WELL-CLIPPED CONES BEHAVE THEMSELVES UNDER ALL FINITE QUOTIENTS, THE CONE CONJECTURE UNDER MOST

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Abstract. We introduce a property of convex cones, being "well-clipped", that is inspired by the work of several complex algebraic geometers on the Morrison-Kawamata cone conjecture. That property is satisfied by movable cones of divisors on various complex projective varieties of Calabi–Yau type, such as abelian varieties and projective hyperkähler manifolds. The property of being well-clipped has the advantage to descend under taking invariants by a finite group action, and to be stable by direct sums. In the class of well-clipped cones, we also provide a simple characterization of those cones that admit a rational polyhedral fundamental domain under some natural group action.

We use this framework to prove the movable Morrison–Kawamata cone conjecture for finite quotients of various projective varieties of Calabi–Yau type, notably products of complex projective primitive symplectic varieties, abelian varieties, and smooth rational surfaces underlying klt Calabi–Yau pairs. This entails Enriques manifolds in the sense of Oguiso–Schröer. We also provide Galois descent statements implying the movable Morrison–Kawamata cone conjecture for abelian varieties over arbitrary perfect fields.

1. Introduction

Birational geometry rarely behaves well under finite covers. However, finite covers offer meaningful ways to relate certain varieties to one another, and appear in numerous geometric constructions and classifications: Fundamental groups and uniformization results, the increasingly popular framework of orbifold pairs, the Enriques–Kodaira classification of complex projective surfaces, and the celebrated Beauville–Bogomolov decomposition theorem [3] all involve finite covers in some way, at the very least in the form of quotients by finite group actions. Modern tools have evolved to tackle this ubiquitous challenge, most notably the equivariant minimal model program (MMP) developed in works by Manin, Iskovskikh, Prokhorov (see notably [42]). From a naive perspective, it is not surprising to be able to detect birational properties of a finite quotient X/G from the initial variety X: After all, a lot of birational information is entailed in the nef and movable cones of a variety, which behave well under finite quotients as

$$\mathrm{Nef}(X/G) = \mathrm{Nef}(X) \cap N^1(X)^G, \quad \overline{\mathrm{Mov}}(X/G) = \overline{\mathrm{Mov}}(X) \cap N^1(X)^G,$$

where $N^1(X)^G$ denotes the G-invariant subspace of the real Néron-Severi space. But this naive perspective soon reaches its limits: By Mori's cone theorem, relevant birational information is typically provided by extremal faces of the nef cone that are K_X-negative. This elicits two questions: What if K_X is not negative? More importantly, how to relate extremal faces of a given cone with extremal faces of its G-invariant slice?

The Morrison–Kawamata cone conjecture was made by Morrison [34], notably studied by Kawamata [20], and reformulated by Totaro [48], and offers a prediction anologous to Mori's cone theorem for projective varieties of Calabi–Yau type (which can be thought of as varieties X with K_X non-positive). It can be summarized as follows.

Definition 1.1. A *pair* (X, Δ) is the data of a normal $\mathbb{Q}$-factorial complex projective variety X and of an effective $\mathbb{Q}$-divisor Δ on X with coefficients in $[0, 1]$. A *Calabi–Yau pair* is a pair (X, Δ) such that the $\mathbb{Q}$-Cartier divisor $K_X + \Delta$ is numerically trivial. A variety X is called *of Calabi–Yau type* if there exists a klt Calabi–Yau pair (X, Δ). (For the definition of a *klt* pair, we refer the reader to [22, Definition 2.34].)

Conjecture 1.2 (Morrison–Kawamata cone conjecture)**.** Let (X, Δ) be a klt Calabi–Yau pair. There exist rational polyhedral fundamental domains both for

(1) (movable cone conjecture) the image of the group $\mathrm{PsAut}(X, \Delta)$ of birational automorphisms of X that are isomorphisms in codimension 1 and preserve the support of Δ, acting on the modified movable cone $\mathrm{Mov}^+(X)$;
(2) (nef cone conjecture) the image of the group $\mathrm{Aut}(X, \Delta)$ of automorphisms of X that preserve the suport of Δ, acting on the modified nef cone $\mathrm{Nef}^+(X)$.

Here, for a cone $\mathscr{C}$ in a vector space V with a rational structure $V_{\mathbb{Q}}$, we define

$$\mathscr{C}^+ := \mathrm{Conv}_{\mathbb{R}}(\overline{\mathscr{C}} \cap V_{\mathbb{Q}}),$$

and the image of $(\mathrm{Ps})\mathrm{Aut}(X)$ refers to the image by the representation

$$\rho : (\mathrm{Ps})\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)).$$

This paper concerns the descent of the movable Morrison–Kawamata cone conjecture under finite quotients, as expressed in the following question.

Question 1.3. Let (X, Δ) be a pair. For a finite subgroup G of $\mathrm{Aut}(X, \Delta)$, consider the *quotient pair* $(X/G, \Delta_G)$, where Δ_G is the $\mathbb{Q}$-divisor such that $K_X + \Delta$ is the pullback of $K_{X/G} + \Delta_G$ by the quotient map. Does the movable cone conjecture for (X, Δ) imply the movable cone conjecture for $(X/G, \Delta_G)$, for every finite subgroup G?

It is worth noting that if (X, Δ) is a klt Calabi–Yau pair, then so is $(X/G, \Delta_G)$.

We give a positive answer to Question 1.3 for a large class of pairs (X, Δ) for which the movable cone conjecture is currently known to be satisfied.

Theorem 1.4. *Let (X, Δ) be a klt Calabi–Yau pair defined over $\mathbb{C}$ that decomposes as*

$$X = A \times \prod_{i=1}^{r} Y_i \times \prod_{j=1}^{s} S_j, \quad \Delta = \sum_{j=1}^{s} p_j^* \Delta_j,$$

where A is an abelian variety, each Y_i is a primitive symplectic variety with canonical singularities and $b_2(Y_i) \geq 5$, and each S_j is a smooth rational surface underlying a klt Calabi–Yau pair (S_j, Δ_j). Then, the movable cone conjecture holds for every quotient pair $(X/G, \Delta_G)$ of the pair (X, Δ) by a finite subgroup G of $\mathrm{Aut}(X, \Delta)$. Moreover, there are finitely many isomorphism classes of pairs (Y, Δ_Y) obtained by small $\mathbb{Q}$-factorial modifications of $(X/G, \Delta_G)$, and the nef cone conjecture holds for each of them.

Primitive symplectic varieties are introduced in [2, Definition 3.1], and generalize hyperkähler manifolds in singular settings. Note that Theorem 1.4 also holds under the weaker assumption that each Y_i is a primitive symplectic variety with canonical singularities that admits a terminalization $\hat{Y}_i$ of second Betti number $b_2(\hat{Y}_i) \geq 5$.

This theorem generalizes the work of Oguiso–Sakurai [39] in dimension 2, which was notably put to use in the proof of the cone conjecture in dimension 2 by Totaro in [48]. It also generalizes the recent results of Pacienza–Sarti on Enriques manifolds of prime index [40], and of Monti–Quedo on generalized hyperelliptic varieties [33].

Our proof starts with an observation: In most instances of pairs for which the cone conjecture has been proved, much more structure was in fact discovered on the cone under study than one may expect: Movable cones of abelian varieties are self-dual homogeneous cones by [41], and so are movable cones of their smooth finite quotients by [33]; Nef cones of K3 surfaces are cut out by the orthogonal hyperplanes to (-2)-curve classes in the hyperbolic cone given by the self-intersection form by [46]; Movable cones of projective hyperkähler manifolds are described very precisely by Markman in [31], and so are movable cones of projective primitive symplectic varieties with terminal $\mathbb{Q}$-factorial singularities by [25].

With all these results in mind, we define a new class of cones, the *well-clipped* cones, meant to subsume all the afore-mentioned algebro-geometric examples, to be stable by direct sum, and to behave better than the Morrison–Kawamata cone conjecture under finite quotients.

Definition 1.5 (= Definition 3.1 later)**.** Let $V = V_{\mathbb{Z}} \otimes \mathbb{R}$ be a finite dimensional real vector space with a preferred lattice. A full-dimensional convex cone $\mathscr{C}$ is *well clipped* if there are a self-dual homgeneous cone $\mathscr{A}$ in V and a set of hyperplanes $(H_i)_{i\in I}$ of V such that

$$\mathring{\mathscr{C}} = \mathscr{A} \cap \bigcap_{i\in I} H_{i,+},$$

where $H_{i,+}$ denotes a connected component of $V \setminus H_i$, and the following three assumptions are satisfied:

(i) Decomposing $\mathscr{A} = \bigoplus_{j\in J} \mathscr{A}_j$ into $\mathbb{R}$-indecomposable summands, every hyperplane H_i is of the form

$$H_i = H_i \cap \operatorname{Span}_{\mathbb{R}} \mathscr{A}_{j(i)} \oplus \bigoplus_{k\neq j(i)} \operatorname{Span}_{\mathbb{R}} \mathscr{A}_k,$$

with the remaining cone $\mathscr{A}_{j(i)}$ of hyperbolic type and defined over $V_{\mathbb{Q}}$.

Fixing an $\operatorname{Aut}(\mathscr{A}, V_{\mathbb{Z}})$-invariant and $V_{\mathbb{Z}}$-integral quadratic form q that is a direct sum of hyperbolic forms on the linear spans of the $(\mathscr{A}_{j(i)})_{i\in I}$ and of a positive definite quadratic form on the other summands' spans,

(ii) For every $i \in I$, the q-orthogonal reflection σ_i fixing the hyperplane H_i preserves the lattice $V_{\mathbb{Z}}$.

(iii) For every $i, k \in I$ such that $H_i \neq H_k$, any element e_i, e_k perpendicular to H_i, H_k and of negative squares satisfy $q(e_i, e_k) \geq 0$.

With this terminology, Theorem 1.4 follows from the following result applied over the field $k = \overline{k} = \mathbb{C}$.

Theorem 1.6. *Let (X, Δ) be a pair defined over a perfect field k. Let $(\overline{X}, \overline{\Delta})$ denote their base change to the algebraic closure $\overline{k}$ of k. Suppose that the movable cone $\operatorname{Mov}(\overline{X})$ is well clipped in a self-dual homogeneous cone preserved by the action of $\operatorname{PsAut}(\overline{X}, \overline{\Delta})$, and that the pair $(\overline{X}, \overline{\Delta})$ satisfies the movable cone conjecture. Then the movable cone conjecture holds for every quotient pair $(X/G, \Delta_G)$ of (X, Δ) by a finite subgroup G of $\operatorname{Aut}_k(X, \Delta)$.*

Note that Theorem 1.6 encompasses is a mixed (geometric and arithmetic) equivariant descent result. It generalizes descent results proven in [7] in order to establish analogues of the Kawamata–Morrison cone conjecture for K3 surfaces over arbitrary base fields of characteristics other than 2, and more recently in [50, Section 2.7, Lemma 23] for Enriques surfaces over non-closed fields of characteristic zero. We make note of the following consequence, used notably by [38, Corollary 8.5] with k a function field:

Corollary 1.7. *Let X be a torsor over an abelian variety over a perfect field k. Then the movable cone conjecture holds for X.*

A key result involved in the proof of Theorem 1.6 can be summarized under the motto "Well-clipped cones behave themselves under finite quotients, and so does the cone conjecture for them." Let us state it more formally.

Theorem 1.8. *Let $\mathscr{C}$ be a well-clipped cone in a self-dual homogeneous cone $\mathscr{A}$ in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Let Γ denote the group $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$. Let G be a subgroup of Γ whose action on V has finite orbits. Then, the invariant cone $\mathscr{C}^G$ is well-clipped. Furthermore, if there exists a rational polyhedral fundamental domain for the action of Γ on $\mathscr{C}^+$, then there exists a rational polyhedral fundamental domain for the action of the centralizer $C_\Gamma(G)$ on ${\mathscr{C}^G}^+$.*

A small technical drawback of our definition of a well-clipped cone is that, although the invariant cone $\mathscr{A}^G$ always is self-dual homogeneous, the cone $\mathscr{C}^G$ often is not well-clipped in $\mathscr{A}^G$, but in a smaller self-dual homogeneous cone.

To prove Theorem 1.8, we use a mix of convex geometry, computations with Coxeter groups generated by hyperplane reflections, and, surprisingly yet unavoidably, the Koecher–Vinberg equivalence of categories between self-dual homogeneous cones and formally real Jordan algebras. Indeed, it seems difficult to show that an invariant subcone $\mathscr{A}^G$ of a self-dual homogeneous cone $\mathscr{A}$ remains homogeneous directly, without using either the Koecher–Vinberg equivalence or the classification of self-dual homogeneous cones (whose proof relies on the Koecher–Vinberg equivalence anyways).

We conclude this introduction by mentioning some limitations to our approach. In principle, Theorem 1.6 may descend the movable cone conjecture under finite quotients for varieties X beyond the scope of Theorem 1.4, for instance for the Wehler varieties introduced in [8] or the blow-ups of $\mathbb{P}^3$ at 8 very general points discussed in [45] (see Example 3.17). These examples are actually uninteresting, because neither they, nor any of their small $\mathbb{Q}$-factorial modifications have any (known) finite quotients. In principle, they may deform to special smooth threefolds that have interesting finite quotients, which could be studied by the recent work of Lutz [28] and Theorem 1.6. In any case, we hope that this remark can inspire a discussion on the cone conjecture for finite quotients of smooth Calabi–Yau threefolds, varieties, and pairs in general.

In light of the Beauville–Bogomolov decomposition theorem [3], a landmark result would be to extend the conclusions of Theorem 1.4 to varieties X of the form

$$X = A \times \prod_{i=1}^{r} Y_i \times \prod_{j=1}^{s} Z_j,$$

where A is an abelian variety, each Y_i is a primitive symplectic variety, and each Z_j is a smooth Calabi–Yau variety. However, the movable cone of a smooth Calabi–Yau threefold is not always well clipped, see Example 3.10. Motivated by the singular

Beauville–Bogomolov decomposition theorem [15, 10, 17], we can ask the same question for Calabi–Yau varieties with canonical singularities in the sense of [15].

A last unfortunate aspect of Definition 3.1 is that most non-simplicial rational polyhedral cones are not even well clipped, see Example 3.9. However, if X is a projective variety with a rational polyhedral movable cone, then the movable cone conjecture clearly descends under any finite group action on X: It is a bit disappointing that this straight-forward case is not directly covered by Theorem 1.6. One could include this situation by studying the smallest class of cones that contains all well-clipped cones, all rational polyhedral cones, and is stable by direct sums and finite quotients, but it feels like a rather marginal improvement.

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2. Preliminaries

All cones considered are non-empty and convex in an ambient real finite-dimensional vector space. A cone is called *non-degenerate* if it contains no line. For a cone $\mathscr{C}$ in a real vector space V with a rational structure $V_{\mathbb{Q}} \subset V$, we set

$$\mathscr{C}^+ := \mathrm{Conv}_{\mathbb{R}}(\overline{\mathscr{C}} \cap V_{\mathbb{Q}}),$$

where $\overline{\mathscr{C}}$ stands for the closure of $\mathscr{C}$ in V in Euclidean topology. A cone is called *rational polyhedral* if it is spanned by a finite set of points of $V_{\mathbb{Q}}$.

For $\mathscr{C}$ a cone in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$, we denote by $\mathrm{Aut}(\mathscr{C})$ the group of linear automorphisms of V that preserve the cone $\mathscr{C}$, by $\mathrm{Aut}(\mathscr{C}, V_{\mathbb{Z}})$ its subgroup that also preserves the lattice V_Z. If $\mathscr{A}$ is another cone in V, we denote by $\mathrm{Aut}(\mathscr{C}, \mathscr{A}, V_{\mathbb{Z}})$ the group of linear automorphisms of V that preserve both cones $\mathscr{C}$ and $\mathscr{A}$, and the lattice V_Z.

If G is a subgroup of $\mathrm{GL}(V)$, we denote by

$$V^G := \{v \in V \mid \forall g \in G,\, g(v) = v\}$$

the *invariant subspace*. For a cone $\mathscr{C}$ in V that is preserved by G, we denote by $\mathscr{C}^G := \mathscr{C} \cap V^G$ the *invariant cone*.

For a normal $\mathbb{Q}$-factorial complex projective variety X, we denote by $N^1(X)$ the finite dimensional real vector space of numerical equivalence classes of $\mathbb{R}$-divisors on X. In this space, we denote by $\mathrm{Mov}(X)$ the closure of the convex cone spanned by classes of movable Cartier divisors on X, and call it the *movable cone* of X. Unless otherwise stated, our lattice of choice in the Néron–Severi space $N^1(X)$ is the lattice of integral Weil divisors, which is preserved by the action of $\mathrm{PsAut}(X)$.

Primitive symplectic varieties are introduced in [2, Definition 3.1]. Recall that a primitive symplectic variety is a normal compact Kähler variety X with $h^1(\mathscr{O}_X) = 0$ and $H^0(X_{\mathrm{reg}}, \Omega^2_{X_{\mathrm{reg}}}) = \mathbb{C} \cdot \sigma$, where σ is a symplectic form on X_{reg} that extends holomorphically (not necessarily as a symplectic form) to any resolution of singularities of X.

For a group Γ and a subgroup $G < \Gamma$, we denote by

$$N_\Gamma(G) := \{\gamma \in \Gamma \mid \gamma G = G\gamma\}, \quad C_\Gamma(G) := \{\gamma \in \Gamma \mid \forall g \in G, \gamma g = g\gamma\}$$

the *normalizer* of G in Γ and the *centralizer* of G in Γ, respectively.

2.A. **Self-dual homogeneous cones.**

Throughout this section, a reference is [11].

Definition 2.1. Let $\mathscr{A}$ be an open non-degenerate cone in a finite dimensional real vector space V. We say that $\mathscr{A}$ is a *self-dual cone* if there exists a positive definite quadratic form $\mathrm{tr} : V \otimes V \to \mathbb{R}$ that induces an identification of $\mathscr{A}$ with its dual cone. We say that $\mathscr{A}$ is a *homogeneous cone* if the group of linear automorphisms of V that preserve $\mathscr{A}$ acts transitively on $\mathscr{A}$. We say that $\mathscr{A}$ is $\mathbb{R}$-*indecomposable* if, for any decomposition $V = V_1 \oplus V_2$ into $\mathbb{R}$-linear subspaces with both $\mathscr{A}_i := \mathscr{A} \cap V_i \neq \varnothing$, we have a strict inclusion $\mathscr{A} \supsetneq \mathscr{A}_1 + \mathscr{A}_2$.

Example 2.2. Let $n \geq 3$. Let q be the quadratic form of signature $(1, n-1)$ on $\mathbb{R}^n$, and pick $h \in \mathbb{R}^n$ such that $q(h) > 0$. The non-degenerate open cone

$$\mathscr{H}_n := \{v \in \mathbb{R}^n \mid q(v) > 0,\, q(h, v) > 0\}$$

is an $\mathbb{R}$-indecomposable self-dual homogeneous cone. A cone that identifies, up to linear isomorphism, with $\mathscr{H}_n$ for some $n \geq 3$ is said to be *of hyperbolic type*, or *hyperbolic* for short.

A classification of $\mathbb{R}$-indecomposable self-dual homogeneous cones was achieved by the 1934 result of Jordan–von Neumann–Wigner [18] on the classification of formally real Jordan algebras, and by the 1970ies Koecher–Vinberg theorem stating an equivalence of categories between formally real Jordan algebras and self-dual homogeneous cones (see e.g. [21], [11, Theorem III.2.1 and Section III.3], and Theorem 2.12 for a precise statement). Here we present the classification.

Theorem 2.3. *Let $\mathscr{A}$ be a self-dual homogeneous cone. Then there is a unique decomposition*

$$\mathscr{A} = \bigoplus \mathscr{A}_j,$$

where each $\mathscr{A}_j$ is an $\mathbb{R}$-indecomposable self-dual homogeneous cone. Moreover, the $\mathbb{R}$-indecomposable self-dual homogeneous cones are classified as:

(1) the positive halfline in $\mathbb{R}$;
(2) the cone of positive definite, real symmetric n by n matrices for some $n \geq 3$;
(3) the cone of positive definite, complex Hermitian n by n matrices for some $n \geq 3$;
(4) the cone of positive definite, quaternionic Hermitian n by n matrices for some $n \geq 3$;
(5) the hyperbolic cone $\mathscr{H}_n$ for some $n \geq 3$;
(6) the cone of positive definite octonionic Hermitian 3 by 3 matrices.

We also mention the following result, usually attributed to Koecher and Vinberg.

Proposition 2.4. *Let $\mathscr{A}$ be a self-dual homogeneous cone. Then the Lie group* $\mathrm{Aut}(\mathscr{A})$ *has finitely many components, and its identity component* $\mathrm{Aut}^\circ(\mathscr{A})$ *is a connected Lie group realised as the real points of an algebraic group.*

We often work with a preferred lattice, or rational structure on the real vector space V. The following definition ensures compatibility of a self-dual homogeneous cone with a given rational structure; We follow [1, Sections 3, 4].

Definition 2.5. Let $\mathscr{A}$ be a self-dual homogeneous cone in a finite dimensional real vector space V. Let $V_{\mathbb{Q}}$ be a rational structure in V. We say that $\mathscr{A}$ is compatible with $V_{\mathbb{Q}}$ if the algebraic group $\mathrm{Aut}^{\circ}(\mathscr{A})$ is defined over $\mathbb{Q}$. For short, we may write that $\mathscr{A}$ be a self-dual homogeneous cone in $V = V_{\mathbb{Q}} \otimes \mathbb{R}$ to indicate compatibility between $\mathscr{A}$ and $V_{\mathbb{Q}}$.

The next lemma is written with a lattice $V_{\mathbb{Z}}$; the corresponding rational structure is of course $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes \mathbb{Q}$.

Lemma 2.6. *Let $\mathscr{A}$ be a self-dual homogeneous cone in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. There exists a positive definite $V_{\mathbb{Z}}$-integral quadratic form* tr *on V that is $\mathrm{Aut}(\mathscr{A}, V_{\mathbb{Z}})$-invariant, and with respect to which $\mathscr{A}$ is self-dual.*

Proof. The existence of a positive definite form tr that is $V_{\mathbb{Q}}$-rational, on which $\mathrm{Aut}(\mathscr{A})$ acts by scaling, and with respect to which $\mathscr{A}$ is self-dual is due to standard facts of Lie theory, see for instance [1, 3.1, Paragraph 2] for an explanation. A large enough multiple of it is $V_{\mathbb{Z}}$-integral and $\mathrm{Aut}(\mathscr{A}, V_{\mathbb{Z}})$-invariant. □

2.B. **Round and simplicial parts.** We introduce a few *ad hoc* definitions.

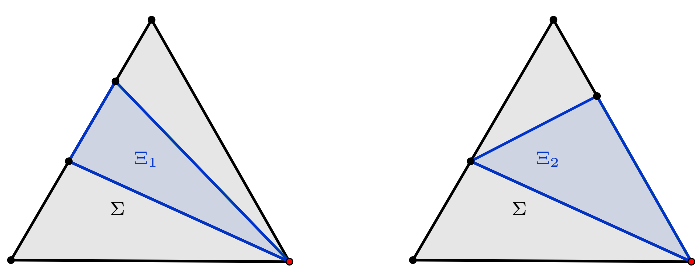
Definition 2.7. Let $\mathscr{A}$ be a self-dual homogeneous cone. We define the *round part* of $\mathscr{A}$, denoted by $\mathrm{rd}\,\mathscr{A}$, to be the cone obtained by removing all halfline summands of the decomposition of $\mathscr{A}$ given by Theorem 2.3. We define the *simplicial part* of $\mathscr{A}$ as the sum of all halfline summands of $\mathscr{A}$ in that same decomposition.

Remark 2.8. The round part and the simplicial part of a self-dual homogeneous cone both are self-dual homogeneous cones in their own linear span.

The next definition defines a preorder on the set of simplicial cones in a vector space with a preferred lattice.

Definition 2.9. Let Σ and Ξ be two simplicial cones in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. We say that Σ *rules* Ξ if the following two conditions both hold

(1) every extremal ray of irrational slope of Ξ is an extremal ray of Σ;
(2) the linear span of all extremal rays of rational slope of Ξ is spanned by a subset of extremal rays of Σ.



With rational points in black and irrational points in red, Σ rules Ξ_1, but not Ξ_2.

These notions will later appear in Lemma 4.11.

2.C. **Formally real Jordan algebras.** As it will be used later, we also state the Koecher–Vinberg theorem, a key ingredient in the classification stated in Theorem 2.3, which provides an equivalence of categories between self-dual homogeneous cones and a certain class of algebraic objects. We loosely follow [21].

Definition 2.10. A *Jordan algebra* is a finite dimensional $\mathbb{R}$-vector space V, an element $e \in V$, and a bilinear operation $\circ : V \times V \to V$ such that

- $\circ$ is commutative;
- e is a neutral element for $\circ$;
- for all $x, y \in V$, the Jordan identity $(x \circ x) \circ (x \circ y) = x \circ ((x \circ x) \circ y)$ holds.

A Jordan algebra $(V, e, \circ)$ is called *formally real* if for all $x_1, \ldots, x_k \in V$, we have

$$x_1 \circ x_1 + \ldots x_k \circ x_k = 0 \Longrightarrow x_1 = \ldots = x_k = 0.$$

A Jordan algebra $(V, e, \circ)$ is called *simple* if any linear subspace of V that is preserved by multiplication by V is equal to $\{0\}$ or V.

Remark 2.11. The notion of a formally real Jordan algebra is closed under isomorphisms, subalgebras, and direct products.

The next result is due to Koecher and Vinberg, see for instance [21, Theorem 15].

Theorem 2.12. *There is an isomorphism of categories between formally real Jordan algebras and pointed self-dual homogeneous cones, given on objects by*

$$F : (V, e, \circ) \mapsto (\mathscr{A}, e), \quad \textit{where } \mathscr{A} := \{x \circ x \mid x \in V\},$$

and on morphisms by $F : \phi \mapsto \phi$.

Remark 2.13. If we decide to not keep track of the neutral element of Jordan algebras, and want to define a functor $\overline{F}$ from the category of formally real Jordan algebras into the category of self-dual homogeneous cones, it remains an equivalence of categories, but fails to be injective on objects.

2.D. **Useful facts from birational geometry.** This subsection gathers a few facts from birational geometry, which we will mostly use in the proof of Theorem 1.4. Our first lemma is very close to a result of Xu for the cone $\mathrm{Mov}^{\mathrm{e}}$ [50, Theorem 24, Remark on Page 26] and Gachet-Lin-Stenger-Wang for the cone $\mathrm{Nef}^{\mathrm{e}}$ [12, Proposition 1.6]. We state it and prove it herefor the cones Mov^+ and Nef^+, under somewhat weaker assumptions than in [50, Theorem 24].

Lemma 2.14. *Let $f : (X, \Delta) \to (Y, \Delta_Y)$ be a crepant birational morphism of klt Calabi–Yau pairs. Assume that the pair (X, Δ) satisfies the movable cone conjecture stated as in Conjecture 1.2. Then the pair (Y, Δ_Y) satisfies the movable cone conjecture.*

Proof. Note that $F := f^*\mathrm{Mov}^+(Y)$ is a face of the cone $\mathrm{Mov}^+(X)$. By [12, Proposition 3.6], since (X, Δ) satisfies the movable cone conjecture, the subgroup $\mathrm{Stab}(F)$ of $\mathrm{PsAut}^*(X, \Delta)$ that stabilizes the face F acts on F with a rational polyhedral fundamental domain. To conclude, it suffices to check that $\mathrm{Stab}(F)$ is a subgroup of $f^{-1} \circ \mathrm{PsAut}^*(Y, \Delta_Y) \circ f$.

For $g \in \mathrm{Stab}(F)$, note that there exists $\alpha : X \dashrightarrow X'$ a small $\mathbb{Q}$-factorial modification such that the relative interiors of $g^* f^* \mathrm{Nef}(Y)$ and $f^* \alpha^* \mathrm{Nef}(Y)$ intersect by [4, Theorem 1.2] (see also [12, Proposition 4.10]). By [12, Lemma 4.2], this means that $f \circ g = \alpha \circ f$, thus $g \in f^{-1} \circ \mathrm{PsAut}^*(Y, \Delta_Y) \circ f$ as wished. ☐

Lemma 2.15. *Let $f : (X, \Delta) \to (Y, \Delta_Y)$ be a crepant birational morphism of klt Calabi–Yau pairs. Assume that finitely many pairs arise as small $\mathbb{Q}$-factorial modifications of (X, Δ), and that each of them satisfies the nef cone conjecture stated as in Conjecture 1.2. Then finitely many pairs arise as small $\mathbb{Q}$-factorial modifications of (Y, Δ_Y), and each of them satisfies the nef cone conjecture.*

Proof. Apply [12, Theorem 1.5 (i), (3) $\Rightarrow$ (4)] to the pair (X, Δ), and restrict to the Mori chambers in $N^1(X)$ associated to small $\mathbb{Q}$-factorial modifications of Y (precomposed with f). □

We also prove a lemma about movable cones of products.

Lemma 2.16. *Let $X = X_1 \times X_2$ be a product of normal projective varieties. Assume that $N^1(X) = p_1^* N^1(X_1) + p_2^* N^1(X_2)$, where p_1, p_2 denote the natural projections. Then we have*

$$\mathrm{Mov}(X) = p_1^*\mathrm{Mov}(X_1) + p_2^*\mathrm{Mov}(X_2).$$

Proof. Let D be a movable divisor on X. We write $D = p_1^* D_1 + p_2^* D_2$. Since D is movable, we can pick two sections $s, s' \in |D|$ that have no common divisorial component. For a component Z of the intersection $s \cap s'$, the restriction of the projection $p_2 : Z \to X_2$ has general fiber that is empty, or of dimension at most $\dim X_1 - 2$.

In particular, for a general point $u \in X_2$, the locus $s \cap s'$ intersects the fiber $X_1 \times \{u\}$ along a closed subscheme of codimension at least 2. Thus, $s|_{X_1 \times \{u\}}$ and $s'|_{X_1 \times \{u\}}$ are well-defined sections of the linear system $|D_1|$ on X_1, and have no common divisorial component. This shows that D_1 is movable, and the same argument works symmetrically for D_2.

Since the movable cone is the closed convex cone spanned by classes of movable divisors, this concludes the proof. □

The next lemma is inspired by [10, Lemma 4.6], and is therein also credited to C. Casagrande.

Lemma 2.17. *Let $X = X_1 \times X_2$ be a product of projective varieties with klt $\mathbb{Q}$-factorial singularities, and with $h^1(X_2) = 0$. Consider a small $\mathbb{Q}$-factorial modification $\alpha : X \dashrightarrow Y$. Then there exist $\alpha_i : X_i \dashrightarrow Y_i$ small $\mathbb{Q}$-factorial modifications for $i = 1, 2$ such that $Y = Y_1 \times Y_2$ and $\alpha = (\alpha_1, \alpha_2)$.*

Proof. By Lemma 2.16, we have a decomposition of the movable cone:

$$\mathrm{Mov}(X) = p_1^*\mathrm{Mov}(X_1) \oplus p_2^*\mathrm{Mov}(X_2).$$

Let H be an ample Cartier divisor on Y, then

$$\alpha^* H = p_1^* M_1 + p_2^* M_2,$$

where M_i is a movable, big Cartier divisor on X_i.

Up to taking large enough multiples, note that M_i is basepoint-free outside of base locus of codimension at least 2 in X_i, thus defines a map $\alpha_i : X_i \dashrightarrow Y_i$, with an ample divisor H_i on Y_i such that $M_i = \alpha_i^* H_i$. In the Néron-Severi space of X, we have a non-empty intersection

$$\alpha^*\mathrm{Amp}(Y) \cap (\alpha_1, \alpha_2)^*\mathrm{Amp}(Y_1 \times Y_2) \neq \varnothing,$$

so by [12, Lemma 4.2] (see also [20, Lemma 1.5] in dimension 3), the birational map

$$(\alpha_1, \alpha_2) \circ \alpha^{-1} : Y \to Y_1 \times Y_2$$

is indeed a biregular isomorphism. □

Corollary 2.18. *Let $X = \prod_{i=1}^r X_i$ be a product of projective varieties with klt $\mathbb{Q}$-factorial singularities, with $h^1(X_i) = 0$ for $i \geq 2$. Then* $\mathrm{PsAut}(X)$ *preserves the decomposition of X, up to permuting factors that are isomorphic in codimension* 1.

Proof. Note that applying [10, Lemma 4.6] to the identity from X to itself, any other decomposition of X as a product differs by a mere permutation. Thus, by [16, Exercise III.12.6] and by Lemma 2.17, for a pseudoautomorphism α of X, there is a permutation σ of the integers between 1 and r and, for every index i, a small $\mathbb{Q}$-factorial modification $\alpha_i : X_i \dashrightarrow X_{\sigma(i)}$, such that

$$\alpha = \prod_{i=1}^r \alpha_i,$$

which indeed, preserves the decomposition of X up to permutation of factors that are isomorphic in codimension 1. □

We conclude with a lemma inspired by [32], [25, Section 7, Page 31], and [12, Proof of Theorem 6.1 (iii) (3a), Page 26-27]. Here, one may think of the $(V_w)_{w\in W}$ as wall hyperplanes corresponding to flops.

Lemma 2.19. *Let (X, Δ) be a klt Calabi–Yau pair that satisfies the movable cone conjecture. Suppose that there exist hyperplanes $(V_w)_{w\in W}$ in $N^1(X)$ such that*

$$\mathrm{Mov}^\circ(X) \setminus \bigcup_{w\in W} V_w = \bigsqcup_{\substack{\alpha: X \dashrightarrow X' \\ \text{SQM}}} \alpha^*\mathrm{Amp}(X')$$

and such that for any rational polyhedral cone $\Pi \subset \mathrm{Mov}^+(X)$, only finitely many of the V_w are intersecting Π°. Then there are finitely many isomorphism classes of pairs (Y, Δ_Y) obtained by small $\mathbb{Q}$-factorial modifications of (X, Δ), and the nef cone conjecture holds for each of them.

Proof. Let Π be a rational polyhedral fundamental domain for $\mathrm{PsAut}^*(X, \Delta)$ acting on the cone $\mathrm{Mov}^+(X)$. By assumption, we can take finitely many small $\mathbb{Q}$-factorial modifications $\alpha_i : X \dashrightarrow X_i$ with $1 \leq i \leq r$ to cover

$$\Pi \quad \subset \quad \bigcup_{i=1}^r \alpha_i^*\mathrm{Nef}(X_i).$$

Since Π is a fundamental domain for $\mathrm{Mov}^+(X)$, this already shows that any small $\mathbb{Q}$-factorial modification of X identifies up to $\mathrm{PsAut}(X, \Delta)$ with one of the α_i. Thus, a pair (Y, Δ_Y) obtained by a small $\mathbb{Q}$-factorial modification from (X, Δ) must be one of the $(X_i, \alpha_{i*}\Delta)$.

As a corollary, for a fixed index i, the chambers of the form $\beta^*\alpha_i^*\mathrm{Amp}(X_i)$ with $\beta \in \mathrm{PsAut}(X, \Delta)$ that intersect Π° are finitely many. We denote them by $\beta_j^*\alpha_i^*\mathrm{Amp}(X_i)$ with $1 \leq j \leq s_i$.

Fix i. We now prove the nef cone conjecture for $(X_i, \alpha_{i*}\Delta)$. Denote by Σ_i the sum of the cones of the form ${\beta_j^{-1}}^*\Pi \cap \mathrm{Nef}(X_i)$ for $1 \leq j \leq s_i$. Note that Σ_i is a itself rational polyhedral: Indeed, by assumption on the wall hyperplanes (V_w), any intersection of the form $\alpha_k^*\mathrm{Nef}(X_k) \cap \Pi$ for $1 \leq k \leq r$ is a rational polyhedral cone. If the translates of Σ_i by $\mathrm{Aut}(X_i, \alpha_{i*}\Delta)$ cover the cone $\mathrm{Amp}(X_i)$, then [27, Proposition 4.1] concludes. Let $D \in \mathrm{Amp}(X_i)$. There exists $\beta \in \mathrm{PsAut}(X, \Delta)$ such that $\beta^*\alpha_i^*D \in \Pi$, in particular it belongs to a chamber $\beta_j^*\alpha_i^*\mathrm{Amp}(X_i)$ for some j. By [12, Lemma 4.2], this forces

$$h := \alpha_i\beta\beta_j^{-1}\alpha_i^{-1}$$

to be a biregular automorphism of X_i, and in fact $h \in \mathrm{Aut}(X_i, \alpha_{i*}\Delta)$. It is now immediate that h^*D belongs to Σ_i. $\square$

3. Well-clipped, neatly clipped, perfectly clipped cones

3.A. **Definition and examples of well-clipped cones.** This definition is the focal point of the whole section.

Definition 3.1. Let $\mathscr{A}$ be a self-dual homogeneous cone in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. A full-dimensional convex cone $\mathscr{C}$ is *well clipped* in $\mathscr{A}$ if there exists a set of hyperplanes $(H_i)_{i\in I}$ of V such that

$$\mathring{\mathscr{C}} = \mathscr{A} \cap \bigcap_{i\in I} H_{i,+},$$

where $H_{i,+}$ denotes a connected component of $V \setminus H_i$, and the following three assumptions hold

(i) Decomposing $\mathscr{A} = \bigoplus_{j\in J} \mathscr{A}_j$ into $\mathbb{R}$-indecomposable summands, every hyperplane H_i is of the form

$$H_i = H_i \cap \mathrm{Span}_{\mathbb{R}}\, \mathscr{A}_{j(i)} \oplus \bigoplus_{k\neq j(i)} \mathrm{Span}_{\mathbb{R}}\, \mathscr{A}_k,$$

where the remaining cone $\mathscr{A}_{j(i)}$ is of hyperbolic type and defined over $V_{\mathbb{Q}}$.

Fixing an $\mathrm{Aut}(\mathscr{A}, V_{\mathbb{Z}})$-invariant and $V_{\mathbb{Z}}$-integral quadratic form q that is a direct sum of hyperbolic forms on the linear spans of the $(\mathscr{A}_{j(i)})_{i\in I}$ and of a positive definite quadratic form as in Lemma 2.6 on the other summands' spans,

(ii) For every $i \in I$, the q-orthogonal reflection σ_i fixing the hyperplane H_i preserves the lattice $V_{\mathbb{Z}}$.

(iii) For every $i, k \in I$ such that $H_i \neq H_k$, any element e_i, e_k perpendicular to H_i, H_k and of negative squares satisfy $q(e_i, e_k) \geq 0$.

We call a cone *well clipped* if there exists a self-dual homogeneous cone $\mathscr{A}$ in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$ in which it is well clipped.

Remark 3.2. Unless otherwise stated, when we describe a well-clipped cone $\mathscr{C}$ in a self-dual homogeneous cone $\mathscr{A}$, we list a set of hyperplanes $(H_i)_{i\in I}$ that is minimal under inclusion for the property of cutting out $\mathscr{C}$.

Notation 3.3. When V is a finite dimensional real vector space, H a hyperplane in V, we denote by H^- and H^+ the two connected components of the complement $V \setminus H$.

Remark 3.4. By Assumption (i), note that the orthogonal reflection σ_i writes:

$$\sigma_i : v \in V \mapsto v - \frac{2q(e_i, v)}{q(e_i, e_i)} e_i,$$

for any element $e_i \in {H_i}^{\perp}$. Unless otherwise stated, we use the convention that e_i denotes the generator of $({H_i}^{\perp} \cap H_{i,-}) \cap V_{\mathbb{Z}}$. The element e_i always has negative q-square. Moreover, Assumption (iii) can be rephrased as $q(e_i, e_j) \geq 0$ for this particular choice of e_i, e_j.

Note that the data of hyperplanes $(H_i)_{i\in I}$ for Definition 3.1 is equivalent to the data of elements $(e_i)_{i\in I}$ satisfying the corresponding conditions we just specified.

If the cone $\mathscr{A}$ is already $\mathbb{R}$-indecomposable, and itself of hyperbolic type, then Assumption (i) is trivially verified, and Assumption (ii) is equivalent to the condition

For every $i \in I$, $q(e_i, e_i)$ divides every element of $2q(e_i, V_{\mathbb{Z}}) \subset \mathbb{Z}$.

In general, Assumption (ii) could be rephrased, in a lattice-theoretic context, by saying that the $(e_i)_{i\in I}$ are *roots* of the lattice $V_{\mathbb{Z}}$, see [5, Section 0.1]. However, we prefer to avoid this terminology altogether, as it is often expected of roots to have norm -2, whilst our $(e_i)_{i\in I}$ can have any negative square.

The next lemma is a clear consequence of the definition.

Lemma 3.5. *A direct sum of well-clipped cones is itself well clipped.*

Proof. Consider a cone $\mathscr{C} = \mathscr{C}_1 \oplus \mathscr{C}_2$, where $\mathscr{C}_u$ is well clipped in a self-dual homogeneous cone $\mathscr{A}_u$ in $V_u = V_{u,\mathbb{Q}} \otimes \mathbb{R}$ for $u = 1, 2$. Note that $\mathscr{A} := \mathscr{A}_1 \oplus \mathscr{A}_2$ is a self-dual homogeneous cone in $V := V_1 \oplus V_2$ with respect to the rational structure $V_{\mathbb{Q}} = V_{1,\mathbb{Q}} \oplus V_{2,\mathbb{Q}}$. Clearly, the pullbacks of the hyperplanes cutting out $\mathscr{C}_1$ in $\mathscr{A}_1$ and $\mathscr{C}_2$ in $\mathscr{A}_2$ still cut out $\mathscr{C}$ in $\mathscr{A}$. Checking the assumptions of Definition 3.1 is immediate as long as we take the split quadratic form $q = q_1 \oplus q_2$ for Assumptions (ii) and (iii). □

We provide examples of well-clipped cones, with an algebro-geometric reader in mind.

Example 3.6. This list of examples is by far not exhaustive.

(1) A simplicial cone in $\mathbb{R}^n$ is always well clipped. Indeed, it decomposes as a direct sum of half-lines over $\mathbb{R}$. In particular, any non-degenerate convex cone in $\mathbb{R}^2$ is well clipped.
(2) The nef cone of a smooth del Pezzo surface Σ is rational polyhedral, and well clipped. For $\rho(\Sigma) = 1$ or 2, it follows from Item (1). Otherwise, we can take the quadratic form to be the intersection form on $N^1(\Sigma)$, and the $(e_i)_{i\in I}$ in Remark 3.4 to be the finitely many classes of (-1)-curves.
(3) The nef cone of a smooth projective K3 surface is well clipped by [46].
(4) The nef cone of a smooth projective surface underlying a klt Calabi–Yau pair is well clipped, see Corollary 3.8 below.
(5) The movable cone of a smooth projective hyperkähler manifold is well clipped by [30, Lemma 6.22] and [31, Theorem 1.1].
(6) The movable cone of a projective primitive symplectic variety with terminal $\mathbb{Q}$-factorial singularities is well clipped [25, Theorem 3.10, Lemma 4.6]; [19, Definition 3.1, Theorem 3.11].
(7) The movable cone of an abelian variety is self-dual homogeneous, thus well clipped [41, Theorem 4.3 and the two paragraphs thereafter].
(8) The movable cones of the Wehler varieties introduced in [8] are of hyperbolic type, thus well clipped.
(9) The nef cone of $\mathbb{P}^2$ blown up at 9 general points and the movable cone of $\mathbb{P}^3$ blown up at 8 very general points are both well clipped, by [35] and [45] respectively.
(10) The movable cone of a product of normal projective varieties
$$X = \prod_{1\leq i\leq k} X_i,$$
with $h^1(\mathscr{O}_{X_i}) = 0$ for $2 \leq i \leq k$ is well clipped if and only if the movable cone of each factor is well clipped. This follows from [16, Exercise III.12.6], Lemmas 2.16 and 3.5.

We prove results backing up Item (4) in Example 3.6 above.

Lemma 3.7. *Let S be a smooth projective surface with $\chi(\mathscr{O}_S) \geq 0$. The cone $\mathrm{Nef}(S)$ is well clipped.*

Proof. We use the intersection form on S to define a hyperbolic cone in the Néron–Severi space $N^1(S)$. By duality between the nef and the Mori cones, it suffices for us to prove that any reduced irreducible curve E of negative square on S induces an integral orthogonal reflection. Showing that $E^2 \in \{-1, -2\}$ would conclude by Remark 3.4. Since S is smooth, E^2 is an integer.

Let E be a reduced irreducible curve of negative square on S. We apply the Riemann-Roch formula

$$h^1(\mathscr{O}_E) = 1 - \chi(\mathscr{O}_S) + \frac{1}{2}E^2 - \frac{1}{2}K_S \cdot E.$$

By contradiction, assume that $E^2 \leq -3$. Then $-K_S \cdot E > 0$. Since E spans an extremal ray of the Mori cone of S, the cone theorem gives an elementary Mori contraction $\varepsilon : S \to S'$ with exceptional divisor E. Computing discrepancies, one checks that E is contracted to a terminal, hence smooth point of S'. Castelnuovo's contraction theorem applies and E has square -1 in S, contradiction. □

The next corollary is easy to derive.

Corollary 3.8. *Let S be a smooth projective surface. If there exists a klt Calabi–Yau pair (S, Δ), then the cone $\mathrm{Nef}(S)$ is well clipped.*

Proof. Running a K_S-MMP, using the criterion of uniruledness by [6], the MRC fibration and the fact that a rationally connected surface is rational, it is easy to check that S is birational to one of the following:

(1) a smooth surface Σ with $K_\Sigma \equiv 0$;
(2) a smooth surface Y that is a $\mathbb{P}^1$-bundle over an elliptic curve E;
(3) a smooth rational surface X.

In these cases, we respectively get

(1) By the Beauville–Bogomolov decomposition [3], either $\chi(\mathscr{O}_\Sigma) \geq 0$ or $S = \Sigma$ is a bielliptic surface. Lemma 3.7 and the fact that bielliptic surfaces have Picard rank 2 respectively conclude.
(2) The 1-forms on Y come from the elliptic curve, so $\chi(\mathscr{O}_Y) = 1 - 1 + h^0(K_Y) \geq 0$.
(3) Since X is rational, $\chi(\mathscr{O}_X) = 1$.

□

3.B. **Non-examples and questions on well-clipped cones.** In this subsection, we describe various cones that cannot be well clipped. Although we personally find these examples instructive, they are independent of the main results of this paper, and can be skipped in a first read.

Example 3.9. We construct a rational polyhedral cone that is not well clipped.

Consider the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$, and pick thirteen distinct points $v_1, \dots, v_{13}$ on it in that order. We take their convex hull and projectivize, to obtain a polyhedral cone $\mathscr{C}$ with 13 extremal rays in $\mathbb{R}^3$.

Arguing by contradiction, assume that $\mathscr{C}$ is well-clipped in some self-dual homogeneous cone $\mathscr{A}$. Since $\mathscr{C}$ is polyhedral in $\mathbb{R}^3$ and not simplicial, $\mathscr{A}$ must be of hyperbolic type. Let q be the hyperbolic quadratic form defining $\mathscr{A}$. Pick an affine hyperplane H in which q restricts to the equation of a circle in $\mathbb{R}^2$. Denote by $\ell_1, \dots, \ell_{13}$ the lines in H that define the sides of the 13-gone $\mathscr{C} \cap H$. Note that in H, the oriented angle measures between these lines satisfy

$$\cos^2(\ell_i, \ell_j) = \frac{q(e_i, e_j)^2}{q(e_i)q(e_j)},$$

since ℓ_i is the polar line to the point e_i with respect to the circle defined by q in H. By Lemma 3.13 below, this implies that

$$(\ell_i, \ell_j) \in \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}\right\}.$$

Summing the angles of the 13-gone $\mathscr{C} \cap H$ and writing $\ell_{14} := \ell_1$, this yields

$$11\pi = \sum_{i=1}^{13} (\ell_i, \ell_{i+1}) \leq 13 \cdot \frac{5\pi}{6},$$

a contradiction.

Example 3.10. We describe the movable cone of a smooth Calabi–Yau threefold that is not well clipped, but nonetheless satisfy the Morrison-Kawamata cone conjecture. Let $X = S_1 \times_{\mathbb{P}^1} S_2$ be a Schoen threefold, as in [43, 36, 14]. It satisfies the nef cone conjecture by [14], as well as the movable cone conjecture by [12]. We claim that neither the nef cone, nor the movable cone of X are well clipped.

Recall that $N^1(X) = p_1^* N^1(S_1) + p_2^* N^1(S_2)$, where $p_i : X \to S_i$ denote the two projections. Recall that there also as a fibration $f : X \to \mathbb{P}^1$ that factors throught both p_i, and satisfies

$$p_1^* N^1(S_1) \cap p_2^* N^1(S_2) = f^* N^1(\mathbb{P}^1).$$

Here, we have $\rho(X) = 19$ and $\rho(S_1) = \rho(S_2) = 10$. Recall also thatthe cones $\mathrm{Nef}(X)$ and $\mathrm{Mov}(X)$ are locally rational polyhedral at a non-zero boundary point v if and only if $v \notin p_1^* N^1(S_1) \cup p_2^* N^1(S_2)$. Let $\mathscr{C}$ denote the nef or the movable cone of X and denote by $\mathfrak{S}$ the set of boundary points swhere $\mathscr{C}$ is not locally rational polyhedral, i.e.,

$$\mathfrak{S} = \partial\mathscr{C} \cap (p_1^* N^1(S_1) \cup p_2^* N^1(S_2)) \setminus \{0\}.$$

Arguing by contradiction, we assume that $\mathscr{C}$ is well clipped in a self-dual homogeneous cone $\mathscr{A}$, which decomposes as

$$\mathscr{A} = \bigoplus_{j \in J} \mathscr{A}_j.$$

If one of the $\mathscr{A}_j$ has dimension 3 or higher, then Theorem 2.3 shows that $\partial\mathscr{A}_j$ is nowhere locally rational polyhedral. If moreover $\mathscr{A}_j$ is not of hyperbolic type, this provides an open set of the boundary $\partial\mathscr{C}$ that is contained in $\mathfrak{S}$, a contradiction. Hence each $\mathscr{A}_j$ is a halfline or a hyperbolic cone.

Let $v \in \mathfrak{S}$. We decompose $v = \sum_{j \in J} v_j$ with $v_j \in \overline{\mathscr{A}_j} \cap \mathscr{C}$ and see that there exists $k \in J$ for which $\mathscr{A}_k$ is of hyperbolic type and $q_k(v_k) = 0$. We then note that near any point of

$$\mathbb{R}_{>0}\, v_k \oplus \bigoplus_{j \in J \setminus \{k\}} \overline{\mathscr{A}_j} \cap \mathscr{C},$$

the cone $\mathscr{C}$ is not locally rational polyhedral. In particular, this provides a subspace of $\mathfrak{S}$ containing v_k of dimension $19 - \dim \mathscr{A}_k + 1$, thus $\dim \mathscr{A}_k \geq 10$.

We apply this inequality for two points v as above, one in each of the linear subspaces $p_i^* \mathrm{Nef}(S_i) \setminus f^* \mathrm{Nef}(\mathbb{P}^1)$ contained in the boundary of $\mathscr{C}$. This provides two indices $k_1 \neq k_2$ such that $\dim \mathscr{A}_{k_i} \geq 10$. But $10 + 10 > 19$, a contradiction.

An important fact to keep in mind, mentioned by Totaro in [48, Section 3], is how Assumption (ii) of Definition 3.1 tends to fail in presence of canonical surface singularities. We present an example inspired by [48, Section 3].

Example 3.11. Let E, F be elliptic curves, let $\Sigma = E \times F/\langle -1\rangle$. Let S be the partial minimal resolution of Σ obtained by resolving all A_1-singularities except for one, say the image of $0 \in E \times F$. The surface S is a K3 surface with canonical singularities in the sense of [15].

Let C denote the strict transform in S of the image of $E \times \{0\}$ in Σ. It is is a smooth rational curve passing through a single A_1-singular point of S, thus $C^2 = -\frac{3}{2}$. Meanwhile, the exceptional divisor D_p over the image of a 2-torsion point $(p, 0)$ satisfies $C \cdot D_p = 1$. The class of C spans an extremal ray of the Mori cone $\overline{\mathrm{NE}}(S)$, and therefore defines a necessary hyperplane of the nef cone $\mathrm{Nef}(S)$ (see Remark 3.2). However, the unique orthogonal reflection with respect to that hyperplane sends the Cartier divisor D_p to $D_p + \frac{4}{3}C$, which is not integral.

The next question, if it had an affirmative answer, would be a great motivation to try and understand cones of positive divisor arising from algebraic geometry in terms of well-clipped cones. However, there seems to me to be no reason to expect a positive answer to it, even for movable cones in dimension 3.

Question 3.12. Is the movable cone of a smooth projective Fano variety well clipped?

3.C. **Fundamental domains and well-clipped cones.** We start with a simple lemma.

Lemma 3.13. *Let $\mathscr{C}$ be a cone in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$ that is well clipped in some self-dual homogeneous cone $\mathscr{A}$ with quadratic form q by hyperplanes $(H_i)_{i \in I}$. Then, for every $i, j \in I$ such that $H_i \neq H_j$, we have*

$$\frac{q(e_i, e_j)^2}{q(e_i, e_i)q(e_j, e_j)} \in \left\{\cos^2\left(\frac{\pi}{n_{ij}}\right) \mid n = 2, 3, 4, 6, \infty\right\} \cup (1, \infty).$$

Proof. By Assumptions (ii) and (iii) of Definition 3.1 and Remark 3.4, we see that

$$\frac{q(e_i, e_j)^2}{q(e_i, e_i)q(e_j, e_j)} \in \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\} \cup (1, \infty).$$

Checking the squares of the appropriate values of cosine concludes the proof. □

Corollary 3.14. *Let $\mathscr{C}$ be a well clipped cone in a self-dual homogeneous cone $\mathscr{A}$ in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Then the cone $\overline{\mathscr{C}}$ is a fundamental domain for the action of the group generated by the orthogonal reflections $(\sigma_i)_{i \in I}$ with respect to the side hyperplanes of $\mathscr{C}$ on the cone $\mathscr{A}^+$.*

Proof. We first check the disjoint interior property. Let $w \in \langle \sigma_i \mid i \in I\rangle$ be such that $\mathscr{C}^\circ \cap w(\mathscr{C}^\circ) \neq \varnothing$. Note that w can be written as the product of finitely many reflections within our set of generators, say $w \in \langle \sigma_1, \dots \sigma_m\rangle$. We pick a rational polyhedral cone $\Pi \subset \overline{\mathscr{C}}$ such that

- The hyperplanes fixed by $\sigma_1, \dots, \sigma_m$ support the cone Π along faces of codimension 1.
- We have $\Pi^\circ \cap w(\Pi^\circ) \neq \varnothing$.

By Lemma 3.13, we can apply [49, Theorem 1, Proposition 6] to the cone Π. We obtain that $w = 1$, which concludes the proof of the disjoint interior property.

We now check the covering property. We fix a point $c \in \mathscr{C}^\circ$. For $a \in \mathscr{A}^+$, the line segment joining a and c is compact. Since the reflection group $W = \langle \sigma_i \mid i \in I \rangle$ is discrete, that line segment only crosses finitely many of the hyperplanes $(P_j)_{j \in J}$ that are fixed by some hyperplane orthogonal reflection of W. We denote by N_a the number of hyperplanes crossed (if a is contained in some of the hyperplanes, we do not count them). We now fix $a \in \mathscr{A}^+$, and choose an element m in the orbit $W \cdot a$ that minimizes N_m.

We claim that $N_m = 0$. Assume by contradiction that $N_m \geq 1$. Then, we can take a crossing hyperplane P_j for the segment $(m, c]$ and denote by $u_j \in P_j^-$ the generator of $P_j^\perp \cap V_{\mathbb{Z}}$. Consider the orthogonal reflection $r_j \in W$ with respect to P_j. Drawing in the plane containing $c, m, r_j(m)$, we have a triangle in $\mathbb{R}^2$. The crossing hyperplanes restrict to lines in $\mathbb{R}^2$; Note that each line must be disjoint from the triangle, contain a vertex of the triangle, or intersect exactly two sides of the triangle in their relative interiors. We deduce three facts:

- The hyperplane P_j does not cross the segment $(r_j(m), c]$.
- If P_k is a hyperplane that does not cross $(m, c]$ but crosses $(r_j(m), c]$, then $q(r_j(u_k), r_j(m)) = q(u_k, m) > 0$ and $q(r_j(u_k), m) = q(u_k, r_j(m)) < 0$, hence the hyperplane $r_j(P_k)$ crosses the segment $(m, c]$ and does not cross the segment $(r_j(m), c]$.
- Conversely, if P_k is a hyperplane that does not cross $(r_j(m), c]$ but crosses $(m, c]$, then $r_j(P_k)$ crosses the segment $(r_j(m), c]$.

The fact that r_j is an involution concludes that $N_{r_j(m)} = N_m - 1$, a contradiction. Since $N_m = 0$, the point m belongs to $\overline{\mathscr{C}}$. This shows the covering property. □

We introduce the notions of a *neatly clipped* and a *perfectly clipped* cone.

Definition 3.15. Let $V = V_{\mathbb{Z}} \otimes \mathbb{R}$ be a finite dimensional vector space and let $\mathscr{A}$ be a self-dual homogeneous cone in V. We say that a full-dimensional non-degenerate convex cone $\mathscr{C}$ is

- *neatly clipped in* $\mathscr{A}$ if $\mathscr{C}$ is well clipped in $\mathscr{A}$, and if there is a subgroup W in $\mathrm{Aut}(\mathscr{A}, V_{\mathbb{Z}})$ such that
  (i) for any $w \in W \setminus \{1\}$, $w(\mathscr{C}^\circ) \cap \mathscr{C}^\circ = \varnothing$;
  (ii) the group W is preserved by conjugation under a subgroup
  $$\Gamma_{\mathscr{C}} < \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}});$$
  (iii) the subgroup $\Gamma = W \rtimes \Gamma_{\mathscr{C}}$ is an arithmetic subgroup of $\mathrm{Aut}(\mathscr{A})$;
  (iv) the subgroup Γ in (iii) contains the reflection group $\langle \sigma_i \mid i \in I \rangle$.
- *perfectly clipped in* $\mathscr{A}$ if $\mathscr{C}$ is well clipped in $\mathscr{A}$, and the orthogonal reflections induced by the hyperplanes delimiting $\mathscr{C}$ are such that
  $$\langle \sigma_i \mid i \in I \rangle \rtimes \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$$
  is an arithmetic subgroup of $\mathrm{Aut}(\mathscr{A})$.

Remark 3.16. Note that a perfectly clipped cone is always neatly clipped with $W = \langle \sigma_i \mid i \in I \rangle$ and $\Gamma_{\mathscr{C}} = \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$. Indeed, the disjoint interior property follows from Corollary 3.14, and the semidirect product structure in the definition of perfectly clipped cone is naturally induced by the fact that $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$ preserves the set of hyperplanes $(H_i)_{i \in I}$.

Let us take the list of Example 3.6, and note that all of those well-clipped cones are in fact perfectly clipped.

Example 3.17. There are many instances of perfectly clipped cones.

(1+7+8) A self-dual homogeneous cone is always perfectly clipped. In particular, any simplicial cone is perfectly clippped, and any non-degenerate convex cone in $\mathbb{R}^2$ is perfectly clipped. For the same reason, the movable cones of abelian varieties and of Wehler varieties are perfectly clipped.

(2+3+4) The nef cone of any smooth surface underlying a klt Calabi–Yau pair is perfectly clipped by Proposition 3.18 below and by the cone conjecture in dimension 2 proven by Totaro [48].

(5) The movable cone of a smooth projective hyperkähler manifold is perfectly clipped by the work of Markman [30, Theorem 6.18 (4+5), Lemma 6.23], notably thanks to the Hodge theoretic Torelli theorem [30, Theorem 1.3]. Please note the words of caution in [30, Paragraph following Definition 1.1, Caution 6.19] and the counterexamples [9, 37, 29], that convinced us to work on descending the movable (rather than the nef) cone conjecture and to take arithmetic subgroups in Definition 3.15 rather than the whole group $\mathrm{Aut}(\mathscr{A}, V_{\mathbb{Z}})$ (compare with [39]).

(6) The movable cone of a projective primitive symplectic variety X with terminal $\mathbb{Q}$-factorial singularities is perfectly clipped. It is clear for $b_2(X) \leq 4$ by Item (1) above, and follows for $b_2(X) \geq 5$ from the work of Bakker-Lehn [2, Theorem 8.2] and Lehn-Mongardi-Pacienza [25, Theorem 5.12], using notably a Torelli theorem [25, Theorem 4.9].

(9) The nef cone of $\mathbb{P}^2$ blown up at 9 general points and the movable cone of $\mathbb{P}^3$ blown up at 8 very general points are both perfectly clipped, by Proposition 3.18 below and by [35] and [45] respectively.

(10) A direct sum of perfectly clipped cones remains perfectly clipped. The proof is essentially the same as that of Lemma 3.5.

Our motivation to introduce these notions of neatly and perfectly clipped cones is the following proposition, which is inspired by the work of Ash [1, Chapter II], Sterk [46], Markman [30], Looijenga [27, Example 4.8], and Lehn–Mongardi–Pacienza [25]. One can view it as a characterization of those well-clipped cones which admit rational polyhedral fundamental domains under some group action.

Proposition 3.18. *Let $\mathscr{C}$ be a well-clipped cone in a self-dual homogeneous cone $\mathscr{A}$ in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Then the following statements are equivalent*

(i) $\mathscr{C}$ is perfectly clipped in $\mathscr{A}$;

(ii) $\mathscr{C}$ is neatly clipped in $\mathscr{A}$;

(iii) there is a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$ on the cone $\mathscr{C}^+$.

In order to prove Proposition 3.18, we start with a technical lemma.

Lemma 3.19. *Let $\mathscr{C}$ be a well-clipped cone in a self-dual homogeneous cone $\mathscr{A}$ in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Then for any arithmetic group $\Gamma < \mathrm{Aut}(\mathscr{A}, V_{\mathbb{Z}})$ that contains the group spanned by the orthogonal reflections $\langle \sigma_i \mid i \in I \rangle$, there exists a rational polyhedral cone $\Pi \subset \mathscr{C}^+$ such that*

(1) We have

$$\mathscr{C}^+ = \bigcup_{\gamma \in \mathfrak{S}} \gamma(\Pi),$$

where $\mathfrak{S}$ denotes the set $\{\gamma \in \Gamma \mid \gamma(\mathscr{C}^\circ) \cap \mathscr{C}^\circ \neq \varnothing\}$.

(2) For $\gamma \neq \gamma' \in \mathfrak{S}$, it holds that $\gamma(\Pi^\circ) \cap \gamma'(\Pi^\circ) = \varnothing$.

If moreover $\mathscr{C}$ is neatly clipped in $\mathscr{A}$, then

(3) *There is a rational polyhedral fundamental domain for the action of the group* $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$ *on* $\mathscr{C}^+$*. Additionally, any group* $\Gamma_{\mathscr{C}}$ *that may appear in Definition 3.15 for the cone* $\mathscr{C}$ *is of finite index in* $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$.

Proof. We start with the self-dual homogeneous cone $\mathscr{A}$. Let Γ be an arithmetic subgroup of $\mathrm{Aut}(\mathscr{A})$ that contains all the orthogonal reflections $(\sigma_i)_{i\in I}$ with respect to the hyperplanes $(H_i)_{i\in I}$ cutting out $\mathscr{C}$. By the work of Vinberg on reduction theory and Siegel sets, and by [1, Chapter II, Theorem 4.1] and [27, Proposition 4.2], there is a rational polyhedral fundamental domain for the action of Γ on the cone $\mathscr{A}^+$. By [27, Application 4.14], we can perform the following construction *à la* Dirichlet–Voronoi: If we choose a point $a \in \mathscr{A} \cap V_{\mathbb{Q}}$ with trivial stabilizer in Γ, the cone

$$\Pi := \{x \in \mathscr{A}^+ \mid \forall \gamma \in \Gamma,\ q(x, \gamma(a)) \geq q(x, a)\}$$

is a rational polyhedral fundamental domain for the action of Γ on $\mathscr{A}^+$. From here on, we fix a to be a point in $\mathscr{C}^\circ \cap V_{\mathbb{Q}}$ that has trivial stabilizer in Γ, and whose Γ-orbit avoids the hyperplanes $(H_i)_{i\in I}$ used to cut out $\mathscr{C}$. (Its existence essentially follows from [27, Theorem 3.8], see [13, Page 8, Proof of Proposition 2.3] and [26, Page 13, Proof of Lemma 3.5, Last Paragraph] for more details.)

We claim that $\Pi \subset \mathscr{C}^+$. Fix $x \in \Pi$. We have

$$q(x, \sigma_i(a)) - q(x, a) = \frac{-2q(a, e_i)q(x, e_i)}{q(e_i, e_i)} \geq 0,$$

which implies, since $a \in \mathscr{C}^\circ$, that $q(x, e_i) \geq 0$. Therefore, we have that $x \in \overline{\mathscr{C}}$. This shows that $\Pi \subset \overline{\mathscr{C}}$, and since Π is rational polyhedral, it proves our claim.

The same argument shows, more generally, that for any $\gamma \in \Gamma$ satisfying $\gamma(a) \in \mathscr{C}^\circ$, we have $\gamma(\Pi) \subset \mathscr{C}^+$. We also note that, for any $\gamma \in \Gamma$ such that $\gamma(a) \notin \mathscr{C}^\circ$, by our assumptions on the point a, there exists $i \in I$ such that $q(\gamma(a), e_i) < 0$, and thus $\gamma(\Pi)$ is contained in the closed halfspace $\overline{H_{i,-}}$. In particular, the intersection $\gamma(\Pi) \cap \mathscr{C}^\circ$ is empty. This is an interesting dichotomic behavior between translates of Π contained in $\mathscr{C}^+$, and translates of Π disjoint from $\mathscr{C}^\circ$.

To conclude this proof, we need to understand whether/how translates of Π cover the cone $\mathscr{C}^+$. Let $x \in \mathscr{C}^\circ$. Since Π is a fundamental domain for the action of Γ on $\mathscr{A}^+$, we can find an element $\gamma \in \Gamma$ such that $\gamma(x) \in \Pi$. This yields that $\gamma^{-1}(\Pi) \cap \mathscr{C}^\circ \neq \varnothing$, hence by the dichotomic behavior explained above, we must have $\gamma^{-1}(\Pi) \subset \mathscr{C}^+$, and $\gamma^{-1}(a) \in \mathscr{C}^\circ$. This shows that

$$\mathscr{C}^+ = \bigcup_{\gamma \in \mathfrak{S}} \gamma(\Pi),$$

where $\mathfrak{S}$ denotes the set $\{\gamma \in \Gamma \mid \gamma(a) \in \mathscr{C}^\circ\}$. This shows Point 1. Point 2 is clear by definition of Π, since a has trivial Γ-stabilizer.

We finally prove Point 3. Assume that $\mathscr{C}$ is neatly clipped, and take Γ to be an arithmetic subgroup that also writes as in Definition 3.15

$$\Gamma = W \rtimes \Gamma_{\mathscr{C}},$$

for some subgroup $\Gamma_{\mathscr{C}} < \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$, and for a group W such that non-trivial W-translates of $\mathscr{C}^\circ$ avoid $\mathscr{C}^\circ$. Note that for $\gamma \in \Gamma$, we have $\gamma(\mathscr{C}^\circ) \cap \mathscr{C}^\circ \neq \varnothing$ if and only if $\gamma \in \Gamma_{\mathscr{C}}$. So $\mathfrak{S} = \Gamma_{\mathscr{C}}$ is a subgroup of $\mathrm{Aut}(\mathscr{A}, V_{\mathbb{Z}})$, and Π transparently is a rational polyhedral fundamental domain for its action. Since $\Gamma_{\mathscr{C}}$ is contained in $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$, there also is a rational polyhedral fundamental domain for the action of the larger group $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_\mathbb{Z})$ on $\mathscr{C}^+$, see for instance [27, Proposition 4.1, Application 4.14]. The fact that $\Gamma_\mathscr{C}$ has finite index in $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_\mathbb{Z})$ is then a consequence of the general fact [27, Proposition 4.6]. □

We can now prove Proposition 3.18.

Proof of Proposition 3.18. By Remark 3.16, Item (i) implies Item (ii). By Lemma 3.19, Item (ii) implies Item (iii).

We now assume Item (iii) and prove Item (i). By the work of Vinberg on reduction theory and Siegel sets, and by [1, Chapter II, Theorem 4.1] and [27, Proposition 4.2], there is a rational polyhedral fundamental domain for the action of the arithmetic group $\mathrm{Aut}(\mathscr{A}, V_\mathbb{Z})$ on the cone $\mathscr{A}^+$. Thus and by [27, Proposition 4.6], the cone $\mathscr{C}$ is perfectly clipped if and only if there also is a rational polyhedral fundamental domain for the action of the group

$$\Gamma := \langle \sigma_i \mid i \in I \rangle \rtimes \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_\mathbb{Z})$$

on the cone $\mathscr{A}^+$. By Item (iii), we have a rational polyhedral fundamental domain $\Pi \subset \mathscr{C}^+$ for the action of $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_\mathbb{Z})$. Since $\mathscr{C}$ is well clipped in $\mathscr{A}$, we have $\mathscr{C}^+ = \overline{\mathscr{C}} \cap \mathscr{A}^+$. Thus, by Corollary 3.14, Π also is a rational polyhedral fundamental domain for the action of Γ on $\mathscr{A}^+$, which shows Item (i). □

4. Descent under finite quotients

4.A. **Descent of self-dual homogeneous cones.** This result is inspired by the work of Monti–Quedo [33]. It is surprisingly hard to write down, for a self-dual homogeneous cone $\mathscr{A}$ and a finite subgroup G of $\mathrm{Aut}(\mathscr{A})$, an explicit group that acts transitively on the cone $\mathscr{A}^G$; That is without the powerful machinery of formally real Jordan algebras and of Theorem 2.12. Our proofs in this subsection highlight this fact. We make the choice to completely forego the use of the classification presented in Theorem 2.3.

Lemma 4.1. *Let $\mathscr{A}$ be a self-dual homogeneous cone in V. Let G be a subgroup of $\mathrm{Aut}(\mathscr{A})$ acting with finite orbits. The invariant cone $\mathscr{A}^G \subset V^G$ is self-dual homogeneous. Moreover, if $\mathscr{A}$ is compatible with a rational structure $V_\mathbb{Q}$ on V, then $\mathscr{A}^G$ is compatible with the rational structure $V_\mathbb{Q}^G$. Furthermore, if $\mathscr{A}$ is of hyperbolic type, then $\mathscr{A}^G$ is a halfline, the direct sum of two halflines, or a cone of hyperbolic type.*

Proof. Since $\mathscr{A}$ is stable by sum and G-invariant, its contains the finite sum of all elements of a given G-orbit. Thus, the invariant cone $\mathscr{A}^G$ is clearly non-empty. By Lemma 2.6, we take an $\mathrm{Aut}(\mathscr{A}, V_\mathbb{Z})$-invariant quadratic form tr on V with respect to which $\mathscr{A}$ is self-dual. Since it is G-invariant, its restriction to V^G remains positive definite and the cone $\mathscr{A}^G$ remains self-dual with respect to it.

Let us take $e \in \mathscr{A}^G$. By Theorem 2.12, there is a formally real Jordan algebra $(V, e, \circ)$ associated to the pointed cone $(\mathscr{A}, e)$. Since $(\mathscr{A}, e)$ is G-invariant, so is that Jordan algebra. Restricting the operation, we obtain the invariant formally real Jordan algebra $(V^G, e, \circ)$ contained in the initial one. By Theorem 2.12, we get a self-dual homogeneous cone $\mathscr{B}$ in V^G, also pointed at e, that is contained in $\mathscr{A}$. We have $\mathscr{B} \subset \mathscr{A}^G$, and both being self-dual with respect to the same quadratic form $\mathrm{tr}|_{V^G}$, that inclusion is an equality. So $\mathscr{A}^G$ is a self-dual homogeneous cone.

Note that $\mathscr{A}$ is compatible with the rational structure $V_\mathbb{Q}$ if and only if the corresponding Jordan operation $\circ$ (for a choice of $e \in \mathscr{A} \cap V_\mathbb{Q}$) restricts to an operation on $V_\mathbb{Q}$. This proves the "Moreover" part of this lemma.

Now, if $\mathscr{A}$ is of hyperbolic type, there is a hyperbolic quadratic form q on V such that $\partial\mathscr{A} \subset \{v \in V \mid q(v) = 0\}$. The boundary of $\mathscr{A}^G$ is also contained in the zero locus of the restriction $q|_{V^G}$. If $\dim V^G \leq 2$, the cone $\mathscr{A}^G$ is a direct sum of one or two halflines, as wished. Assume now that $\dim V^G \geq 3$. If $q|_{V^G}$ is a hyperbolic quadratic form, then we are done. It is not negative definite since $q(e) > 0$, thus it must degenerate. Let $v \in V^G$ be such that $q(v, V^G) = \{0\}$. Since q is G-invariant, we have that

$$|G \cdot w| \cdot q(v, w) = q\left(v, \sum_{g \in G/\mathrm{Stab}(w)} g(w)\right) = 0$$

for any $w \in V$, which contradicts the fact that q is non-degenerate. This concludes. □

Remark 4.2. The proof of Lemma 2.14 involves fixing a G-invariant element e in $\mathscr{A}$. This is essentially the same choice as the one made by Monti–Quedo in [33, Page 13, Paragraph below Lemma 6.6] of a G-invariant polarization on the abelian variety with a G-action.

Lemma 4.3. *Let $\mathscr{A}$ be a self-dual homogeneous cone in V. Let G be a subgroup of* $\mathrm{Aut}(\mathscr{A})$ *acting with finite orbits. Then, the centralizer $C_{\mathrm{Aut}(\mathscr{A})}(G)$ acts transitively on the cone $\mathscr{A}^G$.*

Proof. Let $e, a \in \mathscr{A}^G$. Consider the Jordan algebra $(V, e, \circ)$ associated to the cone $\mathscr{A}$ pointed at e by Theorem 2.12. For $b \in V$, we denote by $L(b)$ the left-multiplication operator by b with respect to the algebra law $\circ$. Following [11, Page 30, Before Proposition II.2.3], we say that an element $b \in V$ is invertible if there exists $y \in \mathbb{R}[b]$ such that $b \circ y = e$. By [11, Proposition II.3.1], if b is invertible, then the quadratic representation $Q(b) := 2L(b)^2 - L(b^2)$ is an invertible linear endomorphism of V. By [11, Proposition III.2.2], the operator $Q(b)$ also preserves the cone $\mathscr{A}$.

Recall that as in the proof of Lemma 4.1, the cone $\mathscr{A}^G$ pointed at e corresponds to the Jordan subalgebra $(V^G, e, \circ)$, and we have

$$\mathscr{A}^G = \{b \circ b \mid b \in V^G \text{ invertible}\},$$

see also [11, Page 48, Last line of Proof of Theorem III.2.1]. In particular, we can take $b \in V^G$ invertible element such that $a = b \circ b$. Then $Q(b)$ belongs to $\mathrm{Aut}(\mathscr{A})$ and sends e to a. The fact that $Q(b)$ commutes with any $g \in G$ follows from the fact that both b and $\circ$ are perserved by the G-action. □

4.B. **Descent of well-clipped cones.** The next proposition is inspired by the work of Oguiso–Sakurai [39]. Before stating it, we need a definition.

Definition 4.4. Let V be a finite dimensional $\mathbb{R}$-vector space, and let $G < \mathrm{GL}(V)$ be a subgroup. Then the invariant subspace V^G is preserved by the action of the normalizer subgroup $N_{\mathrm{GL}(V)}(G)$. We denote by

$$\rho^G : N_{\mathrm{GL}(V)}(G) \to \mathrm{GL}(V^G)$$

the induced representation.

Let us justify the preservation of the invariant subspace in this definition.

Proof. Let $h \in N_{\mathrm{GL}(V)}(G)$ and $v \in V^G$. Let $g \in G$. We have $gh(v) = h(h^{-1}gh)(v) = h(v)$, using the fact that $h^{-1}gh \in G$. This shows that $h(v) \in V^G$, as wished. □

Proposition 4.5. *Let $\mathscr{A}$ be a self-dual homogeneous cone in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Let G be a subgroup of $\operatorname{Aut}(\mathscr{A}, V_{\mathbb{Z}})$ acting with finite orbits. Let $\mathscr{C}$ be a well-clipped cone in $\mathscr{A}$, cut out by hyperplanes $(H_i)_{i\in I}$. Then there exist a subset $I^* \subset I$ and a self-dual homogeneous cone $\mathscr{B} \subset V^G = V_{\mathbb{Z}}^G \otimes \mathbb{R}$ with the same round part as $\mathscr{A}^G$, whose simplicial part is ruled by the simplical part of $\mathscr{A}^G$, such that*

1. *The invariant cone $\mathscr{C}^G$ is well clipped in $\mathscr{B}$, and is cut out by the hyperplanes $h_i = H_i \cap V^G$ indexed by I^*.*
2. *The orthogonal reflection group $\langle \tau_i \mid i \in I^* \rangle$ for the hyperplanes $(h_i)_{i\in I^*}$ is contained in the image $\rho^G(C_{\langle\sigma_i | i\in I\rangle}(G))$ of the centralizer of G in the orthogonal reflection group $\langle \sigma_i \mid i \in I\rangle$ for the hyperplanes $(H_i)_{i\in I}$.*

We refer to Definitions 2.7 and 2.9 on round and simplicial parts. In a first read, the reader might want to focus on the conclusion presented as Item (1); It says that being a well-clipped cone descends under finite quotients. The conclusion reached in Item (2) is equally important, but mostly for technical reasons.

The key of the proof is to extract orthogonal reflections on the invariant space V^G out of a large Weyl group of orthogonal reflections on V, on which G acts by permuting the (infinitely many) generators. Merely keeping the orthogonal reflections that are fixed by this action by permutation of G is not enough: We need to build more. This difficulty is already mentioned in [39, 1.4], and the way we overcome it is quite similar to [39, 1.6]. However, we have to work in the framework of convex cones, which is not as nice as working in the Néron-Severi space of a K3 surface (the set-up of [39]) and bans us from using geometric tools like the Riemann–Roch and Noether formulas.

Proof. By Definition 3.1 and Remark 3.2, we can write

$$\mathring{\mathscr{C}} = \mathscr{A} \cap \bigcap_{i\in I} H_{i,+}, \tag{1}$$

where $({H_i}^+)_{i\in I}$ is a minimal set of open halfspaces delimited by rational hyperplanes H_i satisfying Assumptions (i), (ii), (iii). We choose the index set I to be minimal is in Remark 3.2. For $i \in I$, we set $e_i \in H_{i,-}$ to be the generator of $H_i^\perp \cap V_{\mathbb{Z}}$, and we define $h_i := H_i \cap V^G$, which is a hyperplane in V^G. Note that the finite sum

$$\varepsilon_i := \sum_{g\in G/\operatorname{Stab}(e_i)} g(e_i)$$

is an element of $h_i^\perp \subset V^G$, and of the invariant lattice $V_{\mathbb{Z}}^G$. With the notations of Definition 3.1, we set

$$J_d := \{j \in J \mid \dim {\mathscr{A}_j}^G = d\}$$

for $d \in \mathbb{N}$. We can now define a smaller index set

$$I^* := \{i \in I \mid j(i) \notin J_1 \cup J_2 \text{ and } q(\varepsilon_i) < 0\},$$

and a smaller cone

$$\mathscr{B} := \bigoplus_{j\in J\setminus J_2} {\mathscr{A}_j}^G \oplus \bigoplus_{k\in J_2} \mathscr{B}_k, \quad \text{where for } k \in J_2, \quad \mathscr{B}_k := \mathscr{A}_k^G \cap \bigcap_{i\in j^{-1}\{k\}} h_{i,+}.$$

For $k \in J_2$, the cone $\mathscr{B}_k$ is the interior of a direct sum of two halflines. Thus and by Lemma 4.1, $\mathscr{B}$ is a self-dual homogeneous cone in V^G. It is contained in $\mathscr{A}^G$. It is also worth noting that the following round parts coincide

$$\operatorname{rd} \mathscr{B} = \operatorname{rd} \bigoplus_{j\in J\setminus J_2} {\mathscr{A}_j}^G = \operatorname{rd} \mathscr{A}^G.$$

As for the simplical parts, note that for each $k \in J_2$, the 2-dimensional (thus simplicial) cone $\mathscr{A}_k^G$ rules $\mathscr{B}_k$. Taking direct sums and adding the halflines of the form $\mathscr{A}_j^G$ for $j \in J_1$, it follows that the simplicial part of $\mathscr{A}^G$ rules the simplicial part of $\mathscr{B}$.

We now show that the $(h_i)_{i\in I^*}$ satisfy Assumption (i) of Definition 3.1. Fix $i \in I^*$. Applying Lemma 4.1 to $\mathscr{A}_j$ shows that the cone ${\mathscr{A}_{j(i)}}^G$, which has dimension at least three, must be of hyperbolic type, as wished.

We then show that the $(h_i)_{i\in I^*}$ satisfy Assumption (iii) of Definition 3.1. Fix $i, k \in I^*$ such that $h_i \neq h_k$. Then, by Remark 3.4, it suffices to check that $q(\varepsilon_i, \varepsilon_k) \geq 0$. Since $h_i \neq h_k$, the two orbits $G \cdot e_i$ and $G \cdot e_k$ are distinct, thus disjoint, and thus

$$q(\varepsilon_i, \varepsilon_k) = \sum_{\substack{g \in G/\mathrm{Stab}(e_i) \\ g' \in G/\mathrm{Stab}(e_k)}} q(g(e_i), g'(e_k)) \geq 0,$$

using the fact that the $(H_i)_{i\in I}$ satisfy Assumption (iii).

We now claim that $\mathscr{C}^G$ is cut out by the (possibly non-minimal) set of hyperplanes $(h_i)_{i\in I^*}$ in the cone $\mathscr{B}$. To see this, take Identity (1), intersect with V^G, decompose $\mathscr{A}$ into summands, and regroup indices to see that

$$\mathring{\mathscr{C}}^G = \mathscr{B} \cap \bigcap_{i \in j^{-1}(J_1)} h_{i,+} \cap \bigcap_{\substack{d\geq 3 \\ i \in j^{-1}(J_d)}} h_{i,+}. \tag{2}$$

Note that for $i \in j^{-1}(J_1)$, the cone ${\mathscr{A}_{j(i)}}^G$ is a halfline, so that ${\mathscr{A}_{j(i)}}^G \subset h_{i,+}$. Similarly, assume that $i \in j^{-1}(J_d)$ with $d \geq 3$ is an index satisfying $q(\varepsilon_i) \geq 0$. As shown above, the cone ${\mathscr{A}_{j(i)}}^G$ is of hyperbolic type and in particular, we have $q(\varepsilon_i, v) \geq 0$ for every $v \in {\mathscr{A}_{j(i)}}^G$, so we obtain ${\mathscr{A}_{j(i)}}^G \subset h_{i,+}$. This discussion allows to rewrite Identity (2) as

$$\mathring{\mathscr{C}}^G = \mathscr{B} \cap \bigcap_{i\in I^*} h_{i,+}, \tag{3}$$

and indeed $\mathscr{C}^G$ is cut out by the hyperplanes $(h_i)_{i\in I^*}$ in the cone $\mathscr{B}$.

We are left to show two things: That the $(h_i)_{i\in I^*}$ satisfy Assumption (ii) of Definition 3.1, and that the reflections $(\tau_i)_{i\in I^*}$ belong to the centralizer $\rho^G(C_{\langle \sigma_i | i\in I\rangle}(G))$.

Fix $i \in I^*$. Let $s_i := -q(e_i) \in \mathbb{Z}_{>0}$. Let $\{g_1, \dots g_{d/d_i}\}$ be a set of representatives of the left-cosets $G/\mathrm{Stab}(e_i)$, with $g_1 = 1$. We take note that

$$q(\varepsilon_i) = |G \cdot e_i|\, q(e_i, \varepsilon_i) = |G \cdot e_i| \left(-s_i + \sum_{u=2}^{d/d_i} q(e_i, g_u(e_i)) \right). \tag{4}$$

Since for $u \geq 2$, $g_u(e_i)$ and e_i are distinct, Assumptions (ii) and (iii) in Definition 3.1 show that for $u \geq 2$,

$$q(e_i, g_u(e_i)) \in \frac{s_i}{2}\, \mathbb{Z}_{\geq 0}.$$

In particular, there is at most one $u \geq 2$ such that $q(e_i, g_u(e_i))$ is non-zero, and then it equals $\frac{s_i}{2}$. Furthermore, $q(e_i, \varepsilon_i)$ takes one of the two values $-s_i$, $\frac{-s_i}{2}$.

Since ε_i spans the orthogonal of h_i, we can express the reflection

$$\tau_i : x \in V^G \mapsto x - \frac{2q(e_i, x)}{q(e_i, \varepsilon_i)} \varepsilon_i \in V^G.$$

Since $2q(e_i, V_{\mathbb{Z}}^G) \subset 2q(e_i, V_{\mathbb{Z}}) \subset s_i\mathbb{Z}$, this shows that τ_i is integral with respect to the invariant lattive $V_{\mathbb{Z}}^G$, thus satisfies Assumption (ii) of Definition 3.1.

Let us finally express τ_i as an element of the image of the centralizer $\rho^G(C_{\langle \sigma_i | i \in I\rangle}(G))$. To sum up, we already showed using Inequality (4) that for any element e in the orbit $G \cdot e_i$, there is at most one other element e' in that orbit such that $q(e, e') > 0$, and in that case

$$s_i = -q(e) = -q(e') = -q(e + e') = 2q(e, e'). \tag{5}$$

This allows to decompose the orbit $G \cdot e_i$ into finitely many disjoint subsets (B_v) of size 1 or 2

$$G \cdot e_i = \bigsqcup_{v=1}^{r} B_v,$$

with $e, e' \in B_v$ for some v if and only if $q(e, e') \neq 0$. Since q is G-invariant, this decomposition is preserved by the G-action on $G \cdot e_i$, which is transitive. In particular, all B_v are of the same size, either 1 or 2. To each index v, we associate an element $b_v \in \langle \sigma_i \mid i \in I\rangle$ as follows:

- If $B_v = \{e\}$, we take $b_v = \sigma_e$.
- If $B_v = \{e, e'\}$, we set $b_v = \sigma_e\sigma_{e'}\sigma_e = \sigma_{e'}\sigma_e\sigma_{e'}$, which is given by

$$b_v : x \mapsto x \in V - \frac{2q(e + e', x)}{q(e)}(e + e') \in V.$$

  This can be checked by an explicit computation involving Identity (5) or by a Coxeter group argument in the spirit of Lemma 3.13. Note that this element b_v is fixed by the transposition exchanging e and e'.

Since the B_v are mutually q-orthogonal, it is clear that all of the b_v commute with one another. This ensures that the element $b := \prod_{v=1}^{r} b_v$ is well-defined independently of the order in which the product is taken. In particular, b belongs to the centralizer $C_{\langle \sigma_i | i \in I\rangle}(G)$.

What we claim is that $\tau_i = \rho^G(b)$. Indeed, it is easy to compute the composition of commuting reflections:

- If all B_v are of size 1, we have $q(e_i, \varepsilon_i) = q(e_i)$, and

$$b|_{V^G} : x \in V^G \mapsto x - \frac{2q(e_i, x)}{q(e_i)}\varepsilon_i,$$

  where we recall that ε_i is the sum of the orbit elements in $G \cdot e_i$ (taken once each), and use the fact that $q(e, x) = q(e_i, x)$ for $x \in V^G$ and for any $e \in G \cdot e_i$.
- If all B_v are of size 2, we have $q(e_i, \eta_i) = \frac{1}{2}q(e_i)$, and

$$b|_{V^G} : x \in V^G \mapsto x - \frac{4q(e_i, x)}{q(e_i)}\varepsilon_i.$$

In either case, we have $\tau_i = \rho^G(b)$ as wished, and that concludes this proof. □

4.C. **Centralizer and symmetries of the invariant cone.**

Lemma 4.6. *Let Γ_1 and Γ_2 be two commensurable subgroups of a group Γ, and let G be a subgroup of Γ. Then the centralizers $C_{\Gamma_1}(G)$ and $C_{\Gamma_2}(G)$ are commensurable.*

Proof. It suffices to check that for any finite index subgroup $\Gamma_0 < \Gamma_i$, the inclusion of centralizers $C_{\Gamma_0}(G) < C_{\Gamma_i}(G)$ is of finite index too. But since

$$C_{\Gamma_0}(G) = \Gamma_0 \cap C_{\Gamma_i}(G),$$

it is true (see [47, (3.13)(i)]). □

Lemma 4.7. *Consider a group $\Gamma = W \rtimes Q$ and let G be a subgroup of Q. Then the centralizers satisfy*

$$C_\Gamma(G) = C_W(G) \rtimes C_Q(G).$$

Proof. Let $\gamma \in \Gamma$. We write $\gamma = (w, q)$ with $w \in W$ and $q \in Q$. Take $g \in G$, and view it as $(1, g) \in \Gamma$. The belonging of γ to $C_\Gamma(G)$ is equivalent to

$$(w, qg) = (w, q)(1, g) = (1, g)(w, q) = (g \cdot w, gq) = (gwg^{-1}, gq).$$

This is equivalent to $q \in C_Q(G)$ and $w \in C_W(G)$, as wished. □

The next result is the main point of this section. It compares the centralizer subgroup of a group G acting with finite orbits on a homogeneous cone $\mathscr{A}$ with the symmetries of the invariant cone $\mathscr{A}^G$. Recall that the representation ρ^G was defined in Definition 4.4.

Proposition 4.8. *Let $\mathscr{A}$ be a homogeneous cone in $V = V_\mathbb{Z} \otimes \mathbb{R}$. Let G be a subgroup of $\mathrm{Aut}(\mathscr{A}, V_\mathbb{Z})$ acting with finite orbits. Then the inclusion of groups*

$$\mathrm{Im}(\rho^G : C_{\mathrm{Aut}(\mathscr{A}, V_\mathbb{Z})}(G) \to \mathrm{GL}(V^G)) \quad < \quad \mathrm{Aut}(\mathscr{A}^G, {V_\mathbb{Z}}^G)$$

is of finite index.

Before proving this result, we establish two preparatory lemmas. The first one is a spin-off on Maschke's theorem.

Lemma 4.9. *Let V be a finite dimensional $\mathbb{R}$-vector space and $G < \mathrm{GL}(V)$ be a subgroup acting with finite orbits. Then V^G admits a G-invariant complement $S \subset V$.*

Proof. Let $p : V \twoheadrightarrow V^G$ be any linear projection. We define

$$\overline{p} : v \in V \mapsto \frac{1}{|G \cdot v|} \sum_{w \in G \cdot v} p(w) \in V^G.$$

Clearly, $\overline{p}$ restricts to the identity on V^G. Moreover, for all $g \in G$, we have $\overline{p}(g(v)) = \overline{p}(v)$. Finally, we check that $\overline{p}$ is linear: Since G acts with finite orbits, we have a subgroup inclusion of finite index:

$$H := \mathrm{Stab}(v_1) \cap \mathrm{Stab}(v_2) \quad < \quad \mathrm{Stab}(v_1 + v_2)$$

for any $v_1, v_2 \in V$. This implies:

$$\overline{p}(v_1 + v_2) = \frac{|H|}{|G|} \sum_{g \in G/H} p(g(v_1 + v_2)) = \overline{p}(v_1) + \overline{p}(v_2),$$

as wished. Therefore, $\overline{p}$ is a G-equivariant projection from V to V^G, and its kernel is a G-invariant complement to V^G. □

The following lemma is topological.

Lemma 4.10. *Let $\mathscr{G}$ be a connected Lie group and $\mathscr{H}$ be a closed Lie subgroup of it, both obtained as real points of linear algebraic groups. Let $\mathscr{G}(\mathbb{Z})$ be an arithmetic lattice in $\mathscr{G}$. The image of $\mathscr{G}(\mathbb{Z})$ in the quotient manifold $\mathscr{G}/\mathscr{H}$ is discrete.*

Proof. We work with the Euclidean topology on the real Lie groups. Let (i_n) be a converging sequence in the image of $\mathscr{G}(\mathbb{Z})$ in the quotient $\mathscr{G}/\mathscr{H}$. We claim that it has to be constant after a certain point.

We denote by $f : \mathscr{G} \to \mathscr{G}/\mathscr{H}$ the submersion.

Let U be a small enough open ball centered at the limit of (i_n) and contained in a chart of $\mathscr{G}/\mathscr{H}$ that trivializes the submersion f: We also take a small open ball V in $f^{-1}(U)$ such that $f|_V : V \to U$ is the restriction of a projection $\mathbb{R}^{a+b} \twoheadrightarrow \mathbb{R}^a$. We put the Euclidean distances $d_{\mathscr{G}/\mathscr{H}}$ on $\mathbb{R}^a$ and $d_{\mathscr{G}}$ on $\mathbb{R}^{a+b}$.

For every n large enough, we take a pre-image $g_n \in V$ of i_n such that, for all $m \geq n$, we have

$$d_{\mathscr{G}}(g_n, g_m) < d_{\mathscr{G}/\mathscr{H}}(i_n, i_m) + \frac{1}{m-n}.$$

As a Cauchy sequence in a locally compact space, (g_n) converges. Let $g \in \mathscr{G}$ denote the limit. Write $g_n = u_n h_n$ with $u_n \in \mathscr{G}(\mathbb{Z})$ and $h_n \in \mathscr{H}$. By the theorem of Borel–Harish–Chandra, the arithmetic subgroup $\mathscr{H}(\mathbb{Z}) = \mathscr{G}(\mathbb{Z}) \cap \mathscr{H}$ of $\mathscr{H}$ is cocompact in $\mathscr{H}$. In particular, we may write $h_n = v_n k_n$, with $v_n \in \mathscr{H}(\mathbb{Z})$ and (k_n) taking values in a compact set of representatives in $\mathscr{H}$. Up to extracting a subsequence, we can assume that (k_n) converges to some $k \in \mathscr{H}$.

Note that $(g_n k_n^{-1}) = (u_n v_n)$ converges to gk^{-1} in $\mathscr{G}$. Since for every n, $u_n v_n$ belongs to the discrete subgroup $\mathscr{G}(\mathbb{Z})$, this sequence is eventually constant. Its image in $\mathscr{G}/\mathscr{H}$ is (i_n), which thus is eventually constant too. □

We can now prove Proposition 4.8.

Proof of Proposition 4.8. We consider the centralizer subgroup $C_{\mathrm{Aut}(\mathscr{A})}(G)$ in the Lie group $\mathrm{Aut}(\mathscr{A})$ (see [1, II, Proposition 1.7]), and note that there is an inclusion

$$\rho^G(C_{\mathrm{Aut}(\mathscr{A})}(G)) \quad < \quad \mathrm{Aut}(\mathscr{A}^G). \tag{6}$$

We denote by Q_G the space of corresponding left-cosets. As a quotient of a Lie group by a closed Lie subgroup, Q_G is a smooth manifold [24, Theorem 21.10].

We fix $e \in \mathscr{A}^G$. The transitivity property stated in Lemma 4.3 provides a continuous surjection $\mathrm{Stab}_{\mathrm{Aut}(\mathscr{A}^G)}(e) \twoheadrightarrow Q_G$. Since by [1, II, Proposition 1.7], the stabilizer group $\mathrm{Stab}_{\mathrm{Aut}(\mathscr{A}^G)}(e)$ is compact, we get that Q_G is compact too. By Lemma 4.10 below, the image of $\mathrm{Aut}(\mathscr{A}^G, V_{\mathbb{Z}}^G)$ in Q_G is discrete, hence finite. Therefore, the following inclusion is of finite index

$$\rho^G(C_{\mathrm{Aut}(\mathscr{A})}(G)) \cap \mathrm{GL}(V_{\mathbb{Z}}^G) \quad < \quad \mathrm{Aut}(\mathscr{A}^G, V_{\mathbb{Z}}^G).$$

We claim that the inclusion

$$\rho^G(C_{\mathrm{GL}(V_{\mathbb{Z}})}(G)) \quad < \quad \mathrm{GL}(V_{\mathbb{Z}}^G) \tag{7}$$

also is of finite index. Indeed, since G is a group acting with finite orbits on the rational points of V, by Maschke's theorem (see Lemma 4.9), there is a G-stable complement S, defined over $\mathbb{Q}$, such that $V = V^G \oplus S$. Denoting by $S_{\mathbb{Z}}$ the lattice induced by $V_{\mathbb{Z}}$ in S, we note that the inclusion $V_{\mathbb{Z}}^G \oplus S_{\mathbb{Z}} \subset V_{\mathbb{Z}}$ is of finite index. Computing with block matrices, it is also easy to check that

$$C_{\mathrm{GL}(V_{\mathbb{Z}}^G \oplus S_{\mathbb{Z}})}(G) = \mathrm{GL}(V_{\mathbb{Z}}^G) \times C_{\mathrm{GL}(S_{\mathbb{Z}})}(G),$$

which, after applying ρ^G and using Lemma 4.6, concludes. □

4.D. **Descent of perfectly clipped cones and more.** Before proving a descent result for perfectly clipped cones, we have to address the apparition of an auxiliary self-dual homogeneous cone, *a priori* smaller than the invariant cone, in the descent result for well-clipped cones given by Proposition 4.5. This is what the discussion of round and simplicial parts prepared us to, in particular the following lemma.

Lemma 4.11. *Let $\mathscr{C}$ be a well-clipped cone in a self-dual homogeneous cone $\mathscr{B}$ in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Let $\mathscr{A}$ be a self-dual homogeneous cone in V with the same round part as $\mathscr{B}$. Then $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A})$ is contained in $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{B})$. If moreover the simplical part of $\mathscr{A}$ rules the simplicial part of $\mathscr{B}$, then $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$ is of finite index in $\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{B}, V_{\mathbb{Z}})$, and $\mathrm{Aut}(\mathscr{A}, \mathscr{B}, V_{\mathbb{Z}})$ is of finite index in $\mathrm{Aut}(\mathscr{B}, V_{\mathbb{Z}})$ too.*

Proof. Denote by $\mathscr{R}$ the round part of both our self-dual homogeneous cones, by R its linear span, by k the codimension of R in V. Note that

$$\mathscr{A} = \mathscr{R} \oplus \Sigma_k \quad \text{and} \quad \mathscr{B} = \mathscr{R} \oplus \Xi_k$$

for simplicial cones Σ_k and Ξ_k with the same linear span S in V. Since $\mathscr{C}$ is well-clipped in $\mathscr{B}$, we also have $\mathscr{C} = (\mathscr{C} \cap R) \oplus \Xi_k$.

By uniqueness of the decomposition in Theorem 2.3,

$$\mathrm{Aut}(\mathscr{A}) = \mathrm{Aut}(\mathscr{R}) \times \mathrm{Aut}(\Sigma_k) \quad \text{and} \quad \mathrm{Aut}(\mathscr{B}) = \mathrm{Aut}(\mathscr{R}) \times \mathrm{Aut}(\Xi_k).$$

Hence, the obvious inclusion $\mathrm{Aut}(\Xi_k, \Sigma_k) < \mathrm{Aut}(\Xi_k)$ induces an inclusion

$$\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}) < \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{B})$$

of the same index. It also induces an inclusion $\mathrm{Aut}(\mathscr{A}, \mathscr{B}) < \mathrm{Aut}(\mathscr{B})$ of yet the same index.

Note that the connected component of identity $\mathrm{Aut}^\circ(\Xi_k)$ preserves each extremal ray of Ξ_k individually, and is of index at most $k!$ in $\mathrm{Aut}(\Xi_k)$. Thus, the subgroup $\mathrm{Aut}^\circ(\Xi_k) \cap \mathrm{GL}(V_{\mathbb{Z}})$ acts by the identity on the extremal rays of rational slope of Ξ_k, and remains of finite index in $\mathrm{Aut}(\Xi_k, V_{\mathbb{Z}})$. Since Σ_k rules Ξ_k, we derive that

$$\mathrm{Aut}^\circ(\Xi_k) \cap \mathrm{GL}(V_{\mathbb{Z}}) = \mathrm{Aut}^\circ(\Xi_k, \Sigma_k) \cap \mathrm{GL}(V_{\mathbb{Z}}).$$

Thus, the subgroup $\mathrm{Aut}(\Xi_k, \Sigma_k, V_{\mathbb{Z}})$ indeed has finite index in $\mathrm{Aut}(\Xi_k, V_{\mathbb{Z}})$, as wished. □

The next proposition can be stated in many ways. We choose to state it in a form that is close to the way the proof proceeds.

Proposition 4.12. *Let $\mathscr{C}$ be a perfectly clipped cone in a self-dual homogeneous cone $\mathscr{A}$ in $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. Let $G < \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})$ be a subgroup acting with finite orbits. Then the invariant cone $\mathscr{C}^G$ is neatly clipped in a cone $\mathscr{B}$, and we have an inclusion of finite index*

$$\rho^G(C_{\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})}(G)) \quad < \quad \mathrm{Aut}(\mathscr{C}^{G^\circ}, \mathscr{B}, V_{\mathbb{Z}}^G).$$

Proof. Let Γ denote the arithmetic subgroup given by Definition 3.15, which writes

$$\Gamma = \langle \sigma_i \mid i \in I \rangle \rtimes \mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}}).$$

By Lemma 4.7, we have

$$C_\Gamma(G) = C_{\langle \sigma_i | i \in I \rangle}(G) \rtimes C_{\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})}(G). \tag{8}$$

By Lemma 4.6 and Proposition 4.8, we have an inclusion of finite index

$$\rho^G(C_{\langle \sigma_i | i \in I \rangle}(G)) \rtimes \rho^G(C_{\mathrm{Aut}(\mathscr{C}^\circ, \mathscr{A}, V_{\mathbb{Z}})}(G)) \quad < \quad \mathrm{Aut}(\mathscr{A}^G, V_{\mathbb{Z}}^G). \tag{9}$$

By Item (1) of Proposition 4.5, the cone $\mathscr{C}^G$ is well clipped in a self-dual homogeneous cone $\mathscr{B}$ which has the same round part as $\mathscr{A}^G$, and whose simplicial part is ruled by the simplicial part of $\mathscr{A}^G$. Also note that $\rho^G(C_{\mathrm{Aut}(\mathscr{C}^\circ,\mathscr{A},V_{\mathbb{Z}})}(G))$ is contained in $\mathrm{Aut}(\mathscr{C}^{G^\circ},\mathscr{A}^G)$, thus preserves the cone $\mathscr{B}$ by Lemma 4.11. In particular, by Lemma 4.11 and Inclusion (9), there is an inclusion of finite index

$$W \rtimes \rho^G(C_{\mathrm{Aut}(\mathscr{C}^\circ,\mathscr{A},V_{\mathbb{Z}})}(G)) \quad < \quad \mathrm{Aut}(\mathscr{B},V_{\mathbb{Z}}^G), \tag{10}$$

where W denotes $\rho^G(C_{\langle\sigma_i|i\in I\rangle}(G))\cap\mathrm{Aut}(\mathscr{B})$. By Item (2) of Proposition 4.5, we know that the orthogonal reflection group given by the hyperplane sides of $\mathscr{C}^G$ satisfies $\langle\tau_i \mid i\in I^*\rangle < W$. These facts and an elementary verification of Assumption (i) of Definition 3.15 show that $\mathscr{C}^G$ is in fact neatly clipped in $\mathscr{B}$.

Therefore, we can apply Lemma 3.19. It provides is a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(\mathscr{C}^{G^\circ},\mathscr{B},V_{\mathbb{Z}}^G)$ on $(\mathscr{C}^G)^+$, and shows that the inclusion

$$\rho^G(C_{\mathrm{Aut}(\mathscr{C}^\circ,\mathscr{A},V_{\mathbb{Z}})}(G)) \quad < \quad \mathrm{Aut}(\mathscr{C}^{G^\circ},\mathscr{B},V_{\mathbb{Z}}^G)$$

is of finite index, as wished. □

We deduce the following somewhat more memorable statement.

Corollary 4.13. *Let $\mathscr{C}$ be a perfectly clipped cone in a self-dual homogeneous cone $\mathscr{A}$ in $V=V_{\mathbb{Z}}\otimes\mathbb{R}$. Let $G<\mathrm{Aut}(\mathscr{C}^\circ,\mathscr{A},V_{\mathbb{Z}})$ be a subgroup acting with finite orbits. Then the invariant cone $\mathscr{C}^G$ is perfectly clipped.*

Proof. It follows from Propositions 4.12 and 3.18. □

5. Proof of the main results

We can now prove the theorems stated in the introduction.

Proof of Theorem 1.8. The fact that well-clipped cones descend to well-clipped cones is Proposition 4.5. Assume that there exists a rational polyhedral fundamental domain for the action of $\Gamma := \mathrm{Aut}(\mathscr{C}^\circ,\mathscr{A},V_{\mathbb{Z}})$ on the well-clipped cone $\mathscr{C}$. Then by Proposition 3.18, the cone $\mathscr{C}$ is perfectly clipped. By Propositions 4.12 and 3.18, the cone $\mathscr{C}^G$ is perfectly clipped in a self-dual homogeneous cone $\mathscr{B}$ such that the inclusion $\rho^G(C_\Gamma(G)) < \mathrm{Aut}(\mathscr{C}^{G^\circ},\mathscr{B},V_{\mathbb{Z}}^G)$ is of finite index. Thus, by [27, Proposition 4.6] and Proposition 3.18, the cone $\mathscr{C}^{G+}$ admits a rational polyhedral fundamental domain for the action of $\rho^G(C_\Gamma(G))$. □

Proof of Theorem 1.6. Let G be a finite subgroup of $\mathrm{Aut}(X,\Delta)$ and let $\overline{G}$ be the group $G\times\mathrm{Gal}(\overline{k}/k)$ acting on the base change to the algebraic closure $(\overline{X},\overline{\Delta})$. Note that its induced action on the Néron–Severi space $N^1(\overline{X})_{\mathbb{R}}$ has finite orbits by [23, Paragraphs 16.1, 16.2]. Denoting by $p:\overline{X}\to Y$ the quotient map by the action of $\overline{G}$, note that p^* is injective and

$$N^1(\overline{X})^G = p^*N^1(Y), \quad \mathrm{Mov}(\overline{X})\cap N^1(\overline{X})^G = \quad p^*\mathrm{Mov}(Y),$$
$$\rho^G(C_{\mathrm{PsAut}^*(\overline{X},\overline{\Delta})}(G)) < \quad \mathrm{PsAut}^*(Y,\Delta_G),$$

where PsAut^* denotes the image of the pseudoautomorphism group PsAut in $\mathrm{GL}(N^1(\cdot))$, and (Y,Δ_G) is the quotient pair of (X,Δ) by G. Recall that $\mathrm{PsAut}^*(\overline{X},\overline{\Delta})$ is contained in $\Gamma:=\mathrm{Aut}(\mathrm{Mov}(\overline{X})^\circ,\mathscr{A},N^1(\overline{X})_{\mathbb{Z}-\mathrm{Weil}})$, where $\mathscr{A}$ denotes a self-dual homogeneous cone in which $\mathrm{Mov}(\overline{X})$ is well-clipped.

Since $(\overline{X},\overline{\Delta})$ satisfies the movable cone conjecture, there is a rational polyhedral fundamental domain for the action of Γ on $\mathrm{Mov}^+(\overline{X})$. Thus, by [27, Proposition

4.6], the groups $\mathrm{PsAut}^*(\overline{X}, \overline{\Delta})$ and Γ are commensurable. By Theorem 1.8, there is a rational polyhedral fundamental domain for the action of $\rho^G(C_\Gamma(G))$ on $\mathrm{Mov}^+(Y)$. By Lemmas 4.6 and [27, Proposition 4.6], the cone $\mathrm{Mov}^+(Y)$ thus admits a rational polyhedral fundamental domain for the action of $\rho^G(C_{\mathrm{PsAut}^*(\overline{X},\overline{\Delta})}(G))$, and a fortiori [27, Proposition 4.1, Application 4.14] for the larger group $\mathrm{PsAut}^*(Y, \Delta_G)$. □

Remark 5.1. In principle, one could use that $\mathrm{PsAut}^*(X/G, \Delta_G)$ entails the larger normalizer subgroup $\rho^G(N_{\mathrm{PsAut}^*(\overline{X},\overline{\Delta})}(G))$. However, working with the centralizer subgroup is somewhat simpler.

Proof of Corollary 1.7. Let X be a torsor over an abelian variety over a perfect field k. The base change $\overline{X}$ to the algebraic closure of k is an abelian variety, thus by the work of Prendergast–Smith in [41] has a self-dual homogeneous movable cone and satisfies the movable cone conjecture. Theorem 1.6 concludes. □

Proof of Theorem 1.4. Recall that $X = A \times Y \times \prod_{j=1}^s S_j$, where A is an abelian variety, Y is a product of primitive symplectic varieties with canonical singularities, and each S_j is a smooth rational surface underlying a klt Calabi–Yau pair (S_j, Δ_j). Let G be a finite subgroup of $\mathrm{Aut}(X, \Delta)$. By [10, Lemma 4.6] the G-action preserves the decomposition, up to permutation of potential isomorphic factors.

Regrouping isomorphic factors, we rewrite

$$Y = \prod_{k=1}^m {Y_k}^{n_k}.$$

By [42, Corollary 4.4.4], each Y_k admits a $G \cap \mathrm{Aut}(Y_k)^{n_k}$-equivariant $\mathbb{Q}$-factorial terminalization $\hat{Y_k}$. In particular,

$$\hat{Y} = \prod_{k=1}^m \hat{Y_k}^{n_k}$$

is a G-equivariant $\mathbb{Q}$-factorial terminalization of Y. We define $\hat{X} := A \times \hat{Y} \times \prod_{j=1}^s S_j$. By Lemmas 2.14 and 2.15, it suffices to show the movable cone conjecture, the finiteness of SQMs, and the nef cone conjecture for all SQMs of the pair $(\hat{X}/G, \hat{\Delta}_G)$.

To simplify notations, we write X for $\hat{X}$ and Y for $\hat{Y}$ from here on. All primitive symplectic varieties involved have terminal $\mathbb{Q}$-factorial singularities, and $b_2 \geq 5$.

We now prove the movable cone conjecture for any quotient pair of (X, Δ). By [16, Exercise III.12.6] and Lemma 2.16, the movable cone of X is the direct sum of the movable cones of its factors. Thus and by [48, 41, 25], the pair (X, Δ) satisfies the movable cone conjecture. We checked in Example 3.6, see in particular Items (1), (4), (6), (7) that the movable cone of each factor of X is well clipped, and in fact

- self-dual homogeneous itself for A,
- well clipped in the hyperbolic cone $\mathscr{H}(q_i)$ given by the Beauville–Bogomolov–Fujiki quadratic form q_i for the Y_i,
- and well clipped in the hyperbolic cone given by the intersection form f_j for the S_j.

Item (10) in Example 3.6 and Lemma 3.5 ensure that the movable cone of X is well clipped. By Corollary 2.18, the group $\mathrm{PsAut}(X, \Delta)$ preserves the self-dual homogeneous cone

$$\mathrm{Mov}^\circ(A) \oplus \bigoplus_{i=1}^r \mathscr{H}(q_i) \oplus \bigoplus_{j=1}^s \mathscr{H}(f_j),$$

thus Theorem 1.6 applies, and proves the movable cone conjecture for any quotient pair of (X, Δ).

We now focus on the finiteness of SQMs and the nef cone conjecture for them. Note that

$$\mathrm{Mov}^\circ(X) \setminus \bigcup_{w \in W} w^\perp = \bigsqcup_{\substack{\alpha: X \dashrightarrow X' \\ \mathrm{SQM}}} \alpha^*\mathrm{Amp}(X'),$$

where W is the set of pullbacks of primitive wall divisors of the primitive symplectic terminal $\mathbb{Q}$-factorial factors (see [25, Definition 7.1]). By [25, Proposition 7.7], the set of squares $\{q(w) \mid w \in W\}$ is bounded and contained in $\mathbb{Z}_{<0}$. Restricting to the G-invariant Néron-Severi subspace, we still have

$$\mathrm{Mov}^\circ(X/G) \setminus \bigcup_{w \in W_G} w^\perp = \bigsqcup_{\substack{\alpha: X/G \dashrightarrow Y \\ \mathrm{SQM}}} \alpha^*\mathrm{Amp}(Y),$$

where we define

$$W_G := \left\{ \sum_{g \in G} gw \mid w \in W,\, q(w, \sum_{g\in G} gw) < 0 \right\}.$$

The elements of W_G clearly remain of bounded negative squares, thus by [32, Proposition 3.4], for any rational polyhedral cone Π, only finitely many of their orthogonal hyperplanes intersect Π°. This checks the assumption of Lemma 2.19, and applying it to the pair $(X/G, \Delta_G)$ concludes the proof. □

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