova, L. Survey of finiteness results for hyperkähler manifolds. In *Phenomenological approach to algebraic geometry*, vol. 116 of *Banach Center Publ.* Polish Acad. Sci. Inst. Math., Warsaw, 2018, pp. 77–86.

[97] Kanemitsu, A. Fano n–folds with nef tangent bundle and Picard number greater than $n-5$. *Math. Z. 284* (2016), 195–208.

[98] Kanemitsu, A. Fano 5–folds with nef tangent bundles. *Math. Res. Lett. 24*, 5 (2017), 1453–1475.

[99] Kanemitsu, A., and Watanabe, K. Projective varieties with nef tangent bundles in positive characteristics. arXiv:2012.09419.

[100] Kawamata, Y. On the length of an extremal rational curve. *Invent. Math. 105*, 3 (1991), 609–611.

[101] Kawamata, Y. On the cone of divisors of Calabi-Yau fiber spaces. *Internat. J. Math. 8*, 5 (1997), 665–687.

[102] Kebekus, S. Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron. In *Complex geometry (Göttingen, 2000)*. Springer, Berlin, 2002, pp. 147–155.

[103] Khinich, V. A. On the Gorenstein porperty of the ring of invariants of a Gorenstein ring. *Izv. Akad. Nauk SSSR Ser. Mat. 40*, 1 (1976), 50–56.

[104] Kollár, J. Higher direct images of dualizing sheaves. I. *Ann. of Math. (2) 123*, 1 (1986), 11–42.

[105] Kollár, J. Higher direct images of dualizing sheaves. II. *Ann. of Math. (2) 124*, 1 (1986), 171–202.

[106] Kollár, J. *Flips and abundance for algebraic threefolds – A summer seminar at the university of Utah*, vol. 211. Astérisque (Soc. Math. Française), 1992.

[107] Kollár, J. Shafarevich maps and plurigenera of algebraic varieties. *Invent. Math. 113*, 1 (1993), 177–215.

[108] Kollár, J. *Rational curves on algebraic varieties*, vol. 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1996.

[109] Kollár, J., and Larsen, M. Quotients of Calabi-Yau varieties. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 2009, pp. 179–211.

[110] Kollár, J., Miyaoka, Y., and Mori, S. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom. 36*, 3 (1992), 765–779.

[111] Kollár, J., and Mori, S. *Birational geometry of algebraic varieties*, vol. 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[112] Kollár, J., and Mori, S. *Birational geometry of algebraic varieties*, vol. 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[113] Kopper, J. The nef cone of the Hilbert scheme of points on rational elliptic surfaces and the cone conjecture. *Canad. Math. Bull. 64*, 1 (2021), 216–227.

[114] Kovács, S. J. The cone of curves of a $K3$ surface. *Math. Ann. 300*, 4 (1994), 681–691.

[115] Kreuzer, M., and Skarke, H. LG/cft: LG models, orbifolds, and weighted projective space (n=4,5). online dataset available at http://hep.itp.tuwien.ac.at/~kreuzer/CY/.

[116] Kreuzer, M., and Skarke, H. No mirror symmetry in Landau-Ginzburg spectra! *Nucl. Phys. B 338* (1992), 113–130.

[117] Kreuzer, M., and Skarke, H. Classification of reflexive polyhedra in three dimensions. *Adv. Theor. Math. Phys. 2*, 4 (1998), 853–871.

[118] Kreuzer, M., and Skarke, H. Complete classification of reflexive polyhedra in four dimensions. *Adv. Theor. Math. Phys. 4*, 6 (2000), 1209–1230.

[119] Kubota, K. Ample sheaves. *J. Fac. Sci. Univ. Tokyo 17* (1970), 421–430.

[120] Lazarsfeld, R. *Positivity in Algebraic Geometry, II*, vol. 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2003.

[121] Lazarsfeld, R. *Positivity in algebraic geometry. I*, vol. 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[122] Lazarsfeld, R. *Positivity in algebraic geometry. II*, vol. 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[123] Lazić, V. Around and beyond the canonical class. In *Birational geometry, rational curves, and arithmetic*, Simons Symp. Springer, Cham, 2013, pp. 171–203.

[124] Lazić, V., Oguiso, K., and Peternell, T. The Morrison-Kawamata cone conjecture and abundance on Ricci flat manifolds. In *Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds & Picard-Fuchs equations*, vol. 42 of *Adv. Lect. Math. (ALM)*. Int. Press, Somerville, MA, 2018, pp. 157–185.
[125] Lazić, V., Oguiso, K., and Peternell, T. The Morrison-Kawamata cone conjecture and abundance on Ricci flat manifolds. In *Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds & Picard-Fuchs equations*, vol. 42 of *Adv. Lect. Math. (ALM)*. Int. Press, Somerville, MA, 2018, pp. 157–185.
[126] Lazić, V., and Peternell, T. On the cone conjecture for Calabi-Yau manifolds with Picard number two. *Math. Res. Lett. 20*, 6 (2013), 1103–1113.

[127] Lehn, M., and Sorger, C. A symplectic resolution for the binary tetrahedral group. In *Geometric methods in representation theory. II*, vol. 24 of *Sémin. Congr.* Soc. Math. France, Paris, 2012, pp. 429–435.

[128] Li, D., Ou, W., and Yang, X. On projective varieties with strictly nef tangent bundles. *J. Math. Pures Appl. (9) 128* (2019), 140–151.

[129] Li, Q. Smooth projective horospherical varieties with nef tangent bundles. *Indiana Univ. Math. J. 66*, 6 (2017), 1879–1893.

[130] Li, Z., and Zhao, H. On the relative Morrison-Kawamata cone conjecture. arXiv:2206.13701.

[131] Liu, J., Ou, W., and Yang, X. Projective manifolds whose tangent bundle contains a strictly nef subsheaf. Preprint arXiv:2004.08507, to appear in Journal of Algebraic Geometry.

[132] Looijenga, E. Discrete automorphism groups of convex cones of finite type. *Compos. Math. 150*, 11 (2014), 1939–1962.

[133] Lu, S., and Taji, B. A characterization of finite quotients of abelian varieties. *Int. Math. Res. Not. IMRN*, 1 (2018), 292–319.

[134] Maeda, H. A criterion for a smooth surface to be Del Pezzo. *Math. Proc. Cambridge Philos. Soc. 113*, 1 (1993), 1–3.

[135] Markman, E., and Yoshioka, K. A proof of the Kawamata-Morrison cone conjecture for holomorphic symplectic varieties of $K3^{[n]}$ or generalized Kummer deformation type. *Int. Math. Res. Not. IMRN*, 24 (2015), 13563–13574.

[136] Masley, J., and Montgomery, H. Cyclotomic fields with unique factorization. *J. Reine Angew. Math. 286(287)* (1976), 248–256.

[137] Massarenti, A. Mori Dream Spaces, log Fano varieties and moduli spaces of rational curves. http://mcs.unife.it/alex.massarenti/files/mds.pdf.

[138] Megyesi, G. Generalisation of the Bogomolov-Miyaoka-Yau inequality to singular surfaces. *Proc. London Math. Soc. (3) 78*, 2 (1999), 241–282.

[139] Milnor, J. W., and Stasheff, J. D. *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.

[140] Miyaoka, Y. The Chern classes and Kodaira dimension of a minimal variety. In *Algebraic Geometry, Sendai 1985* (1987), vol. 10, Adv. Stud. Pure Math., pp. 449–476.

[141] Moraga, J. Bounding singular surfaces via Chern numbers. *Math. Z. 295*, 3-4 (2020), 1597–1614.

[142] Mori, S. Projective manifolds with ample tangent bundle. *Annals of Mathematics 110* (1979), 593–606.

[143] Morrison, D. R. Compactifications of moduli spaces inspired by mirror symmetry. In *Journées de géométrie algébrique d'Orsay - Juillet 1992*, no. 218 in Astérisque. Société mathématique de France, 1993.

[144] Morrison, D. R. Beyond the Kähler cone. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)* (1996), vol. 9 of *Israel Math. Conf. Proc.*, Bar-Ilan Univ., Ramat Gan, pp. 361–376.

[145] Muñoz, R., Occhetta, G., Solá Conde, L. E., and Watanabe, K. Rational curves, Dynkin diagrams and Fano manifolds with nef tangent bundle. *Math. Ann. 361*, 3 (2015), 583–609.

[146] Muñoz, R., Occhetta, G., Solá Conde, L. E., Watanabe, K., and Wiśniewski, J. A. A survey on the Campana-Peternell conjecture. *EUT Edizioni Università di Trieste 47* (2015), 127–185.

[147] Mumford, D. Towards an enumerative geometry of the moduli space of curves. *Progr. Math. 36* (1983), 271–326.

[148] Nakayama, N. *Zariski-Decomposition and Abundance*, vol. 14 of *MSJ Memoirs.* Mathematical Society of Japan, 2004.

[149] Namikawa, Y. A note on symplectic singularities. Preprint arXiv:math/0101028.

[150] Namikawa, Y. Periods of Enriques surfaces. *Math. Ann. 270*, 2 (1985), 201–222.

[151] Namikawa, Y. On the birational structure of certain Calabi-Yau threefolds. *J. Math. Kyoto Univ. 31*, 1 (1991), 151–164.

[152] Oguiso, K. On algebraic fiber space structures on a Calabi-Yau 3-fold. *Internat. J. Math. 4*, 3 (1993), 439–465. With an appendix by Noboru Nakayama.

[153] Oguiso, K. Calabi-Yau threefolds of quasi-product type. *Doc. Math. 1* (1996), No. 18, 417–447.

[154] Oguiso, K. On certain rigid fibered Calabi-Yau threefolds. *Math. Z. 221*, 3 (1996), 437–448.

[155] Oguiso, K. On the complete classification of Calabi-Yau threefolds of type III_0. In *Higher-dimensional complex varieties (Trento, 1994)*. de Gruyter, Berlin, 1996, pp. 329–339.

[156] Oguiso, K. Automorphism groups of Calabi-Yau manifolds of Picard number 2. *J. Algebraic Geom. 23*, 4 (2014), 775–795.

[157] Oguiso, K., and Peternell, T. Calabi-Yau threefolds with positive second Chern class. *Comm. Anal. Geom. 6*, 1 (1998), 153–172.

[158] Oguiso, K., and Peternell, T. Calabi-Yau threefolds with positive second Chern class. *Comm. Anal. Geom. 6*, 1 (1998), 153–172.

[159] Oguiso, K., and Sakurai, J. Calabi-Yau threefolds of quotient type. *Asian J. Math. 5*, 1 (2001), 43–77.

[160] Oguiso, K., and Sakurai, J. Calabi-Yau threefolds of quotient type. *Asian J. Math. 5*, 1 (2001), 43–77.

[161] Okonek, C., Schneider, M., and Spindler, H. *Vector Bundles on Complex Projective Spaces*. Progress in Mathematics. Birkhäuser, 1980.

[162] Ottem, J. C. Nef cycles on some hyperkähler fourfolds. In *Facets of algebraic geometry. Vol. II*, vol. 473 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 2022, pp. 228–237.

[163] Paranjape, K., and Ramakrishnan, D. Quotients of E^n by A_{n+1} and Calabi-Yau manifolds. In *Algebra and number theory*. Hindustan Book Agency, Delhi, 2005, pp. 90–98.

[164] Peternell, T., and Serrano, F. Threefolds with nef anticanonical bundles: Dedicated to the memory of fernando serrano. *Collect. Math. 49*, 2-3 (1998), 465–517.

[165] Pirola, G. P. Curves on generic Kummer varieties. *Duke Math. J. 59*, 3 (1989), 701–708.

[166] Prendergast-Smith, A. The cone conjecture for abelian varieties. *J. Math. Sci. Univ. Tokyo 19*, 2 (2012), 243–261.

[167] Prendergast-Smith, A. The cone conjecture for some rational elliptic threefolds. *Math. Z. 272*, 1-2 (2012), 589–605.

[168] Reid, M. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980, pp. 273–310.

[169] Reiner, I. *Maximal orders*. London Mathematical Society Monographs, No. 5. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1975.

[170] Robinson, D. *A course in the theory of groups*, second ed., vol. 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996.

[171] ROCKAFELLAR, R. T. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[172] ROTMAN, J. J. *An introduction to the theory of groups*, fourth ed., vol. 148 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[173] ROUSSEAU, E., AND TAJI, B. Orbifold Chern classes inequalities and applications. Preprint arXiv:1611.06420.

[174] SANO, T. Examples of non-Kähler Calabi-Yau manifolds with arbitrarily large b_2. *J. Topol. 14*, 4 (2021), 1448–1460.

[175] SATO, E. Uniform vector bundles on a projective space. *J. Math. Soc. Japan 28*, 1 (1976), 123–132.

[176] SATO, K., AND SATO, Y. Crepant property of Fujiki-Oka resolutions for Gorenstein abelian quotient singularities. Preprint arXiv:2004.03522.

[177] SCHMITZ, D. On exterior powers of the tangent bundle on toric varieties. arXiv:1811.02603.

[178] SCHOEN, C. Complex multiplication cycles on elliptic modular threefolds. *Duke Math. J. 53*, 3 (1986), 771–794.

[179] SCHOEN, C. On fiber products of rational elliptic surfaces with section. *Math. Z. 197*, 2 (1988), 177–199.

[180] SCHOEN, C. On fiber products of rational elliptic surfaces with section. *Math. Z. 197*, 2 (1988), 177–199.

[181] SERRANO, F. Strictly nef divisors and Fano threefolds. *Journal für die reine und angewandte Mathematik 464* (1995), 187–206.

[182] SHEPHERD-BARRON, N. I., AND WILSON, P. M. H. Singular threefolds with numerically trivial first and second Chern classes. *J. Algebraic Geom. 3*, 2 (1994), 265–281.

[183] SHIBATA, T. Q-abelian and $\mathbb{Q}$-fano finite quotients of abelian varieties. Preprint arXiv:2112.09826.

[184] SHIMURA, G., AND TANIYAMA, Y. *Complex multiplication of abelian varieties and its applications to number theory*, vol. 6 of *Publications of the Mathematical Society of Japan.* The Mathematical Society of Japan, Tokyo, 1961.

[185] STACKS PROJECT AUTHORS, T. Stacks Project. https://stacks.math.columbia.edu/.

[186] STERK, H. Finiteness results for algebraic $K3$ surfaces. *Math. Z. 189*, 4 (1985), 507–513.

[187] SUZUKI, F. Higher-dimensional Calabi-Yau varieties with dense sets of rational points. *Eur. J. Math. 8*, 1 (2022), 193–204.

[188] Suzuki, M. *Group theory. I*, vol. 247 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, 1982. Translated from the Japanese by the author.

[189] Szendrői, B. Some finiteness results for Calabi-Yau threefolds. *J. London Math. Soc. (2) 60*, 3 (1999), 689–699.

[190] Totaro, B. Hilbert's 14th problem over finite fields and a conjecture on the cone of curves. *Compos. Math. 144*, 5 (2008), 1176–1198.

[191] Totaro, B. The cone conjecture for Calabi-Yau pairs in dimension 2. *Duke Math. J. 154*, 2 (2010), 241–263.

[192] Totaro, B. The cone conjecture for Calabi-Yau pairs in dimension 2. *Duke Math. J. 154*, 2 (2010), 241–263.

[193] Uehara, H. Calabi-Yau threefolds with infinitely many divisorial contractions. *J. Math. Kyoto Univ. 44*, 1 (2004), 99–118.

[194] Uehara, H. Calabi-Yau threefolds with infinitely many divisorial contractions. *J. Math. Kyoto Univ. 44*, 1 (2004), 99–118.

[195] Van de Ven, A. On the embedding of abelian varieties in projective spaces. *Ann. Mat. Pura Appl. (4) 103* (1975), 127–129.

[196] Voisin, C. Chow ring and gonality of general abelian varieties. *Ann. H. Lebesgue 1* (2018), 313–332.

[197] Washington, L. *Introduction to cyclotomic fields*, second ed., vol. 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[198] Watanabe, K. Positivity of the exterior power of the tangent bundles. Preprint arXiv:2208.06735.

[199] Watanabe, K. Certain invariant subrings are Gorenstein. I, II. *Osaka Math. J. 11* (1974), 1–8; ibid. 11 (1974), 379–388.

[200] Watanabe, K. Fano 5–folds with nef tangent bundles and Picard numbers greater than one. *Math. Z. 276* (2014), 39–49.

[201] Watanabe, K. Fano manifolds of coindex three admitting nef tangent bundle. *Geom. Dedicata 210* (2021), 165–178.

[202] Watanabe, K. Positivity of the second exterior power of the tangent bundles. *Adv. Math. 385* (2021), Paper No. 107757, 27.

[203] Wilson, P. Boundedness questions for Calabi-Yau threefolds. Preprint arXiv:1706.0126v7, to appear in J. Alg. Geom.

[204] Wilson, P. Calabi-Yau threefolds with Picard number three. Preprint arXiv:2011.12876v3.

[205] Wilson, P. H. The Kähler cone on Calabi-Yau threefolds. *Invent. Math. 107*, 3 (1992), 561–583.

[206] Wiśniewski, J. On contractions of extremal rays of Fano manifolds. *Journal für die reine und angewandte Mathematik 417* (1991), 141–157.

[207] Wiśniewski, J. On Fano manifolds of large index. *Manuscripta Math. 70*, 2 (1991), 145–152.

[208] Xie, Z. Rationally connected threefolds with nef and bad anticanonical divisor. Preprint arXiv:2006.08234.

[209] Yang, Q. Toric Fano manifolds with nef tangent bundles. arXiv:1506.05565.

[210] Yau, S. T. Calabi's conjecture and some new results in algebraic geometry. *Proc. Nat. Acad. Sci. U.S.A. 74*, 5 (1977), 1798–1799.

[211] Zariski, O., and Samuel, P. *Commutative algebra. Vol. II.* Springer-Verlag, New York-Heidelberg, 1975. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29.

[212] Zhang, Q. On projective manifolds with nef anticanonical bundles. *J. Reine Angew. Math. 478* (1996), 57–60.