

THE EFFECTIVE CONE CONJECTURE FOR CALABI–YAU PAIRS

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ABSTRACT. We formulate an *effective cone conjecture* for klt Calabi–Yau pairs (X, Δ) , pertaining to the structure of the cone of effective divisors $\text{Eff}(X)$ modulo the action of the subgroup of pseudo-automorphisms $\text{PsAut}(X, \Delta)$. Assuming the existence of good minimal models in dimension $\dim(X)$, known to hold in dimension up to 3, we prove that the effective cone conjecture for (X, Δ) is equivalent to the Kawamata–Morrison–Totaro movable cone conjecture for (X, Δ) . As an application, we show that the movable cone conjecture unconditionally holds for the smooth Calabi–Yau threefolds introduced by Schoen and studied by Namikawa, Grassi and Morrison. We also show that for such a Calabi–Yau threefold X , all of its minimal models, apart from X itself, have rational polyhedral nef cones.

1. INTRODUCTION

The starting point of this paper is the following conjecture.

Conjecture (Kawamata–Morrison cone conjecture). *Let X be a normal \mathbb{Q} -factorial terminal Calabi–Yau variety. Then*

- (1) *There is a rational polyhedral cone $\Pi \subset \text{Nef}(X)$ such that*

$$\text{Aut}(X) \cdot \Pi = \text{Nef}^e(X) = \text{Nef}^+(X),$$

and for every $g \in \text{Aut}(X)$, $g^\Pi^\circ \cap \Pi^\circ \neq \emptyset$ if and only if $g^* = \text{id}$.*

- (2) *There is a rational polyhedral cone $\Sigma \subset \overline{\text{Mov}}(X)$ such that*

$$\text{PsAut}(X) \cdot \Sigma = \overline{\text{Mov}}^e(X) = \text{Mov}^+(X),$$

and for every $g \in \text{PsAut}(X)$, $g^\Sigma^\circ \cap \Sigma^\circ \neq \emptyset$ if and only if $g^* = \text{id}$.*

Here, for a convex cone \mathcal{C} in $N^1(X)_{\mathbb{R}}$, we denote by \mathcal{C}^e the intersection $\mathcal{C} \cap \text{Eff}(X)$, and by \mathcal{C}^+ the convex cone spanned by the classes of Cartier divisors in $\overline{\mathcal{C}}$. We also denote by $\text{PsAut}(X)$ the group of pseudo-automorphisms of X , i.e., birational self-maps of X which are biregular in codimension 1.

This conjecture has appeared in various forms, notably stated by Morrison, Kawamata, and Totaro in [Mor93, Mor96, Kaw97, Tot08, Tot10] (in order of increasing generality, see Conjecture 1.4 for a more general statement inspired by the work of Totaro). Albeit initially motivated by mirror symmetry, it has attracted much work from birational geometers over the past thirty years, see [LOP18] for a survey.

The main purpose of our work is to understand the role and place of the cone of effective divisors $\text{Eff}(X) \subset N^1(X)_{\mathbb{R}}$ within this conjectural picture.

We start by formulating an effective cone conjecture for klt Calabi–Yau pairs (X, Δ) . Assuming the existence of good minimal models in dimension $\dim X$, we show that a given klt Calabi–Yau pair (X, Δ) satisfies the effective cone conjecture if and only if its movable cone satisfies a generalization of the Kawamata–Morrison cone conjecture to the pair setting due to Totaro (see Definition 1.3 (1) and Theorem 1.5 (iii)). Along the way, we unveil more relations between instances of a cone conjecture for various cones of divisors. These relations are summarized in Theorem 1.5, which constitutes the main result of this paper.

An application of our main theorem concerns the birational geometry of certain smooth Calabi–Yau threefolds introduced by Schoen in [Sch88]. These threefolds were studied by Namikawa, Grassi, and Morrison [Nam91, GM93], and became the first known Calabi–Yau threefolds with infinite automorphism group, whose nef cone satisfied the Kawamata–Morrison cone conjecture. Using our main theorem, we prove that these threefolds’ movable cones satisfy the Kawamata–Morrison cone conjecture as well (see Theorem 1.7).

In order to prove our main result, we provide a chamber decomposition of the cone of effective divisors. Let us introduce it first.

1.1. A chamber decomposition of the effective cone. In the paper [Kaw97], Kawamata notably proved that for a normal \mathbb{Q} -factorial terminal Calabi–Yau threefold X ,

$$(1.1) \quad \overline{\text{Mov}}^e(X) = \bigcup_{\substack{\alpha: X \dashrightarrow X' \\ \text{SQM}}} \alpha^* \text{Nef}^e(X')$$

where $\{\alpha: X \dashrightarrow X'\}$ ranges through all small \mathbb{Q} -factorial modifications of X . The result and its proof generalize and yield a movable cone decomposition for any variety X which underlies a klt Calabi–Yau pair, provided the existence of minimal models for X (see e.g. [SX, Theorem 3.5]). This last condition stems from the minimal model program (MMP); it is briefly discussed in Subsection 2.3.

Under the same assumption, we will prove that the Kawamata decomposition (1.1) can be extended to a similar chamber decomposition of the effective cone $\text{Eff}(X)$. To define the chambers, we reintroduce a notion originating from the influential work of Hu and Keel on Mori dream spaces [HK00]. For any birational contraction $f: X \dashrightarrow Y$ from X to a normal \mathbb{Q} -factorial projective variety Y , we define the *f-Mori chamber* to be

$$\text{Eff}(X; f) := f^* \text{Nef}^e(Y) + \sum_{E \in \text{Exc}(f)} \mathbb{R}_{\geq 0}[E] \subset \text{Eff}(X),$$

where $\text{Exc}(f)$ denotes the set of prime exceptional divisors of f .

Proposition 1.1. *Let (X, Δ) be a klt Calabi–Yau pair. Assume the existence of minimal models for X . We have a chamber decomposition*

$$\text{Eff}(X) = \bigcup_{\substack{f: X \dashrightarrow Y \\ \text{QBC}}} \text{Eff}(X; f)$$

which extends (1.1), where the union runs through all \mathbb{Q} -factorial birational contractions¹ $f: X \dashrightarrow Y$ from X .

It is well worth noting that distinct Mori chambers have disjoint interior. We also note that our minimal model assumption is only needed to ensure that boundary points of the cone $\text{Eff}(X)$ appear in the union of all Mori chambers. A precise unconditional statement for a decomposition of the interior of the effective cone can be found in Proposition 4.10.

1.2. The effective cone conjecture, and other cone conjectures. Let (X, Δ) be a klt Calabi–Yau pair. Following Totaro [Tot10], we introduce the groups

$$(\text{Ps})\text{Aut}(X, \Delta) = \{g \in (\text{Ps})\text{Aut}(X) : g \text{ preserves the support of } \Delta\}.$$

We propose the following conjecture.

Conjecture 1.2 (Effective cone conjecture). *Let (X, Δ) be a klt Calabi–Yau pair. Then there exists a rational polyhedral cone $\Pi \subset \text{Eff}(X)$ such that*

$$\text{PsAut}(X, \Delta) \cdot \Pi = \text{Eff}(X),$$

and for every $g \in \text{PsAut}(X, \Delta)$, $g^\Pi^\circ \cap \Pi^\circ \neq \emptyset$ if and only if $g^* = \text{id}$.*

Before we state our main theorem relating different cone conjectures, we streamline the formulation of various cone conjectures presented in the paper as follows.

Definition 1.3. Let (X, Δ) be a klt Calabi–Yau pair. For a subgroup G of $\text{PsAut}(X)$, let G^* be its image under the natural action

$$\text{PsAut}(X) \rightarrow \text{GL}(N^1(X)_{\mathbb{R}}).$$

We say that the *movable cone conjecture*, the *effective cone conjecture*, the *nef cone conjecture*, respectively the *f -Mori chamber cone conjecture*, holds for the pair (X, Δ) (with its \mathbb{Q} -factorial birational contraction $f: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$) if there exists a rational polyhedral fundamental domain for the action

- (1) (Movable cone conjecture) $\text{PsAut}^*(X, \Delta) \supseteq \overline{\text{Mov}}^e(X)$
- (2) (Effective cone conjecture) $\text{PsAut}^*(X, \Delta) \supseteq \text{Eff}(X)$,
- (3) (Nef cone conjecture) $\text{Aut}^*(X, \Delta) \supseteq \text{Nef}^e(X)$,
- (4) (f -Mori chamber cone conjecture) $\text{PsAut}^*(X, \Delta; f) \supseteq \text{Eff}(X; f)$.

Here, the group $\text{PsAut}^*(X, \Delta; f)$ is the stabilizer of the f -Mori chamber $\text{Eff}(X; f)$ in the group $\text{PsAut}^*(X, \Delta)$; see Subsection 5.1 for an alternative definition.

For future reference, let us state the generalization of the Kawamata–Morrison cone conjecture for Calabi–Yau pairs due to Totaro [Tot10]. We phrase it in the terms of Definition 1.3.

¹We follow [HK00, Definition 1.0], rather than [Kaw97, Definition 1.8], for the definition of a birational contraction.

Conjecture 1.4 (Kawamata–Morrison–Totaro cone conjecture). *Let (X, Δ) be a klt Calabi–Yau pair. The nef cone conjecture and the movable cone conjecture hold for (X, Δ) .*

We can now state our main result. A slightly more detailed and general formulation appears later as Theorem 6.1.

Theorem 1.5. *Let (X, Δ) be a klt Calabi–Yau pair. Consider the following statements.*

- (1) *The movable cone conjecture holds for (X, Δ) .*
- (2) *The effective cone conjecture holds for (X, Δ) .*
- (3)
 - (a) *The nef cone conjecture holds for each klt pair (X', Δ') obtained by a small \mathbb{Q} -factorial modification from (X, Δ) .*
 - (b) *Up to isomorphism of pairs, there are only finitely many (X', Δ') arising as small \mathbb{Q} -factorial modifications of (X, Δ) .*
- (4)
 - (a) *The Mori chamber cone conjecture holds for each \mathbb{Q} -factorial birational contraction $f: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$.*
 - (b) *Up to isomorphism of pairs, there are only finitely many (Y, Δ_Y) arising as \mathbb{Q} -factorial birational contractions of (X, Δ) .*

Then, the following assertions hold.

- (i) *We have $(3) \Leftrightarrow (4) \Rightarrow (2)$.*
- (ii) *Assuming the existence of minimal models for X , we have $(3) \Rightarrow (1)$.*
- (iii) *Assuming the existence of good minimal models in dimension $\dim X$, we have $[(1) \text{ or } (2)] \Rightarrow (3)$, and thus, $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.*

1.3. A descent result for the nef cone conjecture. On the way of proving the implication $(3) \Rightarrow (4)$ in Theorem 1.5, we also prove a *birational descent* result for the nef cone conjecture. This is a higher-dimensional generalization of a result proved by Totaro [Tot10, Lemma 3.4] in dimension two.

Proposition 1.6. *Let (X, Δ) be a klt Calabi–Yau pair. Assume that the nef cone conjecture holds for (X, Δ) . Then, for any regular \mathbb{Q} -factorial birational contraction $f: (X, \Delta) \rightarrow (Y, \Delta_Y)$, the nef cone conjecture holds for (Y, Δ_Y) .*

We will give a slightly more general result in Proposition 5.2.

1.4. Contextualizing Theorem 1.5 and relation to other works. All the numbers in this subsection correspond to the statements in Theorem 1.5.

The implication $(1) \Rightarrow (3)$ was asked by Oguiso in [Ogu22, Question 2.30.(2)]. While $(1) \Rightarrow (3b)$ is proven in [CL14, Theorem 2.14] and [LZ, Proposition 5.3] under the assumption of the existence of good minimal models, the implication $(1) \Rightarrow (3a)$ was still open when we started to work on Theorem 1.5. Shortly before the current version of this paper was ready to be made public, the preprint [Xu] by F. Xu appeared on the arXiv. Independently, the author proves the implication $(1) \Rightarrow (3a)$

under the assumption of the existence of minimal models, and a non-vanishing assumption (see [Xu, Theorem 14]).

Our implication [(1) or (2)] \Rightarrow (3a) is also related to [LZ, Conjecture 1.2(2), Theorem 1.3(3)] in the work by Z. Li and H. Zhao. No clear comparison arises. We also refer the reader to Statement (2') in Theorem 6.1.

The implication (2) \Rightarrow (1) under the assumption of the existence of good minimal models, which we mention in Theorem 1.5(iii), is an immediate consequence of [LZ, Theorem 1.3.(1)].

Finally in the same preprint [Xu], F. Xu proves a reduction result for the movable cone conjecture in [Xu, Theorem 2], of which one direction is the birational descent for the movable cone conjecture. It is also independent from our descent result for the nef cone conjecture (Proposition 1.6).

1.5. Cone conjectures for the Calabi–Yau threefolds introduced by Schoen.

The equivalence of the statements in Theorem 1.5 offers some flexibility to study one of the cone conjectures, and derive results for the others. For instance, if a klt Calabi–Yau pair (X, Δ) of dimension 3 satisfies the effective cone conjecture, then by Theorem 1.5 (and the existence of good minimal models in dimension 3), it also satisfies the more traditional Kawamata–Morrison–Totaro cone conjecture (stated as Conjecture 1.4 above).

This principle works well for the following class of smooth Calabi–Yau threefolds X introduced by Schoen in [Sch88].

Theorem 1.7. *Let X be a smooth projective Calabi–Yau threefold obtained as a fiber product of the form $W_1 \times_{\mathbb{P}^1} W_2$, where for $i = 1, 2$, we consider a relatively minimal rational elliptic surface $\phi_i: W_i \rightarrow \mathbb{P}^1$ with a section. Assume moreover that the generic fibers of ϕ_1 and ϕ_2 are non-isogenous.*

Then, the movable cone conjecture, the effective cone conjecture, as well as the statements (3) and (4) of Theorem 1.5, all hold for X .

A smooth Calabi–Yau threefold X as above has been extensively studied under some general assumptions on the rational elliptic surfaces W_i (see [Nam91] or Corollary 1.8 below for the precise conditions). The nef cone conjecture was established for such X by Grassi and Morrison [GM93]. The finiteness of the minimal models was also proven, and the exact number of isomorphism classes of minimal models was computed to be

$$56, 120, 347, 647, 983, 773, 489 \quad (> 5 \times 10^{19})$$

by Namikawa in [Nam91].

Together with the work of Namikawa, Theorem 1.7 provides the following description of the nef cones of minimal models of X .

Corollary 1.8. *Let X be a smooth projective Calabi–Yau threefold obtained as a fiber product of the form $W_1 \times_{\mathbb{P}^1} W_2$, where for $i = 1, 2$, we consider a relatively minimal rational elliptic surface $\phi_i: W_i \rightarrow \mathbb{P}^1$ with a section, whose singular fibers all are of type I_1 . Assume moreover that the generic fibers of ϕ_1 and ϕ_2 are non-isogenous.*

Then, every minimal model X' of X which is not isomorphic to X has a rational polyhedral nef cone.

To our knowledge, the movable cone conjecture for such a smooth Calabi–Yau threefold X , and the nef cone conjecture for their minimal models are both new results.

Organization of the paper. In Section 2, we set up some notations and review well-known facts about cones of divisors, pushforward and pullback of numerical divisor classes by various types of birational maps, and state some standard conjectures of the minimal model program. In Section 3, we start by reviewing some relevant results about the geometry of convex cones in a finite dimensional vector space with a lattice, often considered with a linear group action preserving the lattice. The work of Looijenga [Loo14] plays a distinct role there. We then proceed to prove preparatory results in this convex geometric set-up, intended for future references throughout the proof of Theorem 1.5. In Section 4, we introduce Mori chambers, recall some results on Shokurov polytopes and the geography of models, and prove various decompositions of cones, including the movable and effective cone decompositions stated in Proposition 1.1. Section 5 is rather short, devoted to the proof of our descent result for the nef cone conjecture. Section 6 is reserved for the proof of Theorem 1.5, or rather of the slightly more detailed Theorem 6.1. Finally, Section 7 is devoted to our results on the smooth Calabi–Yau threefolds introduced by Schoen.

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2. NOTATIONS AND PRELIMINARIES

We refer to [KM98] for standard results of birational geometry, and to [Fuj17] for results and definitions regarding \mathbb{R} -divisors more specifically.

In this paper, a *pair* is the data (X, Δ) of a normal \mathbb{Q} -factorial projective variety² X and of an effective \mathbb{R} -divisor Δ on X . We call a pair (X, Δ) *Calabi–Yau* if the \mathbb{R} -divisor $K_X + \Delta$ is numerically trivial, following [Tot10]. Many pairs considered in this paper are *klt*; for a definition, see [Fuj17, Definition 2.3.4].

²We add \mathbb{Q} -factoriality for convenience.

2.1. Cones of numerical classes of divisors. Let X be a normal projective variety. We write $N^1(X)$ for the free abelian group generated by the classes of Cartier divisors modulo numerical equivalence. Inside the vector space $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$, we denote by $\text{Nef}(X)$ the *nef cone*, i.e., the closure of the ample cone $\text{Amp}(X)$, and by $\text{Eff}(X)$ the *effective cone*, that is the cone generated by the numerical classes of effective Cartier divisors in $N^1(X)_{\mathbb{R}}$. The closure and the interior of $\text{Eff}(X)$ are called the *pseudo-effective cone* $\overline{\text{Eff}}(X)$ and the *big cone* $\text{Big}(X)$ respectively; note that they need not equal $\text{Eff}(X)$ in general. The *nef effective cone* $\text{Nef}^e(X)$ is defined as

$$\text{Nef}^e(X) := \text{Nef}(X) \cap \text{Eff}(X).$$

A Cartier divisor D on a projective variety X is called *movable* if there is a positive integer m such that mD is effective and the base locus of the linear system $|\mathcal{O}_X(mD)|$ does not contain any divisor. We denote by $\text{Mov}(X)$ the convex cone in $N^1(X)_{\mathbb{R}}$ generated by the numerical classes of movable Cartier divisors. In general, the cone $\text{Mov}(X)$ is neither open nor closed. The *closed movable cone* $\overline{\text{Mov}}(X)$ and the *open movable cone* $\text{Mov}^\circ(X)$ are the closure, respectively, the interior of $\text{Mov}(X)$. The *movable effective cone* $\overline{\text{Mov}}^e(X)$ is defined as

$$\overline{\text{Mov}}^e(X) := \overline{\text{Mov}}(X) \cap \text{Eff}(X).$$

We have the following inclusions of cones

$$\begin{aligned} \text{Nef}(X) &\subset \overline{\text{Mov}}(X) \subset \overline{\text{Eff}}(X) \\ \text{Amp}(X) &\subset \text{Mov}^\circ(X) \subset \text{Big}(X). \end{aligned}$$

In particular, each of the cones described here is strictly convex since $\overline{\text{Eff}}(X)$ is. Note that if X is a surface, the horizontal inclusions are all equalities, i.e.,

$$\text{Nef}(X) = \overline{\text{Mov}}(X), \quad \text{Amp}(X) = \text{Mov}^\circ(X).$$

2.2. Some group actions on cones of divisors.

2.2.1. Birational contractions and pseudo-automorphisms. Let X be a normal \mathbb{Q} -factorial projective variety. We define some important types of birational maps. Following [HK00], we call a birational map $f: X \dashrightarrow Y$ to a normal projective variety Y a *birational contraction*, if f^{-1} contracts no divisor. If in addition, Y is \mathbb{Q} -factorial, we call f a *\mathbb{Q} -factorial birational contraction*. For convenience, we will also call (Y, f) a *marked (\mathbb{Q} -factorial) birational contraction with the marking f* . We consider two marked (\mathbb{Q} -factorial) birational contractions (Y_1, f_1) and (Y_2, f_2) of X to be *isomorphic* if the birational map $f_2 \circ f_1^{-1}$ is biregular.

Following [HK00, Definition 1.8], we define a *small \mathbb{Q} -factorial modification* of X as a \mathbb{Q} -factorial birational contraction $\alpha: X \dashrightarrow X'$ such that α contracts no divisor. Note that the birational map α is then an isomorphism in codimension one. The birational maps from X to itself which are isomorphisms in codimension one are called the *pseudo-automorphisms* of X , and they form a group denoted by $\text{PsAut}(X)$. The marked small \mathbb{Q} -factorial modifications and their isomorphisms are defined in the same way.

The following example shows that there can be infinitely many marked small \mathbb{Q} -factorial modifications, but only finitely many isomorphism classes of targets.

Example 2.1. Let X be a very general hypersurface of multidegree $(2, \dots, 2)$ in $(\mathbb{P}^1)^{n+1}$ with $n \geq 3$. Then X is a simply connected smooth Calabi–Yau manifold of dimension n (see, e.g., [CO15, Theorem 3.1]). Every birational self-map f of X is a pseudo-automorphism, and gives a small \mathbb{Q} -factorial modification $(X, f: X \dashrightarrow X)$. By [CO15, Theorem 3.3], the automorphism group $\text{Aut}(X)$ is trivial, while the pseudo-automorphism group $\text{PsAut}(X)$ is infinite. Moreover, by the proof of [CO15, Theorem 3.3.(4)], for every small \mathbb{Q} -factorial modification $\alpha: X \dashrightarrow X'$, we have $X' \cong X$. So there are infinitely many marked small \mathbb{Q} -factorial modifications (X, f) , but they all have the same target variety X .

2.2.2. Pushforwards and pullbacks. We define notions of pushforwards and pullbacks for real divisor classes and for birational maps. Throughout § 2.2.2, we let $f: X \dashrightarrow X'$ be a \mathbb{Q} -factorial birational contraction.

We start with the pushforward: We have a group homomorphism induced by pushforward between the groups of codimension-one cycles

$$f_* : Z^1(X) \rightarrow Z^1(X').$$

By the negativity lemma [KM98, Lemma 3.39], if f is a proper birational morphism, the kernel of the pushforward f_* (at the level of \mathbb{Q} -class groups) is spanned by the set of prime exceptional divisors $\text{Exc}(f)$.

We can also define a pullback by f . We resolve

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ X & \dashrightarrow & X' \\ & f & \end{array}$$

with W a normal \mathbb{Q} -factorial projective variety, and p, q birational morphisms. This yields a pullback group homomorphism between the \mathbb{Q} -class groups ([Har77, II.6, Definition, Page 131])

$$f^* := p_* q^* : \text{Cl}(X')_{\mathbb{Q}} \rightarrow \text{Cl}(X)_{\mathbb{Q}},$$

which is independent of the choice of the resolution (W, p, q) .

The following simple, yet important fact, relates pushforwards and pullbacks. The pushforward f_* preserves rational equivalence, and descends to coincide with $(f^{-1})^*$ at the level of \mathbb{Q} -class groups. Moreover, if (W, p, q) is a resolution of f as above, we have

$$f_* f^* = (f^{-1})^*(f)^* = q_* p^* p_* q^* = \text{id},$$

and therefore f^* is injective and f_* is surjective.

We conclude § 2.2.2 with the decomposition

$$N^1(X)_{\mathbb{R}} = f^* N^1(X')_{\mathbb{R}} \oplus \text{Span}_{\mathbb{R}}(\text{Exc}(f)),$$

which immediately follows from the various facts above.

2.2.3. Pullbacks, pushforwards, and composition. There are instances of pullback functoriality, such as the following lemma.

Lemma 2.2. *Let $f: X \dashrightarrow Y$ and $g: Y \dashrightarrow Z$ be two birational maps between normal \mathbb{Q} -factorial projective varieties. Assume that f^{-1} is a birational contraction. Then*

$$f^*g^* = (gf)^*: \mathrm{Cl}(Z)_{\mathbb{Q}} \rightarrow \mathrm{Cl}(X)_{\mathbb{Q}}.$$

Proof. Choose a common resolution of f and g

$$\begin{array}{ccccc} & & W & & \\ & p \swarrow & \downarrow q & \searrow r & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

where W is a normal \mathbb{Q} -factorial projective variety, and p, q, r are birational morphisms with $\mathrm{Exc}(q) \subset \mathrm{Exc}(p)$. Note that the linear endomorphism $\mathrm{id}_{N^1(W)_{\mathbb{Q}}} - q^*q_*$ is a projector onto the linear subspace $\ker(q_*)$. Since $p_*\ker(q_*) = 0$ by assumption, we have $f^*g^* = p_*q^*q_*r^* = p_*r^* = (gf)^*$, as wished. \square

The following corollary shows an instance of pushforward functoriality.

Corollary 2.3. *Let $f: X \dashrightarrow Y$ and $g: Y \dashrightarrow Z$ be two birational contractions between normal \mathbb{Q} -factorial projective varieties. Then*

$$(gf)_* = g_*f_*: \mathrm{Cl}(X)_{\mathbb{Q}} \rightarrow \mathrm{Cl}(Z)_{\mathbb{Q}}.$$

Proof. Apply Lemma 2.2 to obtain that $[(gf)^{-1}]^* = (g^{-1})^*(f^{-1})^*$, and use that both f and g are birational contractions to identify their inverse pullback with their pushforward. \square

2.2.4. *The action by the groupoid of small \mathbb{Q} -factorial modifications.* Throughout § 2.2.4, we denote by X a normal \mathbb{Q} -factorial projective variety.

For any \mathbb{Q} -factorial birational contraction $f: X \dashrightarrow X'$, it holds

$$f^*N^1(X')_{\mathbb{Q}} \subset N^1(X)_{\mathbb{Q}}, \quad f^*\mathrm{Eff}(X') \subset \mathrm{Eff}(X), \quad f^*\overline{\mathrm{Mov}}(X') \subset \overline{\mathrm{Mov}}(X),$$

with equality if and only if f is a small \mathbb{Q} -factorial modification.

In particular, the group $\mathrm{PsAut}(X)$ acts by pullback on $N^1(X)_{\mathbb{R}}$, yielding a linear representation

$$\mathrm{PsAut}(X) \curvearrowright N^1(X)_{\mathbb{R}},$$

which preserves the lattice of Weil divisor classes $N^1_W(X)$. It also preserves the cones $\overline{\mathrm{Mov}}(X)$ and $\mathrm{Eff}(X)$. The induced representation of the automorphism group $\mathrm{Aut}(X)$ additionally preserves the cone $\mathrm{Nef}(X)$.

2.3. Some standard conjectures of the minimal model program. Part of the main results of this paper are proven under the assumption of *the existence of good minimal models* in dimension n . In this subsection, we recall what this assumption actually means, and provide a reference for the fact that it is satisfied for $n \leq 3$. Note that it is most crucial for the following definition that we allow \mathbb{R} -divisors.

Definition 2.4. We define the following notions:

- (1) Following [KM98, Definition 3.50]), we define a *minimal model* of a klt pair (X, Δ) as the data of a klt pair (W, Δ_W) and a birational contraction $\phi: (X, \Delta) \dashrightarrow (W, \Delta_W)$ such that
 - (i) $\phi_*\Delta = \Delta_W$ as effective \mathbb{R} -divisors;
 - (ii) the \mathbb{R} -divisor $K_W + \Delta_W$ is nef;
 - (iii) for any prime effective Weil divisor $E \subset X$ that is contracted by ϕ , we have the inequality of discrepancies $a(E, X, \Delta) < a(E, W, \Delta_W)$. (See [Fuj17, Lemma 2.3.2] for a definition.)
- (2) A *good minimal model* of a klt pair (X, Δ) is a minimal model (W, Δ_W) such that the \mathbb{R} -divisor $K_W + \Delta_W$ is semiample in the sense of [Fuj17, Definition 2.1.20].
- (3) Let X be a normal \mathbb{Q} -factorial projective variety. We say that we have *the existence of minimal models for X* , respectively *the existence of good minimal models for X* , if for any \mathbb{R} -divisor Δ such that the pair (X, Δ) is klt and the \mathbb{R} -divisor $K_X + \Delta$ is effective, the pair (X, Δ) admits a minimal model, respectively a good minimal model.
- (4) Let n be a positive integer. We say that *the existence of good minimal models holds in dimension n* if for any klt pair (X, Δ) such that X has dimension n and the \mathbb{R} -divisor $K_X + \Delta$ is effective, there exists a good minimal model for the pair (X, Δ) .

The existence of minimal models is known in dimension up to 4 [Bir11], while the existence of good minimal models is known in dimension up to 3 [Sho96].

3. CONVEX GEOMETRY

3.1. General notations for cones and Looijenga's results. In this subsection, we let $V_{\mathbb{Z}}$ be a free \mathbb{Z} -module of finite rank, and $V := V_{\mathbb{Z}} \otimes \mathbb{R}$. A *convex cone* in V is a subset of V that is invariant by multiplication by positive scalars and by sum. For any subset S of V , the convex cone *generated by S* is defined as the smallest convex cone containing S . A convex cone $C \subset V$ is called:

- *strictly convex* if its closure \overline{C} contains no line;
- *polyhedral*, respectively *rational polyhedral* if C is generated by finitely many elements of V , respectively of $V_{\mathbb{Z}}$.

For any convex cone C in V , we define C^+ as the convex cone generated by $\overline{C} \cap V_{\mathbb{Z}}$. Note that any inclusion of convex cones $C_1 \subset C_2$ in V is preserved by this operator, i.e., $C_1^+ \subset C_2^+$.

The following statement is contained in [Loo14, Proposition-Definition 4.1]³

Proposition 3.1 (Looijenga). *Let $C \subset V$ be a strictly convex cone with nonempty interior. Let Γ be a subgroup of $\mathrm{GL}(V_{\mathbb{Z}})$ preserving C . Let Π be a polyhedral cone contained in C^+ such that $C^\circ \subset \Gamma \cdot \Pi$. Then $\Gamma \cdot \Pi = C^+$.*

³More precisely, we apply [Loo14, Proposition-Definition 4.1] to the open cone C° ; note that $\overline{C^\circ} = \overline{C}$ since C is a convex cone with nonempty interior.

Since any rational polyhedral cone $\Pi \subset C$ automatically satisfies $\Pi \subset C^+$, this proposition shows that C^+ is the largest subcone of \overline{C} which may be covered by the Γ -translates of a rational polyhedral cone. This motivates the following definition.

Definition 3.2. Let $C \subset V$ be a strictly convex cone with nonempty interior. Let Γ be a subgroup of $\mathrm{GL}(V_{\mathbb{Z}})$ preserving C . We say that the action $\Gamma \curvearrowright C$ is of *polyhedral type*, if there exists a polyhedral cone $\Pi \subset C^+$ such that $C^{\circ} \subset \Gamma \cdot \Pi$.

This definition is slightly different from [Loo14, Proposition-Definition 4.1], as we do not require the cone C to be open. In fact, we allow some flexibility regarding the boundary of the cone C here, which is convenient in later proofs.

We say that an action $\Gamma \curvearrowright C^+$ as above has a *rational polyhedral fundamental domain* if there exists a rational polyhedral cone $\Pi \subset C^+$ such that

$$\Gamma \cdot \Pi = C^+,$$

and such that for every $\gamma \in \Gamma$, the non-emptiness $\gamma\Pi^{\circ} \cap \Pi^{\circ} \neq \emptyset$ implies $\gamma = \mathrm{id}$. This property is *a priori* stronger than the fact that $\Gamma \curvearrowright C$ is of polyhedral type; they are in fact equivalent by the foundational work [Loo14] of Looijenga.

Proposition 3.3 (Looijenga). *Let $C \subset V$ be a strictly convex cone with nonempty interior. Let Γ be a subgroup of $\mathrm{GL}(V_{\mathbb{Z}})$ preserving C . The following statements are equivalent.*

- (1) $\Gamma \curvearrowright C$ is of polyhedral type.
- (2) $\Gamma \curvearrowright C^+$ has a rational polyhedral fundamental domain.

Proof. Proposition 3.3 is essentially [Loo14, Proposition 4.1, Application 4.14, and Corollary 4.15]; see also [LZ, Lemma 3.5] for more details. \square

3.2. A descent property of actions of polyhedral type. We define a face of a convex cone as follows, which is equivalent to the definition in [Roc70, pp. 162].

Definition 3.4. Let C be a convex cone. A *face* of C is a convex cone $F \subset C$ such that for any closed line segment $I \subset C$ with $I \cap F \neq \emptyset$, it holds

$$I \subset F \quad \text{or} \quad I \cap F = \{ \text{one end point of } I \}.$$

Remark 3.5. Let $C \subset V$ be a convex cone, and let F be a face of the convex cone C^+ . Then F coincides with the convex cone generated by $F \cap V_{\mathbb{Z}}$. Indeed, any element $f \in F \subset C^+$ can be written as a sum $f = \sum_{i=1}^n \lambda_i c_i$ with $c_i \in \overline{C} \cap V_{\mathbb{Z}}$ and $\lambda_i > 0$. Since F is a face of C^+ and $c_i \in C^+$ for all i , this implies that $c_i \in F$ for all i , as wished.

The property of being of polyhedral type descends well from a convex cone to its faces. This is the content of the following proposition. We will apply it to prove a descent result for the nef cone conjecture (see Proposition 5.2).

Proposition 3.6. *Let $\Gamma \curvearrowright C$ be an action of polyhedral type. Then for each face F of C^+ , the action $\mathrm{Stab}_{\Gamma}(F) \curvearrowright F$ is of polyhedral type as well.*

Proof. Applying Proposition 3.1 to the action $\Gamma \curvearrowright C$ of polyhedral type, we obtain a polyhedral cone $\Pi \subset C^+$ such that $\Gamma \cdot \Pi = C^+$. Let $\{F_i\}_{i \in I}$ be the relative interiors of the finitely many non-zero faces of the polyhedral cone Π . Let $\text{ri}(F)$ denote the relative interior of F .

We consider for each index $i \in I$, the set

$$(3.1) \quad \mathfrak{F}_i := \{g \in \Gamma : gF_i \cap \text{ri}(F) \neq \emptyset\} = \{g \in \Gamma : gF_i \subset \text{ri}(F)\}.$$

Here, to see the equality of the two sets, we just note that, if $gF_i \cap \text{ri}(F) \neq \emptyset$, then $gF_i \subset F$ by [Roc70, Theorem 18.1], and thus, $gF_i \subset \text{ri}(F)$ by [Roc70, Corollary 6.5.2]. The set \mathfrak{F}_i is endowed with an action of $\text{Stab}_\Gamma(F)$ by left-multiplication.

Note that for every $g, h \in \mathfrak{F}_i$, we have

$$\emptyset \neq F_i \subset \text{ri}(g^{-1}F) \cap \text{ri}(h^{-1}F),$$

and $\text{ri}(h^{-1}F)$ and $\text{ri}(g^{-1}F)$ both are relative interiors of faces of C^+ . This implies that $\text{ri}(h^{-1}F) = \text{ri}(g^{-1}F)$, because relative interiors of distinct faces of C^+ are disjoint [Roc70, Theorem 18.2]. Thus, the action of $\text{Stab}_\Gamma(F)$ on \mathfrak{F}_i is transitive.

For each index $i \in I$, we choose one element $g_i \in \mathfrak{F}_i$, and let $\Phi_i := g_i F_i \subset \text{ri}(F)$. We have

$$\text{ri}(F) = \bigcup_{g \in \Gamma} (\text{ri}(F) \cap g\Pi) = \bigcup_{i \in I} \bigcup_{g \in \mathfrak{F}_i} gF_i = \bigcup_{i \in I} \text{Stab}_\Gamma(F) \cdot \Phi_i,$$

where the second equality follows from (3.1), and the third equality from the transitivity of the action of $\text{Stab}_\Gamma(F)$ on \mathfrak{F}_i .

We introduce the following convex cone

$$\Pi_F := \sum_{i \in I} \overline{\Phi_i}.$$

As a finite sum of polyhedral cones, it is a polyhedral cone. By [Roc70, Theorem 18.1], it is contained in the face F . It is thus contained in F^+ by Remark 3.5. Moreover, we have

$$\text{ri}(F) = \text{Stab}_\Gamma(F) \cdot \bigcup_{i \in I} \Phi_i \subset \text{Stab}_\Gamma(F) \cdot \sum_{i \in I} \Phi_i \subset \text{Stab}_\Gamma(F) \cdot \Pi_F. \quad \square$$

3.3. Assembling actions of polyhedral type in a chamber decomposition.

We keep the same setting as before: let $C \subset V$ be a strictly convex cone with nonempty interior, and let $\Gamma \leq \text{GL}(V_{\mathbb{Z}})$ be a subgroup preserving C .

Let $\{N_Y\}_{Y \in I}$ be a collection of convex cones contained in C , with nonempty interiors, such that

- the action of Γ on C naturally induces an action by permutations on the set $\{N_Y\}_{Y \in I}$;
- letting $M := \bigcup_{Y \in I} N_Y^+$, we have $C^\circ \subset M \subset C$.

Proposition 3.7. *In this setting, we assume that the action of Γ by permutations on the index set I has finitely many orbits, and that for every $Y \in I$, the action $\text{Stab}_\Gamma(N_Y) \curvearrowright N_Y$ is of polyhedral type. Then, the action $\Gamma \curvearrowright C$ is of polyhedral type as well, and we have $C^+ \subset C$.*

Proof. By assumption, we can pick a finite set of representatives $\{Y_1, \dots, Y_k\}$ for the orbits of $\Gamma \curvearrowright I$. For each Y_i , there exists by Proposition 3.3 a rational polyhedral cone Π_i inside $N_{Y_i}^+$ such that

$$\text{Stab}_\Gamma(N_{Y_i}) \cdot \Pi_i = N_{Y_i}^+.$$

Let Π be the convex cone generated by all of the Π_i , for $1 \leq i \leq k$. It is a rational polyhedral cone contained C by assumption, and in C^+ since it is rational polyhedral. For any N_Y , there are an element $g \in \Gamma$ and an index $i \in \{1, \dots, k\}$ such that

$$N_Y^+ = g \cdot N_{Y_i}^+ = g \cdot \text{Stab}_\Gamma(N_{Y_i}) \cdot \Pi_i \subset \Gamma \cdot \Pi,$$

thus taking the union over I , we obtain $C^\circ \subset M \subset \Gamma \cdot \Pi$. The action $\Gamma \curvearrowright C$ is thus of polyhedral type, and we can apply Proposition 3.1 to show that

$$C^+ = \Gamma \cdot \Pi \subset C. \quad \square$$

4. CHAMBER DECOMPOSITIONS OF CONES OF DIVISORS

4.1. Mori chambers. Let X be a normal \mathbb{Q} -factorial projective variety. For a \mathbb{Q} -factorial birational contraction (Y, f) of X , we define the *f-Mori chamber* $\text{Eff}(X; f: X \dashrightarrow Y)$, or for short $\text{Eff}(X; f)$, as the following cone

$$\text{Eff}(X; f) := f^* \text{Nef}^e(Y) + \sum_{E \in \text{Exc}(f)} \mathbb{R}_{\geq 0}[E] \subset \text{Eff}(X),$$

where $\text{Exc}(f)$ is the (finite) set of prime exceptional divisors of f . Note that it is a strictly convex cone of full dimension.

Remark 4.1. Note that the following facts hold.

- If $\alpha: X \dashrightarrow Y$ is a small \mathbb{Q} -factorial modification and $\mu: Y \dashrightarrow Z$ is a \mathbb{Q} -factorial birational contraction, then $\alpha^* \text{Eff}(X; \mu) = \text{Eff}(X; \mu \circ \alpha)$.
- If $\alpha: X \dashrightarrow Y$ is a small \mathbb{Q} -factorial modification, we have $\text{Eff}(X; \alpha) = \alpha^* \text{Nef}^e(Y)$.

The following lemma inspires the name *Mori chambers*. It generalizes [Kaw97, Lemma 1.5], which describes pullbacks of nef cones of minimal models of a given Calabi–Yau variety X as *chambers* within the movable cone $\overline{\text{Mov}}(X)$.

Lemma 4.2. *Let X be a normal \mathbb{Q} -factorial projective variety. Let (Y_1, f_1) and (Y_2, f_2) be marked \mathbb{Q} -factorial birational contractions of X . Then, the following are equivalent:*

- (1) (Y_1, f_1) and (Y_2, f_2) are isomorphic;
- (2) the two cones $\text{Eff}(X; f_1)$ and $\text{Eff}(X; f_2)$ coincide inside $N^1(X)_\mathbb{R}$;
- (2') the two cones $f_1^* \text{Nef}^e(Y_1)$ and $f_2^* \text{Nef}^e(Y_2)$ coincide inside $N^1(X)_\mathbb{R}$;
- (3) $\text{Eff}^\circ(X; f_1) \cap \text{Eff}^\circ(X; f_2) \neq \emptyset$ in $N^1(X)_\mathbb{R}$, where $\text{Eff}^\circ(X; f_i)$ denotes the interior of $\text{Eff}(X; f_i)$;
- (3') $\text{ri}(f_1^* \text{Nef}^e(Y_1)) \cap \text{ri}(f_2^* \text{Nef}^e(Y_2)) \neq \emptyset$ in $N^1(X)_\mathbb{R}$. Here ri denotes the relative interior.

To prove this lemma, we first recall the following result, stated in [HK00, Lemma 1.7]. It is an application of the negativity lemma and the rigidity lemma.

Lemma 4.3 ([HK00, Lemma 1.7]). *Let $f_i : X \dashrightarrow Y_i$ be two birational contractions of normal \mathbb{Q} -factorial projective varieties. Suppose that we have a numerical equivalence*

$$f_1^* D_1 + E_1 \equiv f_2^* D_2 + E_2$$

of \mathbb{R} -divisors with D_1 ample on Y_1 , D_2 nef on Y_2 , and E_i effective f_i -exceptional. Then $f_1 \circ f_2^{-1} : Y_2 \rightarrow Y_1$ is regular.

Proof of Lemma 4.3. This is essentially [HK00, Proof of Lemma 1.7], with more details. Take a common resolution of f_1 and f_2

$$\begin{array}{ccc} & W & \\ p_1 \swarrow & \downarrow g & \searrow p_2 \\ Y_1 \xleftarrow{f_1} & X & \xrightarrow{f_2} Y_2 \end{array}$$

where W is a normal \mathbb{Q} -factorial projective variety and p_1, p_2, g are birational morphisms. By the negativity lemma [Fuj17, Lemma 2.3.26], applied to g , we note that for both values of i , the divisor $g^* f_i^* D_i - p_i^* D_i = g^* g_*(p_i^* D_i) - p_i^* D_i$ is an effective p_i -exceptional divisor. Thus for $i = 1, 2$, we have

$$g^* f_i^* D_i + g^* E_i = p_i^* D_i + R_i,$$

where R_i is an effective p_i -exceptional divisor.

Denote by L the difference $R_2 - R_1 \in \text{Cl}(W)_{\mathbb{R}}$, and let $L_i := (-1)^{i+1} L$. We have

$$p_{i*} L_i = p_{i*} R_j \in \text{Cl}(Y_i)_{\mathbb{R}},$$

which is effective, where j is the only element in $\{1, 2\}$ different from i . Moreover, we have

$$-L_i \equiv (-1)^i (p_1^* D_1 - p_2^* D_2),$$

which is a p_i -nef class.

The negativity lemma [Fuj17, Lemma 2.3.26] applies, showing that L_i is effective for both $i = 1, 2$. Therefore, $p_1^* D_1 \equiv p_2^* D_2$. This shows that any curve C in W that is contracted by p_2 thus satisfies $D_1 \cdot p_{1*} C = 0$. Since D_1 is ample, any curve that is contracted by p_2 is thus contracted by p_1 . By the rigidity lemma [Deb01, Lemma 1.15], we can then factor p_1 through p_2 , i.e., $f_1 \circ f_2^{-1}$ is regular, as wished. \square

We now prove Lemma 4.2.

Proof of Lemma 4.2. The implications (1) \Rightarrow (2) \Rightarrow (3) and (1) \Rightarrow (2') \Rightarrow (3') are clear. Let us prove that (3) \Rightarrow (1) and (3') \Rightarrow (1).

By [Roc70, Theorem 6.6], we have $\text{ri}(f_i^* \text{Nef}^e(Y_i)) = f_i^* \text{Amp}(Y_i)$. From [Roc70, Corollary 6.6.2], it follows

$$\text{Eff}^\circ(X; f_i) = \text{ri}(\text{Eff}(X; f_i)) = f_i^* \text{Amp}(Y_i) + \text{ri}\left(\sum_{E \in \text{Exc}(f)} \mathbb{R}_{\geq 0}[E]\right)$$

where the first equality holds because the convex cone $\text{Eff}(X; f_i)$ has full dimension.

Assume either (3) or (3'). Then we can take for both $i = 1, 2$, some \mathbb{R} -divisor D_i ample on Y_i and (possibly zero) E_i f_i -exceptional on X such that $f_1^*D_1 + E_1 \equiv f_2^*D_2 + E_2$. Applying Lemma 4.3 twice, symmetrically, we obtain that $f_1 \circ f_2^{-1}: Y_1 \rightarrow Y_2$ is biregular, hence an isomorphism. This shows that (1) holds. \square

4.2. Applications of the Shokurov polytope. We first recall the following result by Shokurov and Birkar (see [Fuj17, Section 4.7] for more details).

Proposition 4.4 ([Bir11, Proposition 3.2.(3)]). *Let X be a normal \mathbb{Q} -factorial projective variety. Let D_1, \dots, D_k be prime divisors on X , and let V be the vector space of \mathbb{R} -divisors spanned by the D_i , for $1 \leq i \leq k$. Then the set*

$$\mathcal{N}(V) := \{B \in V : (X, B) \text{ is log canonical and } K_X + B \text{ is nef}\}$$

is a rational polytope.

We present two applications of this important result.

Proposition 4.5. *Let (X, Δ) be a klt Calabi–Yau pair. Then there is an effective \mathbb{Q} -divisor Δ' such that $K_X + \Delta' \sim_{\mathbb{Q}} 0$, and (X, Δ') is klt.*

Proposition 4.6. *Let (X, Δ) be a klt Calabi–Yau pair. Then the inclusion $\text{Nef}^e(X) \subset \text{Nef}^+(X)$ holds.*

Proof of Propositions 4.5 and 4.6. Proposition 4.5 seems to be well-known to the experts, while Proposition 4.6 is proved by [LOP20, Theorem 2.15] for klt Calabi–Yau varieties and by [LZ, Lemma 5.1.(1)] for pairs. For the sake of completeness, we now propose a proof of the two propositions at once.

Let D be an \mathbb{R} -divisor whose numerical class is in $\text{Nef}^e(X)$. There is some $0 < \varepsilon_0 \ll 1$ such that $(X, \Delta + \varepsilon D)$ is klt for any $\varepsilon \in [0, \varepsilon_0]$. Now fix an arbitrary $\varepsilon \in [0, \varepsilon_0]$. Note that $K_X + \Delta + \varepsilon D \equiv \varepsilon D$ is nef. Let $V \subset \text{Div}_{\mathbb{R}}(X)$ be the vector space spanned by the components of $\Delta + \varepsilon D$. Then $\Delta + \varepsilon D \in \mathcal{N}(V)$. By Proposition 4.4, the set $\mathcal{N}(V)$ is a rational polytope. We have

$$\Delta + \varepsilon D = \sum_{i=1}^n r_i B_i \text{ for some } r_i \in \mathbb{R}_{>0} \text{ with } \sum_{i=1}^n r_i = 1,$$

where B_1, \dots, B_n are some of vertices of $\mathcal{N}(V)$. Adding K_X on both sides, we get

$$\varepsilon D \equiv \sum_{i=1}^n r_i (K_X + B_i).$$

Taking $\varepsilon > 0$, we see that εD is in the convex hull of a finite set of nef \mathbb{Q} -divisors, i.e., $D \in \text{Nef}^+(X)$. This shows Proposition 4.6.

Taking $\varepsilon = 0$, we have $\sum_{i=1}^n r_i (K_X + B_i) \equiv 0$, which implies $K_X + B_i \equiv 0$ for every i . Since (X, Δ) is klt, and since being klt is an open condition, we can perturb the r_i into $r'_i \in \mathbb{Q}_{>0}$ such that $\sum_{i=1}^n r'_i = 1$, and for the effective \mathbb{Q} -divisor $\Delta' := \sum_{i=1}^n r'_i B_i$, the pair (X, Δ') is klt. Clearly, (X, Δ') is also Calabi–Yau. Finally, we can apply the abundance result [Nak04, V.4.9 Corollary] to obtain $K_X + \Delta' \sim_{\mathbb{Q}} 0$. This shows Proposition 4.5. \square

The following result is not used in any of the later proofs, but we include it for the curious reader.

Corollary 4.7. *Let (X, Δ) be a klt Calabi–Yau pair, and let $f: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$ be a \mathbb{Q} -factorial birational contraction. Then $\text{Eff}(X; f) \subset \text{Eff}(X; f)^+$.*

Proof. This is essentially a consequence of Proposition 4.6. Let us spell it out. By definition,

$$\text{Eff}(X; f) = f^* \text{Nef}^e(Y) + \sum_{E \in \text{Exc}(f)} \mathbb{R}_{\geq 0}[E].$$

Since every E is \mathbb{Q} -Cartier, it suffices to show that the cone $f^* \text{Nef}^e(Y)$ is contained in $\text{Eff}(X; f)^+$ to conclude.

By Proposition 4.6 and by injectivity of f^* , we have

$$f^* \text{Nef}^e(Y) \subset \text{Conv}_{\mathbb{R}}(f^* \text{Nef}(Y) \cap f^* N^1(Y)_{\mathbb{Q}}).$$

We clearly have the inclusions $f^* \text{Nef}(Y) \subset \overline{\text{Eff}}(X; f)$, and $f^* N^1(Y)_{\mathbb{Q}} \subset N^1(X)_{\mathbb{Q}}$. Taking intersections and then convex hulls, we obtain the inclusion wished. \square

We conclude this subsection by mentioning the following result, which can be of interest to the reader. We will not use it below. The proof is again an application of Proposition 4.4.

Proposition 4.8 ([LZ, Theorem 2.7]). *Let (X, Δ) be a klt Calabi–Yau pair. Let $\Pi \subset \text{Eff}(X)$ be a polyhedral cone. Then $\Pi \cap \text{Nef}(X)$ is a polyhedral cone as well. Moreover, if the polyhedral cone Π is rational, then $\Pi \cap \text{Nef}(X)$ is also rational.*

Proof. This is claimed in [LZ, Theorem 2.7]. We give a proof here for the sake of completeness. Since $\text{Eff}(X)$ is spanned by numerical classes of \mathbb{Z} -divisors, there exists a rational polyhedral cone Π' such that $\Pi \subset \Pi' \subset \text{Eff}(X)$. If we prove Proposition 4.8 for Π' , then $\Pi \cap \text{Nef}(X) = \Pi \cap (\Pi' \cap \text{Nef}(X))$ is a polyhedral cone, and we obtain Proposition 4.8 for Π as well. Hence, we assume in what follows that Π is rational polyhedral.

By Proposition 4.6, we can assume that Δ is a \mathbb{Q} -divisor. Let us denote by D_1, \dots, D_r the prime effective \mathbb{Z} -divisors whose classes span the extremal rays of the rational polyhedral cone Π . We introduce the cone Π_{div} spanned by the divisors D_i in the vector space $\text{Div}(X)_{\mathbb{R}}$, and we let V be the \mathbb{R} -vector space generated by the irreducible components of Δ and by D_1, \dots, D_r . Then, by Proposition 4.4,

$$\mathcal{N} := \{B \in V : (X, B) \text{ is log canonical and } K_X + B \text{ is nef}\}$$

is a rational polytope. Let us define

$$\mathcal{M} := \{D \in \Pi_{\text{div}} : (X, \Delta + D) \text{ is log canonical and } D \text{ is nef}\}.$$

Then, we clearly have a relation between \mathcal{N} and the translation of \mathcal{M} by the divisor Δ , namely

$$\Delta + \mathcal{M} = \mathcal{N} \cap (\Delta + \Pi_{\text{div}}).$$

Since Δ is a \mathbb{Q} -divisor, and \mathcal{N} and Π_{div} are a rational polytope and a rational cone respectively, we see that \mathcal{M} is a rational polytope. Thus, the projection

$$\mathcal{P} := \{[D] \in \Pi : D \in \Pi_{\text{div}}, (X, \Delta + D) \text{ is log canonical and } D \text{ is nef}\}$$

is a rational polytope in $N^1(X)_{\mathbb{R}}$; cf. [Roc70, Theorem 19.3 and its proof]. The cone over \mathcal{P} is thus a rational polyhedral cone, which we denote by $\text{Cone}(\mathcal{P})$. We claim that $\text{Cone}(\mathcal{P}) = \Pi \cap \text{Nef}(X)$. The direct inclusion is clear. For the reverse inclusion, we take an arbitrary numerical class $[D] \in \Pi \cap \text{Nef}(X)$ with $D \in \Pi_{\text{div}}$. For a sufficiently small rational number $\varepsilon > 0$, we have a klt pair $(X, \Delta + \varepsilon D)$, and thus $\varepsilon[D] \in \mathcal{P}$, as wished. This concludes the proof. \square

4.3. Decompositions of the effective and movable cones. The main result of this subsection is Proposition 4.10, which is an expanded formulation of Proposition 1.1 about the chamber decomposition of the effective cone mentioned in the introduction.

Let us start with decomposing a \mathbb{Q} -factorial birational contraction.

Lemma 4.9. *Let (X, Δ) be a Calabi–Yau pair, and let $f: X \dashrightarrow Z$ be a \mathbb{Q} -factorial birational contraction. Then we have a factorization $f = \mu \circ \alpha$, where $\alpha: X \dashrightarrow Y$ is a small \mathbb{Q} -factorial modification, and $\mu: Y \rightarrow Z$ is a regular \mathbb{Q} -factorial birational contraction or an isomorphism.*

Proof. Let H_Z be an ample divisor on Z , and let $1 \gg \epsilon > 0$ such that the pair $(X, \Delta + \epsilon f^* H_Z)$ is klt. Since $f^* H_Z$ is big, by [BCHM10, Theorem 1.2], we have a minimal model $\alpha: (X, \Delta + \epsilon f^* H_Z) \dashrightarrow (Y, \Delta_Y + \epsilon \alpha_* f^* H_Z)$. Note that since $f^* H_Z$ is a movable and big divisor, the map α does not contract any divisors; it thus is a small \mathbb{Q} -factorial modification. Let $\mu: Y \rightarrow Z$ denote the composition $f \circ \alpha^{-1}$. Note that $\mu^* H_Z = \alpha_* f^* H_Z$ is nef. By Lemma 4.3, this shows that μ is regular. Since Z is \mathbb{Q} -factorial, μ is indeed either an isomorphism or a regular \mathbb{Q} -factorial birational contraction of Y , as wished. \square

Proposition 4.10. *Let (X, Δ) be a Calabi–Yau pair. Then we have the following inclusions:*

$$\begin{array}{ccccc} \text{Big}(X) & \subset & \bigcup_{\substack{(Z,f) \\ \text{QBC}}} \text{Eff}(X; f: X \dashrightarrow Z) & \subset & \text{Eff}(X) \\ \cup & & \cup & & \cup \\ \text{Mov}^\circ(X) & \subset & \bigcup_{\substack{(Y,\alpha) \\ \text{SQM}}} \text{Eff}(X; \alpha: X \dashrightarrow Y) & \subset & \overline{\text{Mov}}^e(X) \end{array}$$

where the indices (Z, f) (resp. (Y, α)) range over all isomorphism classes of marked \mathbb{Q} -factorial birational contractions (resp. marked small \mathbb{Q} -factorial modifications) of X , and the cones featured in the unions have disjoint interiors.

Moreover, assuming the existence of minimal models for X , we obtain decompositions of the cones $\text{Eff}(X)$ and $\overline{\text{Mov}}^e(X)$:

$$\begin{array}{ccc} \bigcup_{\substack{(Z,f) \\ \text{QBC}}} \text{Eff}(X; f: X \dashrightarrow Z) & = & \text{Eff}(X) \\ \cup & & \cup \\ \bigcup_{\substack{(Y,\alpha) \\ \text{SQM}}} \text{Eff}(X; \alpha: X \dashrightarrow Y) & = & \overline{\text{Mov}}^e(X) \end{array}$$

Proof. The claim about “disjoint interiors” follows from Lemma 4.2. Moreover, that the union is contained in $\text{Eff}(X)$ (resp. $\overline{\text{Mov}}^e(X)$) is clear.

We now prove

$$\text{Eff}(X) \subset \bigcup_{\substack{(Z,f) \\ \text{QBC}}} \text{Eff}(Z; f)$$

assuming the existence of minimal models for X , and

$$\text{Big}(X) \subset \bigcup_{\substack{(Z,f) \\ \text{QBC}}} \text{Eff}(Z; f)$$

unconditionally. Let D be an effective \mathbb{R} -divisor on X , and let $0 < \epsilon \ll 1$ such that the pair $(X, \Delta + \epsilon D)$ is klt. By the existence of minimal models for X , or [BCHM10, Theorem 1.2] when D is big, we have a minimal model $f: (X, \Delta + \epsilon D) \dashrightarrow (Z, \Delta_Z)$. Denoting by E_i for $1 \leq i \leq k$ the prime exceptional divisors of f , we have

$$\epsilon D \equiv K_X + \Delta + \epsilon D \equiv f^*(K_Z + \Delta_Z) + \sum_{1 \leq i \leq k} a_i E_i,$$

where we recall that $K_Z + \Delta_Z$ is nef and $a_i = a(E_i; Z, \Delta_Z) - a(E_i; X, \Delta) \geq 0$. Since D is effective, its pushforward $K_Z + \Delta_Z$ is also not just nef, but nef and effective. So D is in the cone $\text{Eff}(Z; f)$, as wished.

The inclusions

$$\overline{\text{Mov}}^e(X) \subset \bigcup_{\substack{\alpha: X \dashrightarrow Y \\ \text{SQM}}} \alpha^* \text{Nef}^e(Y)$$

assuming the existence of minimal models for X , and

$$\text{Mov}^\circ(X) \subset \bigcup_{\substack{\alpha: X \dashrightarrow Y \\ \text{SQM}}} \alpha^* \text{Nef}^e(Y)$$

unconditionally has been proven in many contexts in the literature (see [Kaw97, Theorem 2.3], [Wan22, Propositions 4.6 and 4.7], [SX, Proposition 1.1]). To show it, we proceed as in the case of the effective cone but remark that, starting with a divisor $D \in \overline{\text{Mov}}^e(X)$, we do not contract any divisors by running the minimal model program. \square

The following result is well-known and appears in various contexts in the literature; see e.g., [Kaw97, Proposition 2.4], [LZ, Lemma 5.1.(2)] (assuming the existence of good minimal models), and [SX, Theorem 3.5].

Corollary 4.11. *Let (X, Δ) be a klt Calabi–Yau pair, and assume the existence of minimal models for X . Then $\overline{\text{Mov}}^e(X) \subset \text{Mov}^+(X)$.*

Proof. By Proposition 4.10, we have the decomposition

$$\overline{\text{Mov}}^e(X) = \bigcup_{\substack{(Y, \alpha) \\ \text{SQM}}} \alpha^* \text{Nef}^e(Y),$$

where the union is taken over all the small \mathbb{Q} -factorial modifications (Y, α) of X . For each (Y, α) , we have $\text{Nef}^e(Y) \subset \text{Nef}^+(Y)$ by Proposition 4.6. Moreover, since $\alpha^* \text{Nef}^e(Y) \subset \overline{\text{Mov}}^e(X)$ and since the operator $+$ preserves inclusions of convex cones, we obtain $\alpha^* \text{Nef}^e(Y) \subset \text{Mov}^+(X)$. This concludes. \square

4.4. Induced chamber decompositions of polyhedral cones. In this subsection, we introduce a chamber decomposition for a rational polyhedral cone, which is naturally induced by the Mori chamber decompositions described in Proposition 4.10.

Proposition 4.12. *Let (X, Δ) be a klt Calabi–Yau pair, and assume the existence of good minimal models in dimension $\dim X$. Let $\Pi \subset \text{Eff}(X)$ be a polyhedral cone. Then, there are only finitely many marked \mathbb{Q} -factorial birational contractions $f_i: X \dashrightarrow Y_i$ up to isomorphism such that the interior of the cone $\text{Eff}(X; f_i)$ intersects Π , yielding a finite chamber decomposition:*

$$\Pi = \bigcup_{i=1}^r \Pi \cap \overline{\text{Eff}(X; f_i)},$$

and every closed chamber $\Pi \cap \overline{\text{Eff}(X; f_i)}$ is polyhedral.

Moreover, if Π is rational polyhedral, then every closed chamber $\Pi \cap \overline{\text{Eff}(X; f_i)}$ is rational polyhedral.

In order to prove this statement we apply the following result, due to Kaloghiros, Küronya, and Lazić [KKL16, Theorem 1.1]. That type of result is part of understanding the so-called *geography of models*. A history of this kind of questions in the literature is given in the introduction of the paper [KKL16]; more recently, the similar result [LZ, Theorem 2.6] also draws from this pool of ideas.

Theorem 4.13. *Let X be a normal \mathbb{Q} -factorial projective variety, and let D_1, \dots, D_r be effective \mathbb{Q} -divisors whose numerical classes span $N^1(X)_{\mathbb{R}}$. Assume that the following three conditions hold.*

- (i) *The ring $R(X; D_1, \dots, D_r)$ is finitely generated.*
- (ii) *The convex cone $\Pi \subset N^1(X)_{\mathbb{R}}$ spanned by the D_i , for $1 \leq i \leq r$, contains an ample divisor.*
- (iii) *Every divisor D in the interior of this cone is gen, i.e., for any \mathbb{Q} -divisor D' numerically equivalent to D , the section ring $R(X, D')$ is finitely generated.*

Then there is a finite decomposition

$$\Pi = \bigsqcup_{j \in J} \mathcal{N}_j$$

in $N^1(X)_{\mathbb{R}}$ into convex cones having the following properties:

- (1) each $\overline{\mathcal{N}_j}$ is a rational polyhedral cone;
- (2) for each j , there exists a \mathbb{Q} -factorial birational contraction $\phi_j: X \dashrightarrow X_j$ such that ϕ_j is an optimal model for every divisor in \mathcal{N}_j .

We refer to [KKL16, Definition 2.3.(2)] for the precise definition of an optimal model. We only note that when D is an adjoint divisor, an optimal model for D is just a minimal model in the sense of Definition 2.4.(1) (cf. [KKL16, Remark 2.4.(iv)]).

The convex cones \mathcal{N}_j in Theorem 4.13 and the Mori chambers are related by the following lemma.

Lemma 4.14. *We continue Theorem 4.13, with the same notations and assumptions. For any \mathbb{Q} -factorial birational contraction $f: X \dashrightarrow Y$, we have inclusions*

$$(4.1) \quad \Pi \cap \text{Eff}^\circ(X; f) \subset \bigcup_{j \in J(f)} \overline{\mathcal{N}_j} \subset \Pi \cap \overline{\text{Eff}(X; f)},$$

where $J(f) := \{j \in J \mid \mathcal{N}_j \cap \text{Eff}^\circ(X; f) \neq \emptyset\} = \{j \in J \mid (X_j, \phi_j) \simeq (Y, f)\}$.

If moreover $\Pi \cap \text{Eff}^\circ(X; f) \neq \emptyset$, then

$$\Pi \cap \overline{\text{Eff}(X; f)} = \bigcup_{j \in J(f)} \overline{\mathcal{N}_j}.$$

Proof. We first prove the equality about $J(f)$. By [KKL16, Definition 2.3], by the definitions of Subsection 4.1, and since $\mathcal{N}_j \subset \Pi \subset \text{Eff}(X)$, we see that \mathcal{N}_j is contained in the ϕ_j -Mori chamber $\text{Eff}(X; \phi_j)$. Therefore, if $j \in J(f)$, we have $\text{Eff}^\circ(X; f) \cap \text{Eff}(X; \phi_j) \neq \emptyset$, and thus $\text{Eff}^\circ(X; f) \cap \text{Eff}^\circ(X; \phi_j) \neq \emptyset$ by [Roc70, Corollary 6.3.2]. By Lemma 4.2, this is the case if and only if the marked \mathbb{Q} -factorial birational contractions (Y, f) and (X_j, ϕ_j) are isomorphic. Hence the equality about $J(f)$.

Now we prove (4.1). For the first inclusion, it suffices to show that

$$\mathcal{N}_k \cap \text{Eff}^\circ(X; f) \subset \bigcup_{j \in J(f)} \overline{\mathcal{N}_j}$$

for every $k \in J$. If $k \notin J(f)$, the left handside is empty; if $k \in J(f)$, then $\overline{\mathcal{N}_k}$ appears on the right handside.

The second inclusion follows from

$$\mathcal{N}_j \subset \text{Eff}(X; \phi_j) = \text{Eff}(X; f)$$

for every $j \in J(f)$.

Finally if $\Pi \cap \text{Eff}^\circ(X; f) \neq \emptyset$, then $\Pi^\circ \cap \text{Eff}^\circ(X; f) \neq \emptyset$ by [Roc70, Corollary 6.3.2]. Now [Roc70, Theorem 6.5] implies the equality:

$$\overline{\Pi \cap \text{Eff}^\circ(X; f)} = \Pi \cap \overline{\text{Eff}(X; f)}.$$

Since the union $\bigcup_{j \in J(f)} \overline{\mathcal{N}_j}$ is closed, the inclusions in (4.1) yield the equality in the last assertion. \square

We now proceed to the proof of Proposition 4.12.

Proof of Proposition 4.12. We start with a remark: If we prove Proposition 4.12 for a polyhedral cone $\Pi' \subset \text{Eff}(X)$, then it follows that Proposition 4.12 holds for any polyhedral subcone $\Pi \subset \Pi'$. Indeed, the finiteness for Π follows from the finiteness for Π' , while the polyhedrality for Π comes from the fact that

$$\Pi \cap \overline{\text{Eff}(X; f_i)} = \Pi \cap (\Pi' \cap \overline{\text{Eff}(X; f_i)}),$$

and the intersection of two polyhedral cones is polyhedral. Using this remark, we are reduced to the following essential case: In what follows, we assume that $\Pi \subset \text{Eff}(X)$ is a full-dimensional rational polyhedral cone with $\Pi \cap \text{Amp}(X) \neq \emptyset$. This allows us to apply Theorem 4.13.

Let D_1, \dots, D_r be the effective divisors which generate the rational polyhedral cone Π . Since Π has full dimension, they span $N^1(X)_\mathbb{R}$. Let us verify the three assumptions of Theorem 4.13.

To check Condition (i), we use our assumption of the existence of good minimal models in dimension $\dim X$. We may assume that Δ is an effective \mathbb{Q} -divisor such that $K_X + \Delta \sim_{\mathbb{Q}} 0$ by Proposition 4.5. For a suitably small $\varepsilon \in \mathbb{Q}_{>0}$, the pairs $(X, \Delta + \varepsilon D_i)$ are klt for $i = 1, \dots, r$. Since we assume the existence of good minimal models in dimension $\dim X$, and by [DHP13, Theorem 8.10], the section ring

$$R = R(X, K_X + \Delta + \varepsilon D_1, \dots, K_X + \Delta + \varepsilon D_r)$$

is finitely generated. Since $K_X + \Delta \sim_{\mathbb{Q}} 0$, the section ring $R(X, D_1, \dots, D_r)$ is finitely generated as well.

Condition (ii) clearly holds since $\Pi \cap \text{Amp}(X) \neq \emptyset$.

To check Condition (iii), we recall that (X, Δ) is a klt Calabi-Yau pair. Hence, for any big \mathbb{Q} -divisor D' on X , we can apply [KKL16, Corollary 3.7] and see that $R(X, D')$ is finitely generated, as wished.

Hence, Theorem 4.13 applies, and we have a finite decomposition

$$\Pi = \bigsqcup_{j \in J} \mathcal{N}_j,$$

where each closed cone $\overline{\mathcal{N}_j}$ is rational polyhedral, and we also have \mathbb{Q} -factorial birational contractions $\phi_j: X \dashrightarrow X_j$ that are optimal models for every divisor in the corresponding cone \mathcal{N}_j .

We conclude the proof as follows. In the notations of Lemma 4.14, two Mori chambers $\text{Eff}(X; f)$ and $\text{Eff}(X; g)$ are distinct if and only if the index sets $J(f)$ and $J(g)$ are disjoint. Hence, since J is finite, there are finitely many distinct Mori

chambers $\text{Eff}(X; f)$ with an associated non-empty index set $J(f)$, i.e., finitely many Mori chambers whose interior intersects Π .

For the polyhedrality of the intersections, take a Mori chamber $\text{Eff}(X; f)$ whose interior intersects Π . By Lemma 4.14, we have

$$\Pi \cap \overline{\text{Eff}(X; f)} = \bigcup_{j \in J(f)} \overline{\mathcal{N}_j}.$$

Thus, the union on the right handside is convex. Since $J(f)$ is finite, it is also closed, hence spanned by its extremal rays. Since any extremal ray of the union is an extremal ray in one of the rational polyhedral cones $\overline{\mathcal{N}_j}$, the union on the right handside is a rational polyhedral cone. Hence, the intersection $\Pi \cap \overline{\text{Eff}(X; f)}$ is a rational polyhedral cone, as wished. \square

5. A DESCENT RESULT FOR THE NEF CONE CONJECTURE

5.1. Stabilizers of Mori chambers. Let $f: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$ be a \mathbb{Q} -factorial birational contraction. Define

$$\text{PsAut}(X, \Delta; f) := f^{-1} \circ \text{Aut}(Y, \Delta_Y) \circ f, \text{ and}$$

$$\text{Aut}(X, \Delta; f) := \text{Aut}(X) \cap \text{PsAut}(X, \Delta; f).$$

Corollary 5.1. *Let $f: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$ be a \mathbb{Q} -factorial birational contraction. Then*

$$\begin{aligned} \text{PsAut}(X, \Delta; f) &= \text{Stab}(\text{PsAut}(X, \Delta) \subset f^* \text{Nef}(Y)) \\ &= \text{Stab}(\text{PsAut}(X, \Delta) \subset \text{Eff}(X; f)). \end{aligned}$$

Proof. Let $\gamma \in \text{PsAut}(X, \Delta)$. Applying Lemma 4.2 (and Lemma 2.2) to the two \mathbb{Q} -factorial birational contractions f and $f \circ \gamma: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$ concludes. \square

For a later application, we also define the group

$$\text{Aut}(Y, \Delta_Y; f) := f \circ \text{Aut}(X, \Delta; f) \circ f^{-1}.$$

Using the fact that $\text{Aut}(X, \Delta; f) \leq \text{PsAut}(X, \Delta; f)$, it follows from the definition of $\text{PsAut}(X, \Delta; f)$ that $\text{Aut}(Y, \Delta_Y; f)$ is a subgroup of $\text{Aut}(Y, \Delta_Y)$.

5.2. Descending the nef cone conjecture. The following statement is a descent result for the nef cone conjecture, similar to [Tot10, Lemma 3.4] for surface pairs.

Proposition 5.2. *Let (X, Δ) be a klt Calabi–Yau pair. Assume that the nef cone conjecture holds for (X, Δ) . Then for any regular \mathbb{Q} -factorial birational contraction $f: (X, \Delta) \rightarrow (Y, \Delta_Y)$, the action*

$$\text{Aut}(Y, \Delta_Y; f) \subset \text{Nef}(Y)$$

is of polyhedral type, and $\text{Nef}^e(Y) = \text{Nef}^+(Y)$. In particular, the nef cone conjecture holds for (Y, Δ_Y) .

Proof. By Corollary 5.1, we have

$$\mathrm{Aut}(X, \Delta; f) = \mathrm{Stab}(\mathrm{Aut}(X, \Delta) \subset f^* \mathrm{Nef}(Y)).$$

Applying Proposition 3.6 to the face $f^* \mathrm{Nef}^+(Y)$ of the cone $\mathrm{Nef}^+(X)$, we obtain that $\mathrm{Aut}(X, \Delta; f) \subset f^* \mathrm{Nef}^+(Y)$ is of polyhedral type. This shows that the action $\mathrm{Aut}(Y, \Delta_Y; f) \subset \mathrm{Nef}^+(Y)$ is of polyhedral type as well.

By our assumption on (X, Δ) , we have $\mathrm{Nef}^+(X) = \mathrm{Nef}^e(X)$, so

$$f^* \mathrm{Nef}^+(Y) \subset \mathrm{Nef}^+(X) = \mathrm{Nef}^e(X) \subset \mathrm{Eff}(X).$$

Hence by the negativity lemma, $\mathrm{Nef}^+(Y) \subset \mathrm{Eff}(Y)$. Together with Proposition 4.6, this proves that $\mathrm{Nef}^e(Y) = \mathrm{Nef}^+(Y)$.

Since $\mathrm{Aut}(Y, \Delta_Y; f)$ is a subgroup of $\mathrm{Aut}(Y, \Delta_Y)$, we can apply Proposition 3.3 to see that the nef cone conjecture holds for (Y, Δ) . \square

5.3. Finiteness statement from cone conjecture. It is well-known that the cone conjectures imply various finiteness statements. In this subsection, we recall one of such results for later use. We also provide a proof for the sake of completeness.

Proposition 5.3. *Let (X, Δ) be a klt Calabi–Yau pair. Assume that the nef cone conjecture holds for (X, Δ) . Then, up to isomorphism of pairs, there are only finitely many pairs (Y, Δ_Y) arising from regular \mathbb{Q} -factorial birational contractions $(X, \Delta) \rightarrow (Y, \Delta_Y)$.*

Proof. By assumption, $\mathrm{Aut}(X, \Delta) \subset \mathrm{Nef}^e(X)$ has a rational polyhedral fundamental domain Π . The set of isomorphism classes of pairs (Y_i, Δ_{Y_i}) obtained from (X, Δ) by regular \mathbb{Q} -factorial birational contractions injects into the set of $\mathrm{Aut}(X, \Delta)$ -orbits of closed faces of the closed cone $\mathrm{Nef}(X)$ whose relative interior is contained in the big cone $\mathrm{Big}(X)$. Since any such orbit contains at least one face of the rational polyhedral cone Π , the set of these orbits is finite. This proves the proposition. \square

6. PROOF OF THEOREM 1.5

We state and prove the following theorem, which is slightly more general, and from which Theorem 1.5 directly follows.

Theorem 6.1. *Let (X, Δ) be a klt Calabi–Yau pair. Consider the following statements.*

- (1) *The movable cone conjecture holds for (X, Δ) .*
- (2) *The effective cone conjecture holds for (X, Δ) .*
- (2') *There is a polyhedral cone $\Pi \subset \mathrm{Eff}(X)$ satisfying*

$$\mathrm{Mov}^\circ(X) \subset \mathrm{PsAut}(X, \Delta) \cdot \Pi.$$

- (3) (a) *The nef cone conjecture holds for each klt pair (X', Δ') obtained by a small \mathbb{Q} -factorial modification from (X, Δ) .*
- (b) *Up to isomorphism of pairs, there exist only finitely many (X', Δ') arising as small \mathbb{Q} -factorial modifications of (X, Δ) .*

- (4) (a) *The Mori chamber cone conjecture holds for each \mathbb{Q} -factorial birational contraction $f: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$.*
- (b) *Up to isomorphism of pairs, there exist only finitely many (Y, Δ_Y) arising as \mathbb{Q} -factorial birational contractions of (X, Δ) .*

The following assertions hold.

- (i) *We have (3) \Leftrightarrow (4) \Rightarrow (2) and [(1) or (2)] \Rightarrow (2').*
- (ii) *Assuming the inclusion $\overline{\text{Mov}}^e(X) \subset \text{Mov}^+(X)$, we have (3) \Rightarrow (1).*
- (iii) *Assuming the existence of good minimal models in dimension $\dim X$, we have (2') \Rightarrow (3).*

As a consequence, assuming the existence of good minimal models in dimension $\dim X$, all the statements are equivalent.

To relate Theorem 1.5(ii) and Theorem 6.1 (ii), let us note that the existence of minimal models for X implies the inclusion $\overline{\text{Mov}}^e(X) \subset \text{Mov}^+(X)$, by Corollary 4.11.

Let us note that the implication [(1) or (2)] \Rightarrow (2') is clear. In the next subsections, we prove the remaining implications.

6.1. The equivalence between (3) and (4).

Proof of (3) \Leftrightarrow (4) in Theorem 6.1 (i). The implication (4) \Rightarrow (3) is quite straightforward. Let us explain that. Since small \mathbb{Q} -factorial modifications are \mathbb{Q} -factorial birational contractions, (4b) implies (3b). Let $f: (X, \Delta) \dashrightarrow (X', \Delta')$ be a small \mathbb{Q} -factorial modification. Then $\text{Eff}(X; f) = f^* \text{Nef}^e(X')$. Since

$$\text{PsAut}(X, \Delta; f) = f^{-1} \circ \text{Aut}(X', \Delta') \circ f$$

by definition (see Subsection 5.1), (4a) implies (3a).

Now we assume (3). First we prove (4b). Using (3b), we fix (X_i, Δ_i) for $i = 1, \dots, r$ to be a set of representatives of the (finitely many) klt pairs obtained by small \mathbb{Q} -factorial modifications from (X, Δ) . For each i , we also fix an arbitrary marking $\alpha_i: (X, \Delta) \dashrightarrow (X_i, \Delta_i)$. Since we assume (3a) and by Proposition 5.3, for each $1 \leq i \leq r$, we have finitely many $(Y_{i,j}, \Delta_{i,j})$, $1 \leq j \leq s_i$, arising as regular \mathbb{Q} -factorial birational contractions $\mu_{i,j}: (X_i, \Delta_i) \rightarrow (Y_{i,j}, \Delta_{i,j})$, where we take and fix one $\mu_{i,j}$ for each pair of i, j . We also set $\mu_{i,0}$ to be the identity automorphism on X_i .

We fix an arbitrary \mathbb{Q} -factorial birational contraction $f: (X, \Delta) \dashrightarrow (Y, \Delta_Y)$. By Lemma 4.9, we factorize $f = \mu \circ \alpha$, where $\alpha: (X, \Delta) \dashrightarrow (X', \Delta')$ is a small \mathbb{Q} -factorial modification and $\mu: (X', \Delta') \rightarrow (Y, \Delta_Y)$ is a regular \mathbb{Q} -factorial birational contraction or an isomorphism. By definition of the pairs (X_i, Δ_i) , we can find an index $1 \leq i \leq r$ and an isomorphism $\beta: (X_i, \Delta_i) \xrightarrow{\sim} (X', \Delta')$. Since $\mu \circ \beta: X_i \rightarrow Y$ is a regular \mathbb{Q} -factorial birational contraction, by definition of the $\mu_{i,j}$ we can then find an index $1 \leq j \leq s_i$, and an isomorphism $(Y, \Delta_Y) \xrightarrow{\sim} (Y_{i,j}, \Delta_{i,j})$, which implies (4b).

We now prove (4a). We keep the notations of the previous paragraph, notably the factorization $f = \mu \circ \alpha$. Recall also that

$$\text{PsAut}(X, \Delta; f) = \alpha^{-1} \circ \text{PsAut}(X', \Delta'; \mu) \circ \alpha, \quad \text{and} \quad \alpha^* \text{Eff}(X'; \mu) = \text{Eff}(X; f).$$

By (3a) and Proposition 5.2, we have a rational polyhedral cone Π inside $\text{Eff}(X'; \mu)$ satisfying

$$\text{Aut}(X', \Delta'; \mu) \cdot \Pi = \mu^* \text{Nef}^e(Y) = \mu^* \text{Nef}^+(Y).$$

This implies $\text{Eff}(X'; \mu) = \text{Eff}^+(X'; \mu)$ by definition of a Mori chamber. Define Σ as the convex cone spanned by Π and by the prime exceptional divisors E_1, \dots, E_ℓ of μ . Clearly, Σ is a rational polyhedral cone contained in $\text{Eff}(X'; \mu)$. Since $\text{Aut}(X', \Delta'; \mu)$ permutes the prime exceptional divisors E_1, \dots, E_ℓ , we have

$$\begin{aligned} (6.1) \quad \text{Aut}(X', \Delta'; \mu) \cdot \Sigma &= \text{Aut}(X', \Delta'; \mu) \cdot \left(\Pi + \sum_{k=1}^{\ell} \mathbb{R}_{\geq 0}[E_k] \right) \\ &= \mu^* \text{Nef}^+(Y) + \sum_{k=1}^{\ell} \mathbb{R}_{\geq 0}[E_k] = \text{Eff}(X'; \mu), \end{aligned}$$

and the equality also holds if the group $\text{Aut}(X', \Delta'; \mu)$ gets replaced by the larger group $\text{PsAut}(X', \Delta'; \mu)$. Hence $\text{PsAut}(X', \Delta'; \mu) \supseteq \text{Eff}(X'; \mu)$ satisfies the cone conjecture by Proposition 3.3. \square

6.2. Assembling the cone conjectures of chambers.

Proof of (4) \Rightarrow (2) in Theorem 6.1 (i) and of Theorem 6.1 (ii).

Since (3) and (4) are now proven to be equivalent, let us assume both. We want to prove (2), respectively (1) assuming $\overline{\text{Mov}}^e(X) \subset \text{Mov}^+(X)$. By Proposition 4.10, we have the following inclusions:

$$\begin{aligned} \text{Mov}^\circ(X) &\subset \bigcup_{\substack{(Y, \alpha) \\ \text{SQM}}} \text{Eff}(X; \alpha: X \dashrightarrow Y) \subset \overline{\text{Mov}}^e(X), \\ \text{and } \text{Big}(X) &\subset \bigcup_{\substack{(Z, f) \\ \text{QBC}}} \text{Eff}(X; f: X \dashrightarrow Z) \subset \text{Eff}(X). \end{aligned}$$

Since we assume (4b), we have $\text{Eff}(X; f) = \text{Eff}^+(X; f)$ for every \mathbb{Q} -factorial birational contraction $f: X \dashrightarrow Z$. This puts us in the setting of Subsection 3.3. We can check that the assumptions of Proposition 3.7 are satisfied: Since we are assuming (4a), there are finitely many \mathbb{Q} -factorial birational contractions of X modulo $\text{PsAut}(X, \Delta)$. Our assumption of (4b), together with Corollary 5.1, ensures that each induced action

$$\text{Stab}(\text{PsAut}(X, \Delta) \supseteq \text{Eff}(X; f)) \supseteq \text{Eff}(X; f)$$

is of polyhedral type.

Hence, we can apply Proposition 3.7. It shows that

$$\text{PsAut}(X, \Delta) \supseteq \overline{\text{Mov}}^e(X) \quad \text{and} \quad \text{PsAut}(X, \Delta) \supseteq \text{Eff}(X)$$

are of polyhedral type, and provides the inclusions

$$\text{Mov}^+(X) \subset \overline{\text{Mov}}^e(X) \quad \text{and} \quad \text{Eff}^+(X) \subset \text{Eff}(X).$$

We have the reverse inclusions $\overline{\text{Mov}}^e(X) \subset \text{Mov}^+(X)$ by assumption, and $\text{Eff}(X) \subset \text{Eff}^+(X)$ unconditionally. Now we use Proposition 3.3 to conclude. \square

6.3. From the effective cone to the nef cone conjectures.

Proof of Theorem 6.1 (iii).

We assume (2'): Let $\Pi \subset \text{Eff}(X)$ be a polyhedral cone such that

$$(6.2) \quad \text{Mov}^\circ(X) \subset \text{PsAut}(X, \Delta) \cdot \Pi.$$

By replacing Π with some $\text{PsAut}(X, \Delta)$ -translate, we can assume that Π intersects the ample cone $\text{Amp}(X)$. By Proposition 4.12, the cone Π only intersects the interiors of finitely many Mori chambers. In particular, the cone Π only intersects the interiors of finitely many Mori chambers of the form $\text{Eff}(X; \alpha: X \dashrightarrow Y)$ with α a small \mathbb{Q} -factorial modification. Let us list these chambers as

$$(6.3) \quad \text{Eff}(X; \alpha_1: X \dashrightarrow X_1), \dots, \text{Eff}(X; \alpha_r: X \dashrightarrow X_r).$$

Let us first show (3b): We will show that any klt pair (Y, Δ_Y) arising as a small \mathbb{Q} -factorial modification of (X, Δ_X) satisfies $Y \simeq X_i$ for some $1 \leq i \leq r$. We take an arbitrary small \mathbb{Q} -factorial modification $\alpha: X \dashrightarrow Y$ as a marking. By assumption, there is an element $\gamma \in \text{PsAut}(X, \Delta)$ such that the cone Π intersects the interior of $(\alpha \circ \gamma)$ -Mori chamber:

$$\text{Eff}^\circ(X; \alpha \circ \gamma) = \gamma^* \text{Eff}^\circ(X; \alpha: X \dashrightarrow Y) = \gamma^* \alpha^* \text{Amp}(X) \subset \text{Mov}^\circ(X)$$

(see also Remark 4.1). In our exhaustive list (6.3), there is therefore an index j such that $\text{Eff}(X; \alpha \circ \gamma) = \text{Eff}(X; \alpha_j)$. By Lemma 4.2, this shows that X_j and Y are isomorphic as varieties, which implies (3b).

We now proceed to show (3a). Within the list (6.3), let

$$(6.4) \quad \text{Eff}(X; \gamma_1: X \dashrightarrow X), \dots, \text{Eff}(X; \gamma_s: X \dashrightarrow X)$$

be the chambers of the form $\text{Eff}(X; \gamma: X \dashrightarrow X)$ with $\gamma \in \text{PsAut}(X, \Delta)$. By Proposition 4.12, for each $1 \leq j \leq s$, the intersection

$$(\gamma_j^{-1})^* \Pi \cap \text{Nef}(X) \subset \text{Eff}(X)$$

is a polyhedral cone. Hence, the convex cone Σ generated by these finitely many intersections is a polyhedral cone contained in $\text{Nef}^e(X)$.

By Proposition 3.3, it now suffices to show that

$$\text{Amp}(X) \subset \text{Aut}(X, \Delta) \cdot \Sigma.$$

We take $D \in \text{Amp}(X)$. By the inclusion given in (6.2), there is an element $\beta \in \text{PsAut}(X, \Delta)$ such that $\beta^* D \in \Pi$. In particular $\beta^* \text{Amp}(X) \cap \Pi \neq \emptyset$, so the chamber $\text{Eff}(X; \beta) = \beta^* \text{Nef}^e(X)$ coincides with one of the chambers $\text{Eff}(X; \gamma_i)$ in our exhaustive list (6.4). By Lemma 4.3, (X, β) and (X, γ_i) are isomorphic as marked small \mathbb{Q} -factorial modifications of (X, Δ) , i.e., $\beta \gamma_i^{-1} \in \text{Aut}(X, \Delta)$. We conclude that

$$(\beta \gamma_i^{-1})^* D = (\gamma_i^{-1})^* \beta^* D \in (\gamma_i^{-1})^* \Pi \subset \Sigma,$$

and thus $\text{Amp}(X) \subset \text{Aut}(X, \Delta) \cdot \Sigma$. \square

6.4. Equivalence of cone conjectures in dimension two. In dimension two, the work of Totaro implies the following result.

Corollary 6.2. *Let (X, Δ) be a klt Calabi–Yau pair of dimension 2. Then the four cone conjecture statements given in Theorem 1.5 hold for (X, Δ) .*

In particular, there exists a rational polyhedral fundamental domain for the action of $\text{Aut}(X, \Delta)$ on $\text{Eff}(X)$.

Proof. In dimension two, the nef cone conjecture was proven by Totaro [Tot10, Theorem 4.1] for any klt Calabi–Yau pair (X, Δ) . Since in dimension two, isomorphisms in codimension one are exactly biregular isomorphisms, and since the existence of good minimal models holds in that dimension, this shows that all four statements of Theorem 1.5 hold, for any klt Calabi–Yau surface pair (X, Δ) . \square

Let us make a short comment about the effective cone conjecture for surface pairs. In dimension two, the duality between numerical classes of divisors in $N^1(X)_{\mathbb{R}}$ and of curves in $N_1(X)_{\mathbb{R}}$ allows to identify the cone of effective divisors $\text{Eff}(X)$ with the cone of effective curves $\text{NE}(X)$. Embracing that point of view, the equivalence of Statements (2) and (3) can be partially recovered by a general duality argument (see [Loo14, Proposition-Definition 4.1]), together with the well-known equality $\text{Eff}(X) = \text{Eff}^+(X)$. The latter equality is a consequence of the Zariski decomposition of pseudo-effective \mathbb{R} -divisors on a surface, and of the equality $\text{Nef}^e(X) = \text{Nef}^+(X)$ (given by the nef cone conjecture).

7. CONE CONJECTURES FOR SCHOEN THREEFOLDS

In this final section, we prove Theorem 1.7 and Corollary 1.8 about Schoen threefolds.

7.1. Decomposition of effective cones of fiber products.

Proposition 7.1. *For $i = 1, 2$, let $\phi_i : W_i \rightarrow \mathbb{P}^1$ be a surjective morphism from a projective variety to \mathbb{P}^1 . Assume that*

(1) *the variety $W = W_1 \times_{\mathbb{P}^1} W_2$ is irreducible;*

(2) *we have*

$$p_1^* N^1(W_1)_{\mathbb{R}} + p_2^* N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}}$$

where $p_i : W \rightarrow W_i$ are the natural projections.

Then

$$p_1^* \text{Eff}(W_1) + p_2^* \text{Eff}(W_2) = \text{Eff}(W).$$

Proof. The proof is similar to Namikawa’s argument [Nam91, p. 153]. We draw the commutative diagram here for reader’s convenience:

$$\begin{array}{ccccc}
 & & W := W_1 \times_{\mathbb{P}^1} W_2 & & \\
 & \swarrow p_1 & \downarrow \phi & \searrow p_2 & \\
 W_1 & & & & W_2 \\
 & \searrow \phi_1 & \downarrow & \swarrow \phi_2 & \\
 & & \mathbb{P}^1 & &
 \end{array}$$

First note that actually we have $p_1^*N^1(W_1)_{\mathbb{Q}} + p_2^*N^1(W_2)_{\mathbb{Q}} = N^1(W)_{\mathbb{Q}}$. Take an arbitrary integral element in $\text{Eff}(X)$, represented by an effective line bundle L . By using a multiple of L instead if necessary, we can write $L = p_1^*L_1 \otimes p_2^*L_2$ with $L_i \in \text{Pic}(W_i)$. Then

$$\begin{aligned} \phi_*L &= \phi_{1*}p_{1*}(p_1^*L_1 \otimes p_2^*L_2) \\ &= \phi_{1*}(L_1 \otimes p_{1*}(p_2^*L_2)) \\ &= \phi_{1*}(L_1 \otimes \phi_1^*(\phi_{2*}L_2)) \\ &= \phi_{1*}L_1 \otimes \phi_{2*}L_2, \end{aligned}$$

where the second and the last equalities follow from the projection formula, and the third equality follows from the flat base change theorem. Note that ϕ_*L has a non-zero section as L is effective. Moreover, $\phi_{i*}L_i$ as a vector bundles on \mathbb{P}^1 is a direct sum of line bundles. So there are summands $\mathcal{O}_{\mathbb{P}^1}(a_i)$ of $\phi_{i*}L_i$ such that $a_1 + a_2 \geq 0$. Without loss of generality, we may assume $a_1 \geq 0$. Then both $\phi_{1*}L_1 \otimes \mathcal{O}_{\mathbb{P}^1}(-a_1)$ and $\phi_{2*}L_2 \otimes \mathcal{O}_{\mathbb{P}^1}(a_1)$ have non-zero sections. Now let $M_1 := L_1 \otimes \phi_1^*\mathcal{O}_{\mathbb{P}^1}(-a_1)$ and $M_2 := L_2 \otimes \phi_2^*\mathcal{O}_{\mathbb{P}^1}(a_1)$. Then $L = p_1^*M_1 \otimes p_2^*M_2$, and both M_i are effective as $\phi_{i*}M_i$ have non-zero sections. \square

7.2. Proof of Theorem 1.7. We start by proving the following lemma.

Lemma 7.2. *Let $\phi : W \rightarrow \mathbb{P}^1$ be a relatively minimal rational elliptic surface with a section. There exists a polyhedral cone $\Pi \subset \text{Eff}(W)$ such that*

$$\text{Aut}(W/\mathbb{P}^1) \cdot \Pi = \text{Eff}(W)$$

where $\text{Aut}(W/\mathbb{P}^1) := \{f \in \text{Aut}(W) : \phi \circ f = \phi\}$.

Proof. This is essentially a consequence of Totaro's results [Tot10]. We still provide a proof, because our setting is slightly different than Totaro's. We take smooth fibers F_1, \dots, F_5 of ϕ above five distinct points P_1, \dots, P_5 of \mathbb{P}^1 which are general enough that $\text{Aut}(\mathbb{P}^1, P_1 + \dots + P_5)$ is trivial.

We now apply [Tot10, Theorem 4.1] to the klt Calabi-Yau pair (W, Δ) , where

$$\Delta := \frac{1}{5}F_1 + \dots + \frac{1}{5}F_5.$$

It shows that this pair satisfies the nef cone conjecture. Hence, by Theorem 1.5 (and since any small \mathbb{Q} -factorial modification is biregular, in dimension 2), this pair also satisfies the effective cone conjecture, i.e., there is a rational polyhedral cone $\Pi \subset \text{Eff}(W)$ such that

$$\text{Aut}(W, \Delta) \cdot \Pi = \text{Eff}(W).$$

We can easily check that $\text{Aut}(W/\mathbb{P}^1) = \text{Aut}(W, \Delta)$, which concludes this proof. \square

We can now proceed to the proof of Theorem 1.7.

Proof of Theorem 1.7. Let us start by recalling the setting. We have a smooth projective threefold X obtained as a fiber product of the form $W_1 \times_{\mathbb{P}^1} W_2$, where for $i = 1, 2$, we consider a relatively minimal rational elliptic surface $\phi_i : W_i \rightarrow \mathbb{P}^1$ with a section.

We assume that the generic fibers of ϕ_1 and ϕ_2 are non-isogenous. Hence, by [GLW24, Lemma 5.1] (see also [Nam91, Proof of Proposition 1.1] for a proof under the additional assumption that all singular fibers of the ϕ_i are of type I_1), we have a decomposition

$$p_1^*N^1(W_1)_{\mathbb{R}} + p_2^*N^1(W_2)_{\mathbb{R}} = N^1(X)_{\mathbb{R}},$$

where each $p_i : X \rightarrow W_i$ is the projection.

By Proposition 7.1, we thus have $p_1^*\text{Eff}(W_1) + p_2^*\text{Eff}(W_2) = \text{Eff}(X)$. By Lemma 7.2, there exists a polyhedral cone $\Pi_i \subset \text{Eff}(W_i)$ such that $\text{Aut}(W_i/\mathbb{P}^1) \cdot \Pi_i = \text{Eff}(W_i)$. Let us introduce the polyhedral cone $\Sigma \subset \text{Eff}(X)$ generated by $p_1^*\Pi_1$ and $p_2^*\Pi_2$. Since $\text{Aut}(X)$ contains $\text{Aut}(W_1/\mathbb{P}^1) \times \text{Aut}(W_2/\mathbb{P}^1)$, this yields

$$\text{Aut}(X) \cdot \Sigma = \text{Eff}(X).$$

Thus, by Proposition 3.3, the effective cone conjecture holds for X . Since good minimal models exist in dimension 3, the movable cone conjecture, as well as all of the statements (1) to (4) in Theorem 1.5 holds for X . \square

Proof of Corollary 1.8. The assumptions of Corollary 1.8 are the same as in [Nam91, p. 152]. Hence, [Nam91, p. 162] shows that any minimal model X' of X with $X' \not\cong X$ has finite automorphism group. By Theorem 1.7, the nef cone conjecture holds for X' . Hence $\text{Nef}(X')$, as a union of finitely many rational polyhedral cones, is itself rational polyhedral. \square

Remark 7.3. We may also allow W_i to be a weak del Pezzo surface with a fibration $W_i \rightarrow \mathbb{P}^1$. Take the fiber product $W := W_1 \times_{\mathbb{P}^1} W_2$ and assume that W is smooth. Then we can get klt Calabi–Yau pairs (W, Δ) for suitable Δ , and Statements (1) to (4) in Theorem 1.5 also hold for (X, Δ) .

REFERENCES

- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010. 17, 18
- [Bir11] C. Birkar. On existence of log minimal models II. *J. Reine Angew. Math.*, 658:99–113, 2011. 10, 15
- [CL14] P. Cascini and V. Lazić. On the number of minimal models of a log smooth threefold. *J. Math. Pures Appl.* (9), 102(3):597–616, 2014. 4
- [CO15] S. Cantat and K. Oguiso. Birational automorphism groups and the movable cone theorem for Calabi–Yau manifolds of Wehler type via universal Coxeter groups. *Amer. J. Math.*, 137(4):1013–1044, 2015. 8
- [Deb01] O. Debarre. *Higher-dimensional algebraic geometry*, volume 3. Springer, 2001. 14
- [DHP13] J.-P. Demailly, Ch. Hacon, and M. Păun. Extension theorems, non-vanishing and the existence of good minimal models. *Acta mathematica*, 210(2):203–259, 2013. 21
- [Fuj17] O. Fujino. *Foundations of the minimal model program*, volume 35 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2017. 6, 10, 14, 15
- [GLW24] C. Gachet, H.-Y. Lin, and L. Wang. Nef cones of fiber products and an application to the cone conjecture. *Forum Math. Sigma*, 12:Paper No. e28, 22, 2024. 29
- [GM93] A. Grassi and D. Morrison. Automorphisms and the Kähler cone of certain Calabi–Yau manifolds. *Duke Math. J.*, 71(3):831–838, 1993. 2, 5
- [Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977. 8
- [HK00] Y. Hu and S. Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48(1):331–348, 2000. 2, 3, 7, 14
- [Kaw97] Yu. Kawamata. On the cone of divisors of Calabi–Yau fiber spaces. *Internat. J. Math.*, 8(5):665–687, 1997. 1, 2, 3, 13, 18

- [KKL16] A.-S. Kaloghiros, A. Küronya, and V. Lazić. Finite generation and geography of models. In Minimal models and extremal rays (Kyoto, 2011), volume 70 of Adv. Stud. Pure Math., pages 215–245. Math. Soc. Japan, [Tokyo], 2016. 19, 20, 21
- [KM98] J. Kollár and S. Mori. Birational geometry of algebraic varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. 6, 8, 10
- [Loo14] E. Looijenga. Discrete automorphism groups of convex cones of finite type. Compos. Math., 150(11):1939–1962, 2014. 6, 10, 11, 27
- [LOP18] V. Lazić, K. Oguiso, and Th. Peternell. The Morrison-Kawamata cone conjecture and abundance on Ricci flat manifolds. In Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds & Picard-Fuchs equations, volume 42 of Adv. Lect. Math. (ALM), pages 157–185. Int. Press, Somerville, MA, 2018. 1
- [LOP20] V. Lazić, K. Oguiso, and Th. Peternell. Nef line bundles on Calabi-Yau threefolds, I. Int. Math. Res. Not. IMRN, (19):6070–6119, 2020. 15
- [LZ] Z. Li and H. Zhao. On the relative Morrison-Kawamata cone conjecture. arXiv:2206.13701v3. 4, 5, 11, 15, 16, 18, 19
- [Mor93] D. R. Morrison. Compactifications of moduli spaces inspired by mirror symmetry. In Journées de géométrie algébrique d’Orsay - Juillet 1992, number 218 in Astérisque. Société mathématique de France, 1993. 1
- [Mor96] D. Morrison. Beyond the Kähler cone. In Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), volume 9 of Israel Math. Conf. Proc., pages 361–376. Bar-Ilan Univ., Ramat Gan, 1996. 1
- [Nak04] N. Nakayama. Zariski-decomposition and abundance, volume 14 of MSJ Memoirs. Mathematical Society of Japan, Tokyo, 2004. 15
- [Nam91] Y. Namikawa. On the birational structure of certain Calabi-Yau threefolds. J. Math. Kyoto Univ., 31(1):151–164, 1991. 2, 5, 27, 29
- [Ogu22] K. Oguiso. Problems on algebraic dynamics and related topics (Open problems in complex geometry II). RIMS Kôkyûroku, 2211:165–177, 2022. <https://repository.kulib.kyoto-u.ac.jp/dspace/handle/2433/268853?locale=en>. 4
- [Roc70] R. T. Rockafellar. Convex analysis. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970. 11, 12, 14, 17, 20, 21
- [Sch88] C. Schoen. On fiber products of rational elliptic surfaces with section. Math. Z., 197(2):177–199, 1988. 2, 5
- [Sho96] V. V. Shokurov. 3-fold log models. J. Math. Sci., 81(3):2667–2699, 1996. 10
- [SX] I. Stenger and Z. Xie. Cones of divisors on \mathbb{P}^3 blown up at eight very general points. arXiv:2303.12005v2. 2, 18
- [Tot08] B. Totaro. Hilbert’s 14th problem over finite fields and a conjecture on the cone of curves. Compos. Math., 144(5):1176–1198, 2008. 1
- [Tot10] B. Totaro. The cone conjecture for Calabi-Yau pairs in dimension 2. Duke Math. J., 154(2):241–263, 2010. 1, 3, 4, 6, 22, 27, 28
- [Wan22] L. Wang. Remarks on nef and movable cones of hypersurfaces in Mori dream spaces. J. Pure Appl. Algebra, 226(11):Paper No. 107101, 22, 2022. 18
- [Xu] F. Xu. On the cone conjecture for certain pairs of dimension at most 4. arXiv:2405.20899v1. 4, 5

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