SMOOTH PROJECTIVE SURFACES WITH INFINITELY MANY REAL FORMS

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Abstract. First of all, we confirm a few basic criteria of the finiteness of real forms of a given smooth complex projective variety, in terms of the Galois cohomology set of the discrete part of the automorphism group, the cone conjecture and the topological entropy. We then apply them to show that a smooth complex projective surface has at most finitely many mutually non-isomorphic real forms unless it is either rational or a non-minimal surface birational to either a K3 surface or an Enriques surface. We finally construct an Enriques surface whose blow-up at one point admits infinitely many mutually non-isomorphic real forms, which answers a question of Kondo to us and also shows the three exceptional cases really occur.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers, and refer to [BHPV04] for basic definitions and properties of complex projective surfaces.

The following is our main theorem, which completes the first step of the finiteness problem of real forms of complex projective surfaces initiated by [DO19] and expanded by [DOY21] after Lesieutre [Le18]. We refer to Section 2 for the definition of a real form and other related notions.

Theorem 1.1.

(1) *Let S be a smooth complex projective surface. Assume that S has infinitely many mutually non-isomorphic real forms. Then S is either rational or a non-minimal surface birational to either a K3 surface or an Enriques surface.*

(2) *There is an Enriques surface Z such that a blow-up of Z at one point admits infinitely many mutually non-isomorphic real forms. (See also Remark 4.9.)*

Remark 1.2. Our previous result of [DOY21] shows that there is a smooth projective rational surface S with infinitely many mutually non-isomorphic real forms, which answers a question by [Kh02]. There is also a smooth projective surface S which is a blow-up of some K3 surface at one point such that S admits infinitely many mutually non-isomorphic real forms. Such a surface S is constructed first by [DOY21] after [DO19], answering a question of Mukai to us. Our new result (2), which answers a question by Kondo, shows that the three exceptional cases in (1) all occur.

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Theorem 1.1 (1) should be known to experts at least as a folklore result. We will give a proof, along the line explained by [DIK00] and [CF19] with clarification for the sake of completeness.

Especially, we first show two basic results, Theorems 1.3 and 1.5, in Section 2. Before giving their statements, let us first fix some notations. For a complex projective variety V, $\mathrm{Aut}^0(V)$ denotes the identity component of the automorphism group $\mathrm{Aut}(V)$. Let $\mathrm{NS}(V)$ be the Néron-Severi group of V, which is a finitely generated abelian group. Inside the $\mathbb{R}$-vector space $\mathrm{NS}(V) \otimes_{\mathbb{Z}} \mathbb{R}$, $\mathrm{Amp}(V)$ denotes the ample cone and $\mathrm{Nef}(V)$ denotes the nef cone of V. Let $\mathrm{Nef}^+(V)$ be the rational hull of $\mathrm{Nef}(V)$, that is, the convex hull of the set $(\mathrm{NS}(V) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathrm{Nef}(V)$. We also let

$$\mathrm{Aut}^*(V) := \mathrm{Im}(\mathrm{Aut}(V) \to \mathrm{GL}(\mathrm{NS}(V)/\mathrm{torsion}))$$

be the image under the natural action. Then $\mathrm{Aut}^*(V)$ preserves $\mathrm{Nef}^+(V)$.

Theorem 1.3. *Let V be a complex projective variety with a real form. If*

$$H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(V)/\mathrm{Aut}^0(V))$$

is finite, then V has only finitely many mutually non-isomorphic real forms. Moreover, the number of mutually non-isomorphic real forms of a complex projective variety is at most countable.

Compare with the result of Bot [Bo21] for an affine surface for the last statement.

Theorem 1.3 has the following corollary, which fits within the context of Tits' alternative for $\mathrm{Aut}(V)$ (see [Di12] and references therein). It also generalizes [Be16, Theorem 1] from rational surfaces to arbitrary smooth projective varieties and a result of [Ki20], which is based on [DIK00, Appendix D] (See Remark 1.6 below).

Corollary 1.4. *Let V be a complex projective variety. If $\mathrm{Aut}(V)/\mathrm{Aut}^0(V)$, or more generally the image of the pullback action*

$$\rho : \mathrm{Aut}(V)/\mathrm{Aut}^0(V) \to \mathrm{GL}(\mathrm{NS}(V)/\mathrm{torsion})$$

is virtually solvable, then V has at most finitely many non-isomorphic real forms.

In particular, this is the case where V is a smooth complex projective variety and every automorphism of V has zero entropy.

Theorem 1.5. *Let V be a complex projective variety such that $\mathrm{Nef}^+(V)$ contains a rational polyhedral cone Σ satisfying*

$$\mathrm{Aut}^*(V) \cdot \Sigma \supset \mathrm{Amp}(V).$$

For instance, this is the case when V satisfies the cone conjecture, in the sense that the natural action of $\mathrm{Aut}^*(V)$ on $\mathrm{Nef}^+(V)$ has a rational polyhedral fundamental domain.

Then V has at most finitely many mutually non-isomorphic real forms. In particular, this is the case where V is a minimal surface of Kodaira dimension zero by Sterk [St85]*, Namikawa* [Na85] *and Kawamata* [Ka97, Theorem 2.1] *as well as the case where V is a complex variety of Picard number one, or whose rational hull of the nef cone $\mathrm{Nef}^+(V)$ is a rational polyhedral cone.*

Remark 1.6. Theorem 1.3 was asserted by [DIK00, Appendix D] and essentially the same result as Theorem 1.5 was asserted by [CF19] with key observation [CF19, Proposition 7.4]. However, we have some difficulty to follow their arguments. For this reason, we prove these two basic results in this paper by trying to respect their original arguments as possible as we can. See also Remarks 2.10 and 2.13.

We then show Theorem 1.1 (1) with a slight generalization for smooth complex projective varieties of higher Kodaira dimension. This will be done in Section 3.

We give an explicit construction of an Enrique surface and its blow-up in Theorem 1.1 (2) in Section 4. Here our construction is inspired by [Le18], [DO19], [DOY21] and [Mu10]. Compare also with [Wa21], which is based on an unpublished manuscript [KO19]. We prove Theorem 1.1 (2) in Section 5.

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Notation and convention. In this paper, by a point of a projective variety V over $\mathbb{C}$, we always mean a point of $V(\mathbb{C})$, i.e., a $\mathbb{C}$-valued point of V, except a generic point by which we always mean a generic point in the scheme theoretic sense. A *locally algebraic group* is a group scheme locally of finite type over a field.

For every scheme V over a field $\Bbbk$ (in our paper $\Bbbk$ will be $\mathbb{R}$ or $\mathbb{C}$), we let $\mathrm{Aut}(V/\Bbbk)$ denote the group of biregular automorphisms of V over $\Bbbk$. We also write $\mathrm{Aut}(V) = \mathrm{Aut}(V/\Bbbk)$ if there is no risk of confusion and, unless stated otherwise, we regard $\mathrm{Aut}(V) = \mathrm{Aut}(V/\Bbbk)$ as an abstract group (not as a group scheme). Note that if V is defined over $\mathbb{R}$ and $\mathrm{Aut}(V/\mathbb{C}) = \{\mathrm{id}_V\}$, then the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts trivially on the abstract group $\mathrm{Aut}(V/\mathbb{C})$, whereas it acts as an involution on the group scheme $\mathrm{Aut}(V/\mathbb{C}) \to \mathrm{Spec}\,\mathbb{C}$.

For a complex variety V, we define the decomposition group and the inertia group of subsets $W_1, \dots, W_n \subset V$ by

$$\mathrm{Dec}(V, W_1, \dots, W_n) := \{f \in \mathrm{Aut}(V) \mid \forall i,\ f(W_i) = W_i\},$$
$$\mathrm{Ine}(V, W_1, \dots, W_n) := \{f \in \mathrm{Dec}(V, W_1, \dots, W_n) \mid \forall i,\ f_{W_i} = \mathrm{id}_{W_i}\}.$$

Note then that

$$\mathrm{Dec}(V, W_1, \dots, W_n) \subset \mathrm{Dec}(V, \cup_{i=1}^n W_i),$$

and for an irreducible decomposition $W = \cup_{i=1}^n W_i$ of an algebraic set $W \subset V$,

$$[\mathrm{Dec}(V, \cup_{i=1}^n W_i) : \mathrm{Dec}(V, W_1, \dots, W_n)] \le |S_n| = n!.$$

For an automorphism $f \in \mathrm{Aut}(V)$, we denote the set of fixed point of f by

$$V^f := \{x \in V(\mathbb{C}) \mid f(x) = x\}.$$

We refer to e.g. [Se02, Section I.5] for the basic facts on the group cohomology set $H^1(G, B)$ of a G-group B. In this paper, we only need the non-trivial simplest case where

$$G = G_{\mathbb{C}/\mathbb{R}} := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

2. Two basic criteria of finiteness of real forms

In this section, we first recall the notion of real forms and some classical results due to Borel, Serre, and Weil, in order to fix some notations. We will then prove Theorems 1.3 and 1.5.

2.1. **Real forms and real structures.**
Throughout the paper, $c : \mathbb{C} \to \mathbb{C}$ denotes the complex conjugate, so

$$G_{\mathbb{C}/\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{id}_{\mathbb{C}}, c\}\,.$$

Let V be a scheme over $\mathbb{C}$ and let $\pi : V \to \operatorname{Spec}\mathbb{C}$ be the structural morphism.

Definition 2.1.

(1) A *real form* of V is a scheme W over $\mathbb{R}$ such that

$$V \simeq W \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$$

over $\operatorname{Spec}\mathbb{C}$.

(2) A *real structure* of V is an anti-holomorphic involution

$$\imath : V \to V,$$

namely $\imath$ is an automorphism over $\operatorname{Spec}\mathbb{R}$ such that

$$\imath^2 = \mathrm{id}_V \quad \text{and} \quad \pi \circ \imath = c \circ \pi.$$

Two real forms W and W' are equivalent if they are isomorphic over $\operatorname{Spec}\mathbb{R}$. Two real structures $\imath$ and $\imath'$ on V are said to be equivalent if $\imath' = h \circ \imath \circ h^{-1}$ for some $h \in \mathrm{Aut}(V/\mathbb{C})$.

The real structure associated to a real form W of scheme V over $\mathbb{C}$ is defined as

$$\imath_W := \mathrm{id}_W \times c : V \to V,$$

if one fixes an identification $V = W \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. Assume that V is a quasi-projective variety. As a consequence of Galois descent, the map $W \mapsto \imath_W$ defines a one-to-one correspondence

$$\{\text{Real forms on } V\}\,/\simeq \;\longleftrightarrow\; \{\text{Real structures on } V\}\,/\simeq\,. \tag{2.1}$$

Example 2.2.

(1) Let W be a real form of a complex scheme V. Then $G_{\mathbb{C}/\mathbb{R}}$ acts naturally on the group scheme $\mathrm{Aut}(V/\mathbb{C})$ by

$$c \cdot f = \imath_W \circ f \circ \imath_W, \tag{2.2}$$

which we fix throughout the paper. If V is a projective complex variety, then $\mathrm{Aut}(V/\mathbb{C})$ is a locally algebraic group over $\mathbb{C}$ and $\mathrm{Aut}(W/\mathbb{R})$ is a real form of it [MO67, Theorem 3.7]. See also [FGIKNV, Section 5.6]. The associated real structure on $\mathrm{Aut}(V)$ is defined by (2.2).

(2) Let $V_{\mathbb{R}}$ be a real scheme and let V be its complexification. Let $\imath : V \to V$ be the associated real structure. For every $f \in \mathrm{Aut}(V/\mathbb{C})$ such that

$$c \cdot f := \imath \circ f \circ \imath = f^{-1}, \tag{2.3}$$

the composition

$$\imath \circ f : V \to V$$

defines a real structure on V. Condition (2.3) is equivalent to the property that

$$\phi : G_{\mathbb{C}/\mathbb{R}} \to \mathrm{Aut}(V)$$

defined by $\phi(\mathrm{id}_{\mathbb{C}}) = \mathrm{id}_{\mathbb{C}}$ and $\phi(c) = f$ is a 1-cocycle where the $G_{\mathbb{C}/\mathbb{R}}$-action on $\mathrm{Aut}(V)$ is defined by (2.2). We call $\imath \circ f$ the real structure twisted by ϕ, and let

V_ϕ denote the complex scheme V endowed with the new $G_{\mathbb{C}/\mathbb{R}}$-action defined by $c \cdot v := \imath(f(v))$ for all $v \in V$. We also let $V_{\mathbb{R},\phi}$ denote the corresponding real form.

(3) We continue the above example, and assume moreover that $V_\mathbb{R}$ is a real group scheme: then V is a complex group scheme. We verify that the group laws of V_ϕ, viewed as morphisms over $\mathbb{C}$, are $G_{\mathbb{C}/\mathbb{R}}$-equivariant, so they descend to group laws on the real form $V_{\mathbb{R},\phi}$, giving it a group scheme structure over $\mathbb{R}$. Finally, note that if $V_\mathbb{R}$ (or equivalently V) is an algebraic group, then so is $V_{\mathbb{R},\phi}$. Moreover, since for algebraic groups, the property of being linear (resp. connected) does not depend on the base field, if $V_\mathbb{R}$ is linear (resp. connected) then so is $V_{\mathbb{R},\phi}$.

We can also describe the set of real forms up to equivalence using Galois cohomology [Se02, Page 124, Proposition 5].

Theorem 2.3. *Let V be a complex quasi-projective variety having a real form W with real structure $\imath_W$. Then there are natural bijective correspondences between the following three sets:*

(1) *The set of real forms of V up to isomorphism as varieties over $\mathbb{R}$;*
(2) *The set of real structures on V up to equivalence;*
(3) *The Galois cohomology set*

$$H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(V)),$$

where the action of $G_{\mathbb{C}/\mathbb{R}}$ on $\mathrm{Aut}(V)$ is given by $f \mapsto \imath_W \circ f \circ \imath_W$.

For later use, we say that a subvariety W on V (resp. a morphism $f : V \to U$) is defined over $\mathbb{R}$ with respect to the real form $V_\mathbb{R}$ (resp. real forms $V_\mathbb{R}$ and $U_\mathbb{R}$) if there is an object $W_\mathbb{R}$ on $V_\mathbb{R}$ (resp. a morphism $f_\mathbb{R} : V_\mathbb{R} \to U_\mathbb{R}$) such that $W = W_\mathbb{R} \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$ (resp. $f = f_\mathbb{R} \times \mathrm{id}_{\mathrm{Spec}\,\mathbb{C}}$ for some morphism $f_\mathbb{R} : V_\mathbb{R} \to U_\mathbb{R}$). We say that a subvariety W on V is defined over $\mathbb{R}$ with respect to a real structure of V, if W defined over $\mathbb{R}$ with respect to the corresponding real form. Similarly, we have the definition of a morphism $f : V \to U$ defined over $\mathbb{R}$ with respect to two real structures of V and U. When a real structure $\imath$ of V is fixed, by abuse of terminology, a complex point x of V is called a real point if $x \in V^\imath$, i.e., if the support of x is fixed under $\imath$. Note that $V(\mathbb{C})^\imath = V_\mathbb{R}(\mathbb{R})$ as sets.

2.2. **Some finiteness results of Galois cohomology.**

Recall that a group H is said to be *polycyclic* if it is solvable and every subgroup of H is finitely generated.

The following proposition is well-known. In our applications, the G-group H in Proposition 2.4 will be mostly a subgroup or a quotient group of $\mathrm{Aut}(V)$ of a complex projective variety V having a real form V_0 with real structure c_0, to which the action of c_0 by conjugation restricts or extends.

Proposition 2.4. *Set $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let H be a G-group.*

(1) *Suppose that the G-group H is arithmetic, in the sense that there exists a linear G-group $L_\mathbb{Q}$ over $\mathbb{Q}$ such that H embeds G-equivariantly into $L_\mathbb{Q}$ as an arithmetic subgroup. Then $H^1(G, H)$ is finite.*
(2) *If H has a filtration consisting of normal G-subgroups N_i of H*

$$\{1_H\} = N_s \leq N_{s-1} \leq \ldots \leq N_1 \leq N_0 = H$$

such that $H^1(G, N_i/N_{i+1})$ is finite for any G-action on N_i/N_{i+1} (this is the case when e.g. N_i/N_{i+1} is either a finitely generated abelian group or a finite group), then $H^1(G, H)$ is a finite set.

(3) *Let H be a G-group which is virtually polycyclic, namely, H admits a finite index polycyclic subgroup $N \leq H$ (without assuming that the G-action preserves N), then $H^1(G, H)$ is a finite set.*

(4) *Assume that the G-action on H is trivial. Then the cardinality of $H^1(G, H)$ coincides with the cardinality of the set of conjugacy classes of involutions with 1_H in H.*

Proof. (1) is proved by [BS64, Théorème 6.1]. (2) is stated by [DIK00, D.1.7, Appendix D] and rigorously restated and proved by [CF19, Lemma 4.9].

Now we prove (3). Suppose N is a polycyclic subgroup of H of finite index. Up to replacing N by

$$\bigcap_{h \in H} h^{-1} N h,$$

which is still a finite index subgroup of H, we can assume that N is normal in H. Up to replacing H by

$$\bigcap_{g \in G} g \cdot N,$$

we can further assume that N is a polycyclic G-subgroup. Since N is solvable, the derived sequence $N^{(i)}$ of N gives a sequence of normal G-subgroups of H

$$\{1_H\} = N^{(m)} \leq \cdots \leq N^{(1)} \leq N^{(0)} = N \leq H,$$

and the finite generation assumption (for all subgroups of N) implies that the quotient abelian groups $N^{(i)}/N^{(i+1)}$ are all finitely generated. Hence (3) follows from (2).

(4) is clear by the definition of the Galois cohomology set. To our best knowledge, Lesieutre [Le18, Lemma 13] is the first who explicitly mentioned (4) and effectively applied (4) for the existence of a smooth projective variety with infinitely many real forms. □

2.3. **Proof of Theorem 1.3.**

In the subsection, we prove Theorem 1.3 which is restated as Theorem 2.9 below. Let us start from some lemmas.

Lemma 2.5. *Let $f : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be a Lie group automorphism of order 2. Let $G := \langle f \rangle \leq \mathrm{Aut}(\mathbb{R}^n/\mathbb{Z}^n)$ act naturally on $\mathbb{R}^n/\mathbb{Z}^n$. Then $H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$ is finite.*

Here we provide two different proofs of this lemma.

First proof of Lemma 2.5. Since the Lie group $\mathbb{R}^n$ is the universal covering of $\mathbb{R}^n/\mathbb{Z}^n$, it follows that f can be lifted to a Lie group automorphism g of $\mathbb{R}^n$. Note that g is a linear map. In fact, since g preserves addition in $\mathbb{R}^n$ and $g(\mathbb{Z}^n) = \mathbb{Z}^n$, it follows that g is $\mathbb{Q}$-linear on $\mathbb{Q}^n$. Since g is a diffeomorphism (in particular, continuous), we have that g is $\mathbb{R}$-linear on $\mathbb{R}^n$. The restriction $g|_{\mathbb{Z}^n} : \mathbb{Z}^n \to \mathbb{Z}^n$ is an automorphism of the free abelian group $\mathbb{Z}^n$ of order at most 2. We may and will view $\mathbb{R}^n$ and $\mathbb{Z}^n$ as G-groups via g and $g|_{\mathbb{Z}^n}$ respectively. Thus we have the following exact sequence of G-groups

$$0 \to \mathbb{Z}^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \to 0.$$

As these are abelian groups, hence G-modules, we have the following long exact sequence of cohomology groups

$$H^1(G, \mathbb{R}^n) \to H^1(G, \mathbb{R}^n/\mathbb{Z}^n) \to H^2(G, \mathbb{Z}^n) \to H^2(G, \mathbb{R}^n).$$

By Comessatti's Lemma (see [Si82, Proposition 2]), it suffices to prove the finiteness of $H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$ in the following three cases:

(1) $n = 1$, $g|_{\mathbb{Z}} = \mathrm{id}_{\mathbb{Z}}$;
(2) $n = 1$, $g|_{\mathbb{Z}} = -\mathrm{id}_{\mathbb{Z}}$;
(3) $n = 2$, $g|_{\mathbb{Z}^2}(a, b) = (a + b, -b)$ for any $(a, b) \in \mathbb{Z}^2$.

By [HS97, Chapter VI, Proposition 7.1] and the above long exact sequence, $H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$ in the three cases is $\mathbb{Z}/2\mathbb{Z}$, 0, 0 respectively. □

Second proof of Lemma 2.5. Since $T = \mathbb{R}^n/\mathbb{Z}^n$ is a commutative G-group, we have group isomorphisms

$$Z^1(G, \mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{\sim} \mathrm{Ker}(f + \mathrm{id}_T) \subset T,$$

and

$$B^1(G, \mathbb{R}^n/\mathbb{Z}^n) \xrightarrow{\sim} \mathrm{Im}(f - \mathrm{id}_T) \subset T,$$

where both maps are defined by $\sigma \mapsto \sigma(f)$. Since $\mathrm{Ker}(f + \mathrm{id}_T)$ is a Lie subgroup of T and T is compact, it has only finitely many connected components. Thus to show that $H^1(G, \mathbb{R}^n/\mathbb{Z}^n)$ is finite, it suffices to show that

$$\dim \mathrm{Ker}(f + \mathrm{id}_T) = \dim \mathrm{Im}(f - \mathrm{id}_T). \tag{2.4}$$

Let $T_f : \mathbb{R}^n \to \mathbb{R}^n$ be the tangent map of f at the origin. Since $T_f^2 = \mathrm{id}_T$, we have

$$\mathbb{R}^n = \mathrm{Ker}(T_f + \mathrm{id}_T) \oplus \mathrm{Ker}(T_f - \mathrm{id}_T).$$

Hence

$$\dim \mathrm{Ker}(T_f + \mathrm{id}_T) = \dim \mathrm{Im}(T_f - \mathrm{id}_T),$$

which implies (2.4). □

Lemma 2.6. *Let $A_{\mathbb{R}}$ be a real abelian variety and let $A = A_{\mathbb{R}} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. Then $H^1(G_{\mathbb{C}/\mathbb{R}}, A)$ is finite.*

Proof. Recall that $G_{\mathbb{C}/\mathbb{R}}$ acts on A via the anti-holomorphic involution $\imath := \mathrm{id}_{A_{\mathbb{R}}} \times c$ of A. Moreover, $\imath$ is a group homomorphism of A. Then as real Lie groups, we may identify A with $\mathbb{R}^{2d}/\mathbb{Z}^{2d}$ where $d = \dim A$, and $\imath$ corresponds to a Lie group automorphism of $\mathbb{R}^{2d}/\mathbb{Z}^{2d}$ of order 2. By Lemma 2.5, $H^1(G_{\mathbb{C}/\mathbb{R}}, A)$ is finite. □

Lemma 2.7. *Let $A_{\mathbb{R}}$ be a connected algebraic group over $\mathbb{R}$ and let $A = A_{\mathbb{R}} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. Then $H^1(G_{\mathbb{C}/\mathbb{R}}, A)$ is finite.*

Proof. By Barsotti–Chevalley's structure theorem [Mi17, Theorem 8.27, Notes 8.30], $A_{\mathbb{R}}$ (resp. A) has a unique normal connected linear algebraic subgroup $N_{\mathbb{R}}$ (resp. $N := N_{\mathbb{R}} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$) such that the quotient $P_{\mathbb{R}} := A_{\mathbb{R}}/N_{\mathbb{R}}$ (resp. $P := P_{\mathbb{R}} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$) is an abelian variety. Then we have an exact sequence

$$H^1(G_{\mathbb{C}/\mathbb{R}}, N) \to H^1(G_{\mathbb{C}/\mathbb{R}}, A) \to H^1(G_{\mathbb{C}/\mathbb{R}}, P),$$

as pointed sets, induced from the exact sequence of $G_{\mathbb{C}/\mathbb{R}}$-groups

$$1 \to N \to A \to P \to 1.$$

By Lemma 2.6, $H^1(G_{\mathbb{C}/\mathbb{R}}, P)$ is finite. Thus, by [Se02, Page 53, Corollary 3], it suffices to show that $H^1(G_{\mathbb{C}/\mathbb{R}}, N_\phi)$ is finite for any $\phi \in Z^1(G_{\mathbb{C}/\mathbb{R}}, A)$ (see Example 2.2 (2) for the definition of N_ϕ). As we mentioned in Example 2.2 (3), since $N_\mathbb{R}$ is a linear algebraic group over $\mathbb{R}$, so is the real form $N_{\mathbb{R},\phi}$. It follows from [Se02, Page 144, Theorem 4; Page 143, Examples] that $H^1(G_{\mathbb{C}/\mathbb{R}}, N_\phi)$ is finite. $\square$

For a locally compact field k of characteristic 0 and a so-called k-group A of type (ALA), Borel and Serre ([BS64, Théorème 6.1]) show that $H^1(k, A)$ is finite. For $k = \mathbb{R}$, the following result is in some sense a generalization of [BS64, Théorème 6.1].

Theorem 2.8. *Let $A_\mathbb{R}$ be a locally algebraic group over $\mathbb{R}$ and let $A = A_\mathbb{R} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. Let A^0 denote the identity component of A. If $H^1(G_{\mathbb{C}/\mathbb{R}}, A/A^0)$ is finite (resp. countable), then $H^1(G_{\mathbb{C}/\mathbb{R}}, A)$ is finite (resp. countable) as well. In particular, $H^1(G_{\mathbb{C}/\mathbb{R}}, A)$ is finite if $A_\mathbb{R}$ is an algebraic group over $\mathbb{R}$.*

Proof. We have an exact sequence

$$H^1(G_{\mathbb{C}/\mathbb{R}}, A^0) \to H^1(G_{\mathbb{C}/\mathbb{R}}, A) \to H^1(G_{\mathbb{C}/\mathbb{R}}, A/A^0),$$

as pointed sets, induced from the exact sequence of $G_{\mathbb{C}/\mathbb{R}}$-groups

$$1 \to A^0 \to A \to A/A^0 \to 1.$$

Let $A^0_\mathbb{R}$ denote the identity component of $A_\mathbb{R}$. We have $A^0 = A^0_\mathbb{R} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. Since $A^0_\mathbb{R}$ is a connected algebraic group, so is the real form which underlies A^0_ϕ for all $\phi \in Z^1(G_{\mathbb{C}/\mathbb{R}}, A)$ by Example 2.2. Thus $H^1(G_{\mathbb{C}/\mathbb{R}}, A^0_\phi)$ is finite by Lemma 2.7. The first claim then follows from [Se02, Page 53, Corollary 3].

If $A_\mathbb{R}$ is an algebraic group, then A/A^0 is finite. Hence $H^1(G_{\mathbb{C}/\mathbb{R}}, A/A^0)$ is finite by definition, and the second claim follows from the first one. $\square$

Theorem 2.9. *Let V be a complex projective variety with a real form. Then the number of mutually non-isomorphic real forms of V is at most countable. If*

$$H^1(G_{\mathbb{C}/\mathbb{R}}, \operatorname{Aut}(V)/\operatorname{Aut}^0(V)) \tag{2.5}$$

is finite, then V has only finitely many real forms up to equivalence.

Proof. The first statement follows from Theorem 2.8, as the group $\operatorname{Aut}(V)/\operatorname{Aut}^0(V)$, hence, the set $H^1(G_{\mathbb{C}/\mathbb{R}}, \operatorname{Aut}(V)/\operatorname{Aut}^0(V))$, is countable. According to Example 2.2, $\operatorname{Aut}(V)$ is a locally algebraic group admitting a real form, so (2.5) makes sense, and we can apply Theorem 2.8 with $A = \operatorname{Aut}(V)$. The finiteness of (2.5) then implies that $H^1(G_{\mathbb{C}/\mathbb{R}}, \operatorname{Aut}(\mathrm{V}))$ is finite, thus V has only finitely many real forms by Theorem 2.3. $\square$

Remark 2.10. As we mentioned in the introduction, Theorem 2.9 was asserted in [DIK00, Corollary D.1.10]. However, they claimed there that $\operatorname{Aut}^0(V)$ is a linear algebraic group and did not consider the abelian variety factor of $\operatorname{Aut}^0(V)$. As we believe that Theorem 2.9 is fundamental, we gave a proof here.

Proof of Corollary 1.4. It is clear that if $\operatorname{Aut}(V)/\operatorname{Aut}^0(V)$ is virtually solvable, then so is $\operatorname{Im}(\rho)$. By Fujiki-Lieberman's theorem [Br18, Theorem 2.10], $\operatorname{Ker}(\rho)$ is finite. As $\operatorname{Im}(\rho)$ embeds into $\operatorname{GL}(\operatorname{NS}(V)/\text{torsion})$, $\operatorname{Im}(\rho)$ is virtually polycyclic by Malcev's theorem [Se83, Page 26, Corollary 1]. It follows from Proposition 2.4 (3), then (2), that $H^1(G_{\mathbb{C}/\mathbb{R}}, \operatorname{Aut}(V)/\operatorname{Aut}^0(V))$ is finite. The first statement then follows from Theorem 1.3.

The second statement follows from the first one together with [DLOZ22, Proposition 2.6 (1)]. ☐

2.4. **Cone conjecture and real structures.**

Now we prove Theorem 1.5 mentioned in the introduction by clarifying some arguments of [CF19]. First we prove the following finiteness result, which is claimed in [Be17, Lemma 2.5] without proof. We prove it here for the sake of completeness (see also [CF19, Section 9]).

Lemma 2.11. *Let Γ be a $\mathbb{Z}/2\mathbb{Z}$-group. If the semidirect product $\Gamma \rtimes \mathbb{Z}/2\mathbb{Z}$ induced by the $\mathbb{Z}/2\mathbb{Z}$-action on Γ contains only finitely many conjugacy classes of elements of order 2, then $H^1(\mathbb{Z}/2\mathbb{Z}, \Gamma)$ is finite.*

Proof. Here we identify the elements of $\mathbb{Z}/2\mathbb{Z}$ with $\{\overline{0}, \overline{1}\}$. Note that conjugation by $(1_\Gamma, \overline{1})$ makes $\Gamma \rtimes \mathbb{Z}/2\mathbb{Z}$ into a $\mathbb{Z}/2\mathbb{Z}$-group, in a way that we have the following exact sequence of $\mathbb{Z}/2\mathbb{Z}$-groups:

$$1 \to \Gamma \to \Gamma \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

where the induced action on $\mathbb{Z}/2\mathbb{Z}$ is trivial. This induces an exact sequence of pointed sets

$$\{\pm 1\} \to H^1(\mathbb{Z}/2\mathbb{Z}, \Gamma) \to H^1(\mathbb{Z}/2\mathbb{Z}, \Gamma \rtimes \mathbb{Z}/2\mathbb{Z}).$$

By [Se02, Page 53, Corollary 3], it suffices to show that $H^1(\mathbb{Z}/2\mathbb{Z}, \Gamma \rtimes \mathbb{Z}/2\mathbb{Z})$ is finite.

Since $\mathbb{Z}/2\mathbb{Z}$ acts on $\Gamma \rtimes \mathbb{Z}/2\mathbb{Z}$ by conjugation, we have

$$H^1(\mathbb{Z}/2\mathbb{Z}, \Gamma \rtimes \mathbb{Z}/2\mathbb{Z}) \simeq H^1(\mathbb{Z}/2\mathbb{Z}, (\Gamma \rtimes \mathbb{Z}/2\mathbb{Z})_{\mathrm{triv}})$$

where $(\Gamma \rtimes \mathbb{Z}/2\mathbb{Z})_{\mathrm{triv}}$ is the $\mathbb{Z}/2\mathbb{Z}$-group $\Gamma \rtimes \mathbb{Z}/2\mathbb{Z}$ with the trivial $\mathbb{Z}/2\mathbb{Z}$-action. The group cohomology $H^1(\mathbb{Z}/2\mathbb{Z}, (\Gamma \rtimes \mathbb{Z}/2\mathbb{Z})_{\mathrm{triv}})$ is in bijection with the set of elements of order 1 or 2 in $\Gamma \rtimes \mathbb{Z}/2\mathbb{Z}$ modulo conjugation, which is finite by assumption. ☐

Let V be a smooth complex projective variety. The Klein automorphism group $\mathrm{KAut}(V)$ of V, is defined as the group of holomorphic and anti-holomorphic automorphisms of a scheme $V \to \operatorname{Spec} \mathbb{C}$ over $\operatorname{Spec} \mathbb{R}$ to itself. If V admits a real structure $\imath$, then

$$\mathrm{KAut}(V) \simeq \mathrm{Aut}(V/\mathbb{C}) \rtimes \langle \imath \rangle.$$

Since $\imath$ is an automorphism of a scheme V, we have

$$\imath^* : \mathcal{O}_V(U) \simeq \mathcal{O}_V(\imath^{-1}(U))$$

for any Zariski open subset $U \subset V$. Then for $f \in \mathcal{O}_V(U)$ and for any $x \in \imath^{-1}(U)(\mathbb{C})$, we have

$$(\imath^* f)(x) = c(f(\imath(x))) = \overline{f(\imath(x))},$$

as by definition, the value $(\imath^* f)(x) \in \mathbb{C} = \mathcal{O}_{V,x}/\mathfrak{m}_{V,x}$ is uniquely determined by the condition

$$\imath^* f - (\imath^* f)(x) \in \mathfrak{m}_{V,x}.$$

(See for instance [MO15, Section 4.2].) This naturally extends for the pull-back of rational functions of V. Let D be a Cartier divisor on V with local equations (f_U, U). We define the Cartier divisor $\overline{D}$ on V by the local equations $(\imath^* f_U, \imath^{-1}(U))$. Then the contravariant $\mathrm{Aut}(V)$-action on $\mathrm{Pic}(V)$ extends to a contravariant $\mathrm{KAut}(V)$-action by $\imath^*(\mathcal{O}_V(D)) = \mathcal{O}_V(\overline{D})$. It induces a contravariant $\mathrm{KAut}(V)$-action on $\mathrm{NS}(V)$, which preserves the ample cone. Note that, by the definition of $H^0(V, \mathcal{O}_V(D))$ and $H^0(V, \mathcal{O}_V(\overline{D}))$ (as vector subspaces of the rational function field of V), the linear system $|\mathcal{O}_V(D)|$ is free (resp. very ample) if and only if so is $|\mathcal{O}_V(\overline{D})|$.

Let $\mathrm{Aut}^*(V)$ and $\mathrm{KAut}^*(V)$ denote respectively the images of $\mathrm{Aut}(V)$ and $\mathrm{KAut}(V)$ in $\mathrm{GL}(\mathrm{NS}(V)/\mathrm{torsion})$. We have

$$\mathrm{KAut}^*(V) = \langle \mathrm{Aut}^*(V), \imath^* \rangle.$$

Proposition 2.12. *Let V be a complex projective variety and let Γ be a subgroup of $\mathrm{GL}(\mathrm{NS}(V)/\mathrm{torsion})$ such that Γ contains $\Gamma \cap \mathrm{Aut}^*(V)$ as a finite index subgroup and preserves $\mathrm{Amp}(V)$ (e.g. $\Gamma = \mathrm{Aut}^*(V)$ or $\mathrm{KAut}^*(V)$). Suppose that the rational hull $\mathrm{Nef}^+(V)$ of the nef cone $\mathrm{Nef}(V)$ contains a rational polyhedral cone Σ satisfying*

$$(\Gamma \cap \mathrm{Aut}^*(V)) \cdot \Sigma \supset \mathrm{Amp}(V).$$

Then Γ has only finitely many finite subgroups, up to conjugation under $\Gamma \cap \mathrm{Aut}^*(V)$.

Proof. Since $[\Gamma : \Gamma \cap \mathrm{Aut}^*(V)] < \infty$, by Fujiki-Lieberman's theorem (see e.g. [Br18, Theorem 2.10]) for each $v \in \mathrm{Amp}(V) \cap (\mathrm{NS}(V)/\mathrm{torsion})$, the stabilizer group of v

$$\{g \in \Gamma \mid g(v) = v\}$$

is a finite group. In particular, for any subset $F \subset \mathrm{NS}(V) \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$F \cap \mathrm{Amp}(V) \cap (\mathrm{NS}(V)/\mathrm{torsion}) \neq \varnothing,$$

the pointwisely stabilizer group of F

$$Z_\Gamma(F) := \{g \in \Gamma \mid g(v) = v, \ \forall\, v \in F\}$$

is a finite group as well.

Thus, by the Siegel property [Lo14, Theorem 3.8], for any two polyhedral cones Π_1 and Π_2 in $\mathrm{Nef}^+(V)$, which are not necessarily of maximal dimension nor of the same dimension, the set

$$\{g \in \Gamma \mid g(\Pi_1^\circ) \cap \Pi_2^\circ \cap \mathrm{Amp}(V) \neq \varnothing\}$$

is a finite set as $Z_\Gamma(F_i)$ in [Lo14, Theorem 3.8] is a finite group as mentioned above. Here and hereafter, Π° is the relative interior of Π.

Let Δ be the set of all faces of Σ. Here Σ itself is also considered as a face as did in [Lo14, Section 1]. Since Σ is a rational polyhedral cone, Δ is a finite set. Hence

$$\mathcal{S} := \{g \in \Gamma \mid g(\Pi_i^\circ) \cap \Pi_i^\circ \cap \mathrm{Amp}(V) \neq \varnothing \text{ for some } \Pi_i \in \Delta\}$$

is also a finite set.

Let $H \subset \Gamma$ be a finite subgroup. Choose $v \in \mathrm{Amp}(V) \cap (\mathrm{NS}(V)/\mathrm{torsion})$. Then

$$v_H := \sum_{g \in H} g(v) \in \mathrm{Amp}(V) \cap (\mathrm{NS}(V)/\mathrm{torsion})$$

as Γ preserves $\mathrm{Amp}(V)$ and $\mathrm{NS}(V)/\mathrm{torsion}$. Since $(\Gamma \cap \mathrm{Aut}^*(V)) \cdot \Sigma \supset \mathrm{Amp}(V)$, there is then an element $a \in \Gamma \cap \mathrm{Aut}^*(V)$ such that

$$u_H := a(v_H) \in \Sigma \cap \mathrm{Amp}(V) \cap (\mathrm{NS}(V)/\mathrm{torsion}).$$

As $g(v_H) = v_H$ whenever $g \in H$, it follows that

$$a \circ g \circ a^{-1}(u_H) = a \circ g(v_H) = a(v_H) = u_H$$

for all $g \in H$. Hence, considering the (unique) face Π of Σ such that $u_H \in \Pi^\circ$, we deduce that

$$a \circ H \circ a^{-1} \subset \mathcal{S}.$$

Since $\mathcal{S}$ is a finite set, it contains only finitely many finite subgroups of Γ. Thus finite subgroups of Γ are at most finite up to conjugation under $\Gamma \cap \mathrm{Aut}^*(V)$. □

Proof of Theorem 1.5. We may and will assume that V has a real structure $\imath$. By Theorem 2.9, it suffices to show that $H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(V)/\mathrm{Aut}^0(V))$ is finite. Recall that we have an exact sequence of $G_{\mathbb{C}/\mathbb{R}}$-groups

$$1 \to N \to \mathrm{Aut}(V)/\mathrm{Aut}^0(V) \to \mathrm{Aut}^*(V) \to 1$$

for some finite $G_{\mathbb{C}/\mathbb{R}}$-group N by Fujiki-Lieberman's theorem. It follows that $H^1(G_{\mathbb{C}/\mathbb{R}}, N_\phi)$ is finite for all $\phi \in Z^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(V)/\mathrm{Aut}^0(V))$. By [Se02, Page 53, Corollary 3], it suffices to show that $H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}^*(V)) = H^1(\langle \imath^* \rangle, \mathrm{Aut}^*(V))$ is finite.

First we assume that $\imath^* \in \mathrm{Aut}^*(V)$. Then $\mathrm{KAut}^*(V) = \mathrm{Aut}^*(V)$. Since the $\imath^*$-action on $\mathrm{Aut}^*(V)$ is the conjugation by $\imath^*$, the set $H^1(\langle \imath^* \rangle, \mathrm{Aut}^*(V))$ is in bijection with the set of conjugacy classes of involutions of $\mathrm{Aut}^*(V) = \mathrm{KAut}^*(V)$, which is finite by Proposition 2.12. Now assume that $\imath^* \notin \mathrm{Aut}^*(V)$, then $\mathrm{Aut}^*(V) \rtimes \langle \imath^* \rangle = \mathrm{KAut}^*(V)$, and it follows from again Proposition 2.12, together with Lemma 2.11, that $H^1(\langle \imath^* \rangle, \mathrm{Aut}^*(V))$ is finite. □

Remark 2.13. Proof of [CF19, Section 9, Proof of Theorem 1.1] is correct modulo the proof of [CF19, Proposition 7.4] which is crucial to conclude. For instance, in the proof of [CF19, Proposition 7.4], it is unclear in general if $\{g^*(\Sigma)\}_{g^* \in \mathrm{Aut}(V)^*}$ form a fan or not. Therefore, it is in general unclear if $g^*(\Sigma) \cap \Sigma$ is a face of both Σ and $g^*(\Sigma)$ or not, either. Even if this would be the case, it is yet unclear if the one-dimensional ray R of both Σ and $g^*(\Sigma)$ in the proof of [CF19, Proposition 7.4] is *inside* $\mathrm{Amp}(V)$ or not. Indeed, if R is on the boundary of $\mathrm{Amp}(V)$, then the set of $g^* \in \mathrm{Aut}(V)^*$ such that

$$R \subset \Sigma \cap g^*(\Sigma)$$

could be an infinite set. For instance, this is the case where g is an element of the Mordell-Weil group of an elliptic K3 surface $V \to \mathbb{P}^1$ of infinite order. For this reason and the importance of Theorem 1.5, we again gave a complete proof under a slightly more general setting.

3. Proof of Theorem 1.1 (1)

We will prove Theorem 1.1 (1) at the end of this section. Let us begin with the following corollary of Theorem 1.3, originally proven by Silhol [Si82, Proposition 7].

Corollary 3.1. *Let A be an abelian variety. Then A, as a complex variety, has at most finitely many non-isomorphic real forms.*

Proof. The proof of [Si82, Proposition 7] is more precise, in that it enumerates the number of real forms. Here we only show the finiteness. Since the $G_{\mathbb{C}/\mathbb{R}}$-group

$$\mathrm{Aut}(A)/\mathrm{Aut}^0(A)$$

is arithmetic [BS64, Exemples 3.5],

$$H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(A)/\mathrm{Aut}^0(A))$$

is finite by Proposition 2.4 (1). Thus the result follows from Theorem 1.3. □

Proposition 3.2. *Let V be a smooth complex projective variety. Assume that $\kappa(V) \geq \dim(V) - 1$. Then every automorphism of V has zero entropy. As a consequence, V has at most finitely many non-isomorphic real forms.*

Proof. The first statement is well-known. Here we provide a proof for reader's convenience. Consider the pluricanonical map

$$\Phi := \Phi_{|mK_V|} : V \dashrightarrow B.$$

Let $f \in \mathrm{Aut}(V)$ be an automorphism of V. By the finiteness of the pluricanonical representation [Ue75, Theorem 14.10], the action $f_{\tilde{B}}$ of f on an equivariant resolution $\tilde{B}$ of B is finite. Thus, all the dynamical degrees of $f_{\tilde{B}}$ equal 1. Since a general fiber of Φ is of dimension at most 1, the relative dynamical degrees of f are also 1. Hence the first dynamical degree of f is 1 and f has zero entropy by the product formula ([DN11, Theorem 1.1] or [Tr20]). Proposition 3.2 then follows from Corollary 1.4. □

Recall that a minimal surface S with $\kappa(S) = 0$ is either a K3 surface, an Enriques surface, an abelian surface or a hyperelliptic surface. Recall also that an irrational surface S with $\kappa(S) = -\infty$ admits a genus 0 fibration $\pi : S \to B$, which is nothing but the Albanese morphism of S, over a smooth projective curve B of genus $g(B) \geq 1$.

Proposition 3.3. *Let S be a smooth complex projective surface birational to an irrational ruled surface or a hyperelliptic surface. Then every automorphism of S has zero entropy. As a consequence, S has at most finitely many non-isomorphic real forms.*

The first statement of Proposition 3.3 is also well-known; see [Ca99, Proposition 1] for a more general statement. As the proof is simple, we include it here for reader's convenience.

Proof. Let $S \to B$ be the Albanese morphism, which is a fibration with $\dim B = 1$ in each case. By the universal property, every automorphism of S preserves this fibration. Since the base and general fibers of the fibration are curves, by the product formula ([DN11, Theorem 1.1] or [Tr20]), every automorphism of S has zero entropy. Proposition 3.3 then follows from Corollary 1.4. □

Proposition 3.4. *Let S be a smooth complex projective surface which is birational to an abelian surface A. Then S has at most finitely many non-isomorphic real forms.*

Proof. It suffices by Theorem 2.3 to show that $H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(S))$ is finite.

By running the minimal model program, S is obtained by a sequence of blow-ups

$$\pi : S = S_k \to \cdots \to S_1 \to S_0 = A$$

at $k \geq 0$ reduced points. If $k = 0$, then Proposition 3.4 is contained in Corollary 3.1. Suppose that $k = 1$, then we can choose the origin of A to be the blow-up center o of $\pi : S \to A$, and

$$\mathrm{Aut}(S) \simeq \mathrm{Dec}(A, o) = \mathrm{Aut}_{\mathrm{group}}(A).$$

Since $\mathrm{Aut}_{\mathrm{group}}(A)$ is an arithmetic $G_{\mathbb{C}/\mathbb{R}}$-group, $H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(S))$ is finite by Proposition 2.4 (1).

Now assume that $k \geq 2$. Let $E_1, \dots, E_k$ be the irreducible components of the exceptional set of π. Then

$$H := \mathrm{Dec}(S, E_1, \dots, E_k)$$

is a finite index subgroup of $\mathrm{Aut}(S)$ and H descends to a subgroup of $\mathrm{Dec}(A, \Sigma)$. Here $\Sigma \subset A$ is the blow-up center of $S_2 \to A$, which is a subscheme of length 2, and $\mathrm{Dec}(A, \Sigma)$ is the decomposition group of the *closed subscheme* $\Sigma \subset A$. We choose a point o in the support of Σ as the origin of A.

Case 1: Σ is supported at one point $o \in A$.

In this case, we have

$$\mathrm{Dec}(A, \Sigma) = \{ f \in \mathrm{Aut}_{\mathrm{group}}(A) \mid [(df)_o(v)] = [v] \in \mathbb{P}(T_{A,o}) \text{ for some } v \in T_{A,o} \} .$$

Claim 3.5. $\mathrm{Dec}(A, \Sigma)$ is a solvable group.

Proof. By assumption, there is a $\mathbb{C}$-basis $\langle v, u\rangle$ of $T_{A,o}$ such that the action of $f \in \mathrm{Dec}(A, \Sigma)$ on the tangent space $T_{A,o}$ is of the form

$$(df)_o = \begin{pmatrix} c(f) & a(f) \\ 0 & b(f) \end{pmatrix} \ (c(f), b(f) \in \mathbb{C}^\times, a(f) \in \mathbb{C})$$

with respect to the basis $\langle v, u\rangle$. Thus $\mathrm{Dec}(A, \Sigma)$ is solvable, as the representation

$$\mathrm{Aut}_{\mathrm{group}}(A) = \mathrm{Dec}(A, o) \to \mathrm{GL}(T_{A,o}), \quad f \mapsto (df)_o$$

is faithful. □

Consider the natural faithful representation

$$\rho : \mathrm{Dec}(A, \Sigma) \subset \mathrm{Aut}_{\mathrm{group}}(A) \hookrightarrow \mathrm{GL}(H^1(A, \mathbb{Z})).$$

Since $\mathrm{Dec}(A, \Sigma)$ is solvable by Claim 3.5, and since $H^1(A, \mathbb{Z})$ is a free abelian group of finite rank, $\mathrm{Dec}(A, \Sigma)$ is then a polycyclic group by Malcev's theorem [Se83, Page 26, Corollary 1]. It follows that H is polycyclic as well, and $\mathrm{Aut}(S)$ is virtually polycyclic. Thus $H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(S))$ is finite by Proposition 2.4 (3).

Case 2: Σ is supported at two points $o, P \in A$ such that P is not torsion.

Let B be the irreducible component of the Zariski closure of $\{nP \mid n \in \mathbb{Z}\}$ containing the origin o:

$$o \in B \subset \overline{\{nP \mid n \in \mathbb{Z}\}}^{\mathrm{Zar}}.$$

Since P is not a torsion point, B is either an elliptic curve E (with the origin o) or A.

Claim 3.6. $\mathrm{Dec}(A, o, P)$ is a finite group.

Proof. Since $\mathrm{Dec}(A, o, P)$ acts trivially on $\{nP \mid n \in \mathbb{Z}\}$, and therefore on B, the result follows if $B = A$. Consider the case where $B = E$. Consider the elliptic curve $C := A/E$ and the quotient morphism $p : A \to C$. We choose $p(o) \in C$ as the origin of the elliptic curve C. Then $\mathrm{Dec}(A, o, P)$ embeds into $\mathrm{Aut}_{\mathrm{group}}(C)$. Since C is an elliptic curve, the group $\mathrm{Aut}_{\mathrm{group}}(C)$ is finite. Thus the result follows also in the case where $B = E$. □

Recall that $H \subset \mathrm{Dec}(A, o, P)$ and H is a finite index subgroup of $\mathrm{Aut}(S)$, Claim 3.6 implies that $\mathrm{Aut}(S)$ is finite, hence $H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(S))$ is finite.

Case 3: Σ is supported at two points $o, P \in A$ such that P is torsion.

This is the last case we need to consider. Thanks to the first two cases, up to rearranging the blow-up sequence, we can reduce to the case where $S \to A$ is the blow-up at finitely many distinct torsion points, including the origin o, of A. Then

$$\mathrm{Ine}_{\mathrm{group}}(A, A[N]) \subset H \subset \mathrm{Dec}_{\mathrm{group}}(A, A[N]) = \mathrm{Aut}_{\mathrm{group}}(A)$$

for some $N > 0$, where $A[N] \simeq (\mathbb{Z}/N)^4$ is the subgroup of torsion points of order dividing N. Here we note that $A[N]$ is preserved by $\mathrm{Aut}_{\mathrm{group}}(A)$ and

$$[\mathrm{Dec}_{\mathrm{group}}(A, A[N]) : \mathrm{Ine}_{\mathrm{group}}(A, A[N])] < \infty.$$

Since $\mathrm{Aut}_{\mathrm{group}}(A)$ is arithmetic, it follows that H and hence $\mathrm{Aut}(S)$ are also arithmetic. Therefore, by Proposition 2.4 (1), $H^1(G_{\mathbb{C}/\mathbb{R}}, \mathrm{Aut}(S))$ is a finite set. Hence S has at most finitely many real forms by Theorem 2.3. $\square$

Proof of Theorem 1.1 (1). Let S be a smooth complex projective surface with infinitely many mutually non-isomorphic real forms. We may assume that S is not rational. Then by Propositions 3.2, 3.3 and 3.4, S is birational to a K3 surface or an Enriques surface.

Suppose that S is minimal. Then S is a K3 surface or an Enriques surface. By [Ka97, Theorem 2.1] (see also [St85] and [Na85]), the cone conjecture holds for S, that is, there exists a rational polyhedral fundamental domain for the action of $\mathrm{Aut}^*(S)$ on the cone $\mathrm{Nef}^+(S)$. By Theorem 1.5, S has at most finitely many non-isomorphic real forms. This is a contradiction and therefore, S is non-minimal. $\square$

Remark 3.7. Let S be a smooth projective surface. Then the group $\mathrm{Aut}(S)/\mathrm{Aut}^0(S)$ is finitely generated unless S is either rational or non-minimal and birational to an abelian surface, a K3 surface or an Enriques surface. Indeed, our proof of Theorem 1.1 (1) shows that the group $\mathrm{Aut}(S)/\mathrm{Aut}^0(S)$ is either a polycyclic group or an arithmetic group, up to finite kernel and cokernel, or satisfies the cone conjecture. In the first two cases $\mathrm{Aut}(S)/\mathrm{Aut}^0(S)$ is clearly finitely generated. In the last case one can deduce from [Lo14, Corollary 4.15] that $\mathrm{Aut}(S)/\mathrm{Aut}^0(S)$ is finitely generated as well. It would be interesting to study relations between finiteness of real forms and finite generation of the group $\mathrm{Aut}(S)/\mathrm{Aut}^0(S)$ more closely.

4. Surfaces with infinitely many real forms

The goal of this section is Theorem 4.8, which gives an explicit description of a surface with infinitely many real forms, obtained as a one-point blow-up of an Enriques surface. We will prove Theorem 4.8 in Section 5.

4.1. **Kummer surfaces of product type.**

Let E be the projective elliptic curve given by the affine Weierstrass equation

$$y^2 = x(x-1)(x-t) \tag{4.1}$$

and F be the projective elliptic curve given by the affine Weierstrass equation

$$y_2^2 = x_2(x_2-1)(x_2-s). \tag{4.2}$$

Note that $E/\langle -1_E \rangle = \mathbb{P}^1$, the associated quotient map $E \to \mathbb{P}^1$ is given by $(x, y) \mapsto x$ and the points 0, 1, t and ∞ of $\mathbb{P}^1$ are exactly the branch points of this quotient map. The same holds for F if we replace t by s.

Throughout this paper, we make the following assumptions on s and t:

Assumption 4.1. $t, s \in \mathbb{R}$ are two real numbers which are algebraically independent over $\mathbb{Q}$.

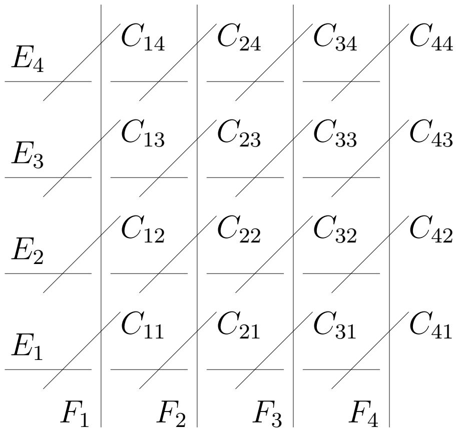

FIGURE 1. Curves E_i, F_j and C_{ij}

There are many such s and t. By the assumption that $t, s \in \mathbb{R}$, the two elliptic curves E and F, and thus $E \times F$, have obvious real structures, which we will denote by $\imath_E$, $\imath_F$ and $\imath_{E\times F} = \imath_E \times \imath_F$. By the algebraically independent assumption of s, t over $\mathbb{Q}$, the elliptic curves E and F are not isogenous with no complex multiplication, that is,

$$\mathrm{End}_{\mathrm{group}}(E) \simeq \mathrm{End}_{\mathrm{group}}(F) \simeq \mathbb{Z}.$$

Let

$$X := \mathrm{Km}(E \times F)$$

be the Kummer K3 surface associated to the product abelian surface $E \times F$, that is, the minimal resolution of the quotient surface $E \times F/\langle -1_{E\times F}\rangle$. We write $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$. By Assumption 4.1, X has a natural real structure $\imath_X$ induced from $\imath_{E\times F}$.

Let $\{a_i\}_{i=1}^4$ and $\{b_i\}_{i=1}^4$ be the 2-torsion subgroups of F and E respectively. Then X contains 24 smooth rational curves which form the so-called double Kummer pencil on X, as in Figure 1. Here smooth rational curves E_i, F_i $(1 \le i \le 4)$ are arising from the elliptic curves $E \times \{a_i\}$, $\{b_i\} \times F$ on $E \times F$. Smooth rational curves C_{ij} $(1 \le i, j \le 4)$ are the exceptional curves over the A_1-singularities of the quotient surface $E \times F/\langle -1_{E\times F}\rangle$.

Note that all these 24 curves are defined over $\mathbb{R}$ with respect to $\imath_X$.

We denote the unique point $E_j \cap C_{ij}$ by P_{ij} and the unique point $F_i \cap C_{ij}$ by P'_{ij}. These are real points with respect to $\imath_X$. We can use the same x (resp. x_2) as in the defining equations of E and F, the affine coordinate of E_j and F_i so that

$$x(P_{1j}) = t, \quad x(P_{2j}) = 1, \quad x(P_{3j}) = \infty, \quad x(P_{4j}) = 0 \tag{4.3}$$

on E_j with respect to the coordinate x and

$$x_2(P'_{i1}) = s, \quad x_2(P'_{i2}) = 1, \quad x_2(P'_{i3}) = \infty, \quad x_2(P'_{i4}) = 0 \tag{4.4}$$

on F_i with respect to the coordinate x_2.

Note that the coordinate values of points are *different* from the ones in [DO19] and [DOY21] as we found that the current ones are more convenient to study the Enriques surface Z defined in the next subsection, whereas the previous ones were more convenient to study the rational surface T there.

Set

$$\theta := [(1_E, -1_F)] = [(-1_E, 1_F)] \in \mathrm{Aut}(X).$$

Then θ is an automorphism of X of order 2. The following lemma is essentially the same as [Og89, Lemmas 1.3, 1.4, 1.11] and [DO19, Lemma 3.3].

Lemma 4.2. *Under Assumption 4.1, the following assertions hold.*

(1) *The Picard number $\rho(X)$ of X is* 18 *and the image of canonical representation of* $\mathrm{Aut}(X)$
$$\mathrm{Aut}(X) \to \mathrm{GL}(H^0(X, \Omega_X^2)) = \mathbb{C}^\times$$
is $\{\pm 1\}$.

(2) *The action of $\imath_X$ on* $\mathrm{Pic}(X) \simeq \mathrm{NS}(X)$ *is trivial and* $\imath_X \circ f \circ \imath_X = f$ *for all* $f \in \mathrm{Aut}(X)$. *In particular, every $f \in \mathrm{Aut}(X)$ is defined over $\mathbb{R}$ with respect to $\imath_X$.*

(3) $\theta^* = \mathrm{id}$ *on* $\mathrm{Pic}(X)$ *and* $\theta^*\omega_X = -\omega_X$.

(4) $f \circ \theta = \theta \circ f$ *for all* $f \in \mathrm{Aut}(X)$.

(5) *Let X^θ be the fixed locus of θ. Then* $X^\theta = \cup_{i=1}^4 (E_i \cup F_i)$.

(6) $\mathrm{Aut}(X) = \mathrm{Dec}(X, \cup_{i=1}^4 (E_i \cup F_i))$.

Proof. By Assumption 4.1, We have
$$\mathrm{End}_{\mathrm{group}}(E \times F) \simeq \mathrm{End}_{\mathrm{group}}(E) \times \mathrm{End}_{\mathrm{group}}(E) \simeq \mathbb{Z} \times \mathbb{Z}.$$
This implies that $E \times F$ is of Picard number 2 and the first assertion of (1) follows from this. The second assertion of (1) is proved by [Og89, Lemma 1.11]. The first assertion of (2) is then clear, as the 24 smooth rational curves are invariant under $\imath_X$ and they generate $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the first assertion of (1). Thus for $g \in \mathrm{Aut}(X)$
$$\imath_X \circ g \circ \imath_X = g$$
on $\mathrm{Pic}(X)$. Since $g^*\omega_X = \pm\omega_X$ by the second assertion (1) and $\imath_X^*\omega_X = \omega_X$, it also follows that
$$\imath_X \circ g \circ \imath_X = g$$
on $H^0(X, \Omega_X^2)$. Since $\imath_X \circ g \circ \imath_X, g \in \mathrm{Aut}(X)$, it follows from the global Torelli theorem for K3 surfaces that
$$\imath_X \circ g \circ \imath_X = g$$
in $\mathrm{Aut}(X)$ as in [DO19, Lemma 3.3]. This proves (2). The last four assertions (3), (4), (5) and (6) are proved by [Og89, Lemmas 1.3, 1.4]. ☐

4.2. **Mukai's Enriques surfaces.** We use the same notation as in Subsection 4.1. By Assumption 4.1, the two sets
$$\{P'_{i1}, P'_{i2}, P'_{i3}, P'_{i4}\} \subset F_i \cong \mathbb{P}^1, \quad \{P_{1j}, P_{2j}, P_{3j}, P_{4j}\} \subset E_j \cong \mathbb{P}^1$$
are not in the same orbit of the action of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2, \mathbb{C})$ on $\mathbb{P}^1$.

In this subsection, following Mukai [Mu10], with necessary remarks about real structure, we recall the construction of an Enriques surface Z with a natural real structure $\imath_Z$ from our $X = \mathrm{Km}(E \times F)$.

Let
$$T := X/\langle\theta\rangle, \quad \text{and} \quad q : X \to T$$
be the quotient surface and the quotient morphism. Then T is a smooth projective surface such that $q(C_{ij})$ $(1 \le i, j \le 4)$ is a smooth rational curve with self-intersection number -1.

Then T is obtained by the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the 16 real points defined over $\mathbb{R}$ (with respect to the natural real structure)

$$p_{ij} \in \mathbb{P}^1 \times \mathbb{P}^1 \ (1 \le i, j \le 4).$$

By construction, C_{ij} contracts to p_{ij} under the composite morphism

$$X \to T \to \mathbb{P}^1 \times \mathbb{P}^1.$$

Note also that the real structure $\imath_{\mathbb{P}^1 \times \mathbb{P}^1}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ induced from $\imath_X$ is the same as the real structure of $\mathbb{P}^1 \times \mathbb{P}^1$ induced from

$$\operatorname{Proj} \mathbb{C}[y_0, y_1] = \operatorname{Proj} \mathbb{R}[y_0, y_1] \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}.$$

Let us consider the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3,$$

and identify $\mathbb{P}^1 \times \mathbb{P}^1$ with a smooth quadric surface Q in $\mathbb{P}^3$ defined over $\mathbb{R}$ with respect to $\imath_{\mathbb{P}^1 \times \mathbb{P}^1}$ above and $\imath_{\mathbb{P}^3}$ below. Since the four points $p_{11}, p_{22}, p_{33}, p_{44} \in Q$ are not coplanar in $\mathbb{P}^3$, we may and will choose the *real* homogeneous coordinates $[w_1 : w_2 : w_3 : w_4]$ of

$$\mathbb{P}^3 = \operatorname{Proj} \mathbb{R}[w_1, w_2, w_3, w_4] \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$$

so that

$$p_{11} = [1 : 0 : 0 : 0], \quad p_{22} = [0 : 1 : 0 : 0], \quad p_{33} = [0 : 0 : 1 : 0], \quad p_{44} = [0 : 0 : 0 : 1].$$

Then, up to multiplying w_i by some real numbers if necessarily, the equation of Q is written in the form

$$\alpha_1 w_2 w_3 + \alpha_2 w_1 w_3 + \alpha_3 w_1 w_2 + (w_1 + w_2 + w_3) w_4 = 0 \tag{4.5}$$

for some *non-zero real* numbers α_i satisfying the smoothness (non-degeneration) condition

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 2\alpha_1\alpha_2 - 2\alpha_1\alpha_3 - 2\alpha_2\alpha_3 \neq 0.$$

Then the Cremona involution of $\mathbb{P}^3$

$$\tilde{\tau}' : [w_1 : w_2 : w_3 : w_4] \mapsto [\alpha_1 w_2 w_3 w_4 : \alpha_2 w_1 w_3 w_4 : \alpha_3 w_1 w_2 w_4 : \alpha_1 \alpha_2 \alpha_3 w_1 w_2 w_3]$$

is defined over $\mathbb{R}$ with respect to $\imath_{\mathbb{P}^3}$ above and satisfies $\tilde{\tau}'(Q) = Q$. Hence we obtain a birational automorphism of Q

$$\tau' := \tilde{\tau}'|_Q \in \operatorname{Bir}(Q)$$

which is defined over $\mathbb{R}$ with respect to the real structure $\imath_Q$ of Q induced from $\imath_{\mathbb{P}^1 \times \mathbb{P}^1}$ and $\imath_{\mathbb{P}^3}$. Let $I(\tau')$ be the indeterminacy locus of τ'. By the definition of τ', one can readily check the following ([Mu10, Section 2]):

Lemma 4.3. (1) *$I(\tau') = \{p_{ii}\}_{i=1}^4$ and τ' contracts the conic curve $C_i' := Q \cap (w_i = 0)$ to p_{ii} $(1 \le i \le 4)$.*

(2) *τ' interchanges the two lines through p_{ii} for each $i = 1, 2, 3, 4$.*

(3) *$\mu^{-1} \circ \tau' \circ \mu \in \operatorname{Aut}(B)$, where $\mu : B \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up at the four points p_{ii} $(1 \le i \le 4)$.*

Hence B has a real structure $\imath_B$ induced from $\imath_{\mathbb{P}^1\times\mathbb{P}^1}$ by Lemma 4.3 (3).

By Lemma 4.3 (2), $\tau'(p_{ij}) = p_{ji}$ if $1 \le i \ne j \le 4$. Therefore τ' lifts to

$$\tau \in \mathrm{Aut}(T),$$

which is defined over $\mathbb{R}$ with respect to $\imath_T$.

Since $q : X \to T$ is the finite double cover branched along the unique anti-bicanonical divisor

$$\sum_{i=1}^{4}(q(E_i) + q(F_i)) \in |-2K_T|,$$

defined over $\mathbb{R}$ with respect to $\imath_T$ (especially by the uniqueness), it follows that τ lifts to an involution

$$\epsilon \in \mathrm{Aut}(X), \tag{4.6}$$

which is again defined over $\mathbb{R}$ with respect to $\imath_X$. A priori, there are exactly the two choices of the lifting ϵ; if we denote one lifting by ϵ_0 then the other is $\theta \circ \epsilon_0$. Recall that $\theta^*\omega_X = -\omega_X$ and $g^*\omega_X = \pm\omega_X$ for $g \in \mathrm{Aut}(X)$ by Lemma 4.2 (1). Thus, we may and will choose the unique lift ϵ with $\epsilon^*\omega_X = -\omega_X$. Let

$$Z := X/\langle\epsilon\rangle, \quad \text{and} \quad \pi : X \to Z \tag{4.7}$$

be the quotient surface and the quotient morphism.

By construction, Z has a natural real structure $\imath_Z$ induced from $\imath_X$ and thus from $\imath_{E\times F}$.

The following theorem, which is crucial for us, was found by Mukai [Mu10, Proposition 2].

Theorem 4.4. *The involution ϵ acts on X freely and Z is an Enriques surface.*

Note that the involution ϵ does not come from any involution of the Kummer quotient $E \times F/\langle -1_{E\times F}\rangle$, since it does not preserve the set of exceptional divisors of the birational map $X \to E \times F/\langle -1_{E\times F}\rangle$. Set

$$C_i := \epsilon(C_{ii}) \subset X \ (i = 1,\ 2,\ 3,\ 4).$$

As the image of C_{ii} through the morphism $X \to T \to B$ is the exceptional divisor over $p_{ii} \in \mathbb{P}^1 \times \mathbb{P}^1$, the curve C_i is the proper transform of the curve C_i' in Lemma 4.3 (1) under the morphism

$$X \to T \to B \to \mathbb{P}^1 \times \mathbb{P}^1 = Q.$$

Corollary 4.5. (1) *$\epsilon(E_i) = F_i$, $\epsilon(F_i) = E_i$ for all $i = 1,\ 2,\ 3,\ 4$.*

(2) *$\epsilon(C_{ij}) = C_{ji}$ for all i, j such that $i \ne j$.*

(3) *$(C_i.E_i)_X = (C_i.F_i)_X = 1$, $(C_i.C_{ii})_X = 0$, $(C_i.C_{kj})_X = 0$ for all i, j, k such that $k \ne j$.*

(4) *$(C_i.E_j)_X = (C_i.F_j)_X = 0$ for all $i \ne j$.*

Proof. (1) and (2) follow from the description of τ and ϵ. Then (3) and (4) follow from $\epsilon(C_{ii}) = C_i$ and (1) and (2). Here $(C_i.C_{ii})_X = 0$ follows from the fact that the conic curve $C_i' \subset Q$ that is contracted to p_{ii} by τ' does not pass through p_{ii} (See Lemma 4.3 (1)). $\square$

4.3. **Surface birational to an Enriques surface with infinitely many real forms.** In this subsection, we assume that E and F are the elliptic curves defined by Equations (4.1), (4.2), the parameters s and t satisfy Assumption 4.1, $X = \mathrm{Km}(E \times F)$ is the Kummer K3 surface, and $Z = X/\langle \epsilon \rangle$, with the quotient morphism $\pi : X \to Z$, is the Enriques surface defined in Subsection 4.2.

We use the following notation for curves and points on Z:

$$H_j := \pi(E_j), \quad D_{ij} := \pi(C_{ij}), \quad Q_{ij} := \pi(P_{ij}).$$

The smooth rational curves H_j, D_{ij} and the points Q_{ij} are defined over $\mathbb{R}$ with respect to $\imath_Z$.

The next lemma, which is similar to [DOY21, Lemma 2.4], is also crucial in this paper.

Assumption 4.6. Let A be a point of D_{31} which satisfies the following three conditions:

(1) A is a real point of D_{31} in the sense that $A \in D_{31}^{\imath_Z}$;
(2) $A \notin D_{31} \cap C$ for any irreducible curve $C \subset Z$ with $C \neq D_{31}$ and $(C^2)_Z < 0$;
(3) $A \notin D_{31}^f$ for any $f \in \mathrm{Dec}(Z, D_{31}) \setminus \mathrm{Ine}(Z, D_{31})$.

Lemma 4.7. *There are uncountably many points $A \in D_{31}$ satisfying Assumption 4.6.*

Proof. Note that there are at most countably many irreducible curves $C \neq D_{31}$ on Z with $(C^2)_Z < 0$ and thus the points $B \in D_{31}$ which are in the union of $D_{31} \cap C$ (for all such curves C) are at most countable. Note also that $\mathrm{Aut}(Z)$ is discrete and hence countable and D_{31}^f is at most two points for each $f \in \mathrm{Dec}(Z, D_{31}) \setminus \mathrm{Ine}(Z, D_{31})$, as $D_{31} \simeq \mathbb{P}^1$. Therefore the points $B \in D_{31}$ which are in the union of D_{31}^f (for all $f \in \mathrm{Dec}(Z, D_{31}) \setminus \mathrm{Ine}(Z, D_{31})$) are also at most countable. On the other hand, $D_{31}^{\imath_Z} = \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$, as a set, is uncountable. So, there are uncountably many points $A \in D_{31}$ satisfying Assumption 4.6. □

Our main theorem is the following:

Theorem 4.8. *Let s, t be as in Assumption 4.1 and $A \in D_{31} \subset Z$ be as in Assumption 4.6. Let $\mu : Y \to Z$ be the blow-up of Z at A. Then Y has infinitely many mutually non-isomorphic real forms.*

Then Theorem 1.1 (2) follows from Theorem 4.8.

Remark 4.9. By construction, Y in Theorem 4.8 is parametrized by the three real parameters

$$(s, t, A)$$

which move in a dense subset of $\mathbb{R}^3$.

5. Proof of Theorem 4.8 and Theorem 1.1 (2)

In this section, we prove Theorem 4.8 and thus complete the proof of Theorem 1.1 (2). Theorem 4.8 will follow from Propositions 5.1, 5.2 and 5.3 below. More precisely, first we will reduce the proof to a problem on the existence of a set of involutions on X with certain property (Proposition 5.1 (3)) and we will then solve this problem in Propositions 5.2 and 5.3 (2).

We will use the same notation as in Section 4.

Note that $\mathrm{Bir}(Z) = \mathrm{Aut}(Z)$ as Z is a minimal projective smooth surface. Let E_A be the exceptional curve of the blow-up $\mu : Y \to Z$. Then $|2K_Y| = \{2E_A\}$. Thus under the natural inclusion

$$\mathrm{Aut}(Y) \subset \mathrm{Bir}(Z) = \mathrm{Aut}(Z),$$

induced from μ, we have

$$\mathrm{Aut}(Y) = \mathrm{Dec}(Y, E_A) = \mathrm{Dec}(Z, A).$$

Note that $\epsilon(C_{13}) = C_{31}$ by Corollary 4.5 (2). Thus, if $f \in \mathrm{Dec}(Z, D_{31})$, then f lifts in two ways to $\mathrm{Aut}(X)$. Namely, if we write one of them by $\tilde{f}$, then they are $\tilde{f}$ and $\epsilon \circ \tilde{f}$. Moreover, $\tilde{f}$ satisfies either $\tilde{f}(C_{31}) = C_{13}$ or $\tilde{f}(C_{31}) = C_{31}$, and thus, $\epsilon \circ \tilde{f}(C_{31}) = C_{31}$ or $\epsilon \circ \tilde{f}(C_{31}) = C_{13}$, respectively. Hence, there is a unique lift of each element of $\mathrm{Dec}(Z, D_{31})$ to $\mathrm{Aut}(X)$ so that

$$\mathrm{Dec}(Z, D_{31}) \subset \mathrm{Dec}(X, C_{31}) \subset \mathrm{Aut}(X),$$

and therefore

$$\mathrm{Ine}(Z, D_{31}) \subset \mathrm{Ine}(X, C_{31}) \subset \mathrm{Aut}(X).$$

Proposition 5.1. (1) *We have*

$$\mathrm{Aut}(Y) = \mathrm{Dec}(Z, A) = \mathrm{Ine}(Z, D_{31}) \subset \mathrm{Ine}(X, C_{31}).$$

(2) *The conjugate action of the real structure $\imath_Y$ on Y, which is naturally induced from the original real structure $\imath_{E\times F}$, is trivial on* $\mathrm{Aut}(Y)$

(3) *Assume that there is a set $\mathcal{S} \subset$* $\mathrm{Ine}(X, C_{31})$ *consisting of some involutions on X such that the set of conjugacy classes of $\mathcal{S}$ in* $\mathrm{Ine}(X, C_{31})$ *is an infinite set and $\mathcal{S} \subset$* $\mathrm{Aut}(Y)$ *under the inclusion in (1). Then Y has infinitely many mutually non-isomorphic real forms.*

Proof. First we show (1). As already remarked, we have

$$\mathrm{Aut}(Y) = \mathrm{Dec}(Z, A), \quad \mathrm{Ine}(Z, D_{31}) \subset \mathrm{Ine}(X, C_{31}).$$

So, it suffices to show that $\mathrm{Dec}(Z, A) = \mathrm{Ine}(Z, D_{31})$.

Since $A \in D_{31}$, we have $\mathrm{Ine}(Z, D_{31}) \subset \mathrm{Dec}(Z, A)$.

Let us show the reverse inclusion $\mathrm{Dec}(Z, A) \subset \mathrm{Ine}(Z, D_{31})$. Let $f \in \mathrm{Dec}(Z, A)$. Then $A \in D_{31} \cap f(D_{31})$. Since

$$(f(D_{31})^2)_Z = (D_{31}^2)_Z = -2 < 0,$$

it follows that $f(D_{31}) = D_{31}$ by the choice of A (Assumption 4.6 (2)). Thus $f \in \mathrm{Dec}(Z, D_{31})$. Note that $f(A) = A$ by $f \in \mathrm{Dec}(Z, A)$ and therefore $A \in D_{31}^{f|_{D_{31}}}$. Thus $f|_{D_{31}} = \mathrm{id}_{D_{31}}$ by the choice of A (Assumption 4.6 (3)). Hence $\mathrm{Dec}(Z, A) \subset \mathrm{Ine}(Z, D_{31})$ as claimed.

(2) now follows from the facts that $\imath_X$ acts on $\mathrm{Aut}(X)$ as identity (Lemma 4.2 (2)) and the inclusion in (1) which commutes with $\imath_X$ and $\imath_Y$ by the definition of $\imath_Y$ and $\imath_Z$.

Let us show (3). Let $\mathcal{S}_Y$ be the set consisting of the involutions in $\mathrm{Aut}(Y)$ and id_Y. By (2), the number of real forms on Y is the same as the cardinality of conjugacy classes of $\mathcal{S}_Y$ with respect to $\mathrm{Aut}(Y)$ by Proposition 2.4 (4). Since $\mathcal{S} \subset \mathcal{S}_Y$ by the assumption made in (3) and $\mathrm{Aut}(Y) \subset \mathrm{Ine}(X, C_{31})$ by (1), the cardinality of the conjugacy classes of $\mathcal{S}_Y$ with respect to $\mathrm{Aut}(Y)$ is larger than or equal to the cardinarily of the conjugacy classes of $\mathcal{S}$ with respect to $\mathrm{Ine}(X, C_{31})$, which is infinite by the assumption in (3). Hence Y has infinitely many mutually non-isomorphic real forms. $\square$

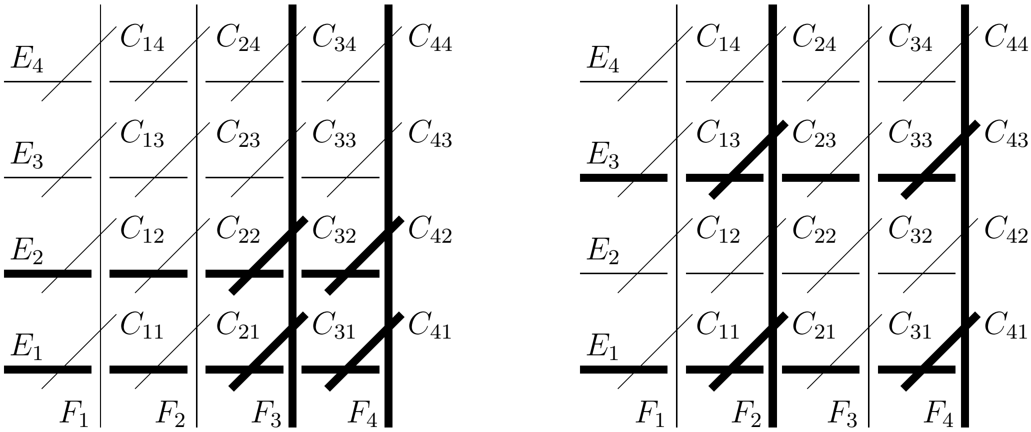

FIGURE 2. Divisors D_1 and D_2

Now it suffices to find such a set $\mathcal{S}$ as in Proposition 5.1 (3). We will use the notation of curves in Figure 1.

We consider two different elliptic fibrations

$$\Phi_{|D_i|} : X \to B_i := \mathbb{P}^1 \ (i = 1, 2),$$

where D_i ($i = 1$, 2) are divisors on X of Kodaira type I_8 defined by:

$$D_1 := C_{31} + E_1 + C_{41} + F_4 + C_{42} + E_2 + C_{32} + F_3,$$

$$D_2 := C_{21} + E_1 + C_{41} + F_4 + C_{43} + E_3 + C_{23} + F_2.$$

See Figure 2. Note that $\Phi_{|D_i|}$ ($i = 1$, 2) are Type $\mathcal{I}_1$ in [Og89, Theorem 2.1].

Almost by definition, a smooth rational curve C on X is a section of $\Phi_{|D_i|}$ if and only if $(C.D_i)_X = 1$. In particular, $\Phi_{|D_1|}$ has sections C_{21}, C_{12}, C_{43} and C_{34}, which we will use, and $\Phi_{|D_2|}$ has a section C_{31}, which we will also use.

Let $F_{1,\eta}$ be the generic fiber of $\Phi_{|D_1|}$. Then $(F_{1,\eta}, F_{1,\eta} \cap C_{21})$ is an elliptic curve with the origin $F_{1,\eta} \cap C_{21}$ over the function field $\mathbb{C}(B_1)$. The group of translation f of the elliptic curve $(F_{1,\eta}, F_{1,\eta} \cap C_{21})$ over $\mathbb{C}(B_1)$ is called the Mordell-Weil group of $\Phi_{|D_1|}$ and we will denote it by $\mathrm{MW}(\Phi_{|D_1|})$. The group $\mathrm{MW}(\Phi_{|D_1|})$ is an abelian group and it bijectively corresponds to the set of sections of $\Phi_{|D_1|}$ in an obvious manner. Moreover

$$\mathrm{MW}(\Phi_{|D_1|}) \subset \mathrm{Bir}(X/B_1) = \mathrm{Aut}(X/B_1) \subset \mathrm{Aut}(X).$$

Let

$$f \in \mathrm{MW}(\Phi_{|D_1|}) \text{ defined by } f(F_{1,\eta} \cap C_{21}) = F_{1,\eta} \cap C_{43}.$$

Since $f^*\omega_{F_{1,\eta}} = \omega_{F_{1,\eta}}$ for a global one-form $\omega_{F_{1,\eta}}$ of the elliptic curve $F_{1,\eta}$ and f acts on the base space B_1 as identity, it follows that $f^*\omega_X = \omega_X$.

Let $F_{2,\eta}$ be the generic fiber of $\Phi_{|D_2|}$. Then $(F_{2,\eta}, F_{2,\eta} \cap C_{31})$ is an elliptic curve with the origin $F_{2,\eta} \cap C_{31}$ over $\mathbb{C}(B_2)$. Consider the inversion ψ of the elliptic curve $(F_{2,\eta}, F_{2,\eta} \cap C_{31})$. Then we have

$$\psi \in \mathrm{Bir}(X/B_2) = \mathrm{Aut}(X/B_2) \subset \mathrm{Aut}(X).$$

Since $\psi^*\omega_{F_{2,\eta}} = -\omega_{F_{2,\eta}}$ for a global one-form $\omega_{F_{2,\eta}}$ of the elliptic curve $F_{2,\eta}$ and ψ acts on the base space B_2 as identity, it follows that $\psi^*\omega_X = -\omega_X$. Set

$$\psi_n := f^{-4n} \circ \psi \circ f^{4n} \in \mathrm{Aut}(X)$$

and

$$\mathcal{S} := \{\psi_n \,|\, n \in \mathbb{Z}\} \subset \mathrm{Aut}(X).$$

Proposition 5.2. (1) *ψ_n are involutions and $\mathcal{S} \subset \mathrm{Ine}(X, C_{31})$.*
(2) *$\psi_n \neq \psi_m$ if $n \neq m$.*
(3) *ψ_n descends to $\mathrm{Aut}(Y)$, i.e., $\mathcal{S} \subset \mathrm{Aut}(Y)$ under the natural inclusion in Proposition 5.1 (1).*

Proof. Let us first show (1). Since ψ is of order two by definition, so are ψ_n. By [Ko63, Theorem 9.1], the group structure of the smooth part of the singular fiber D_1 of $\Phi_{|D_1|}$ is

$$\mathbb{C}^\times \times \mathbb{Z}/8\mathbb{Z},$$

where

$$\mathbb{C}^\times = E_1 \setminus (C_{41} \cup C_{31}) \tag{5.1}$$

which is compatible with the multiplication with respect to the affine coordinate x of E_1, as

$$x(E_1 \cap C_{31}) = \infty, \quad x(E_1 \cap C_{41}) = 0, \quad x(E_1 \cap C_{21}) = 1$$

for the fixed zero section C_{21}, and where $\mathbb{Z}/8\mathbb{Z}$ is the cyclic group of consisting of the 8 irreducible components.

Then, the component E_1, hence also C_{31}, is stable under f^4. In particular, $f^{4n} \in \mathrm{Dec}(X, C_{31})$ for all n. Combining this with $\psi \in \mathrm{Ine}(X, C_{31})$ by the definition of ψ, we deduce that

$$\psi_n = f^{-4n} \circ \psi \circ f^{4n} \in \mathrm{Ine}(X, C_{31}).$$

This proves (1).

Let us then show (2). Since $f^*\omega_X = \omega_X$ (which is nowhere vanishing) and $x(E_1 \cap C_{31}) = \infty$, we have

$$f^{4n}|_{E_1}(x) = r^n \cdot x, \quad f^{4n}|_{C_{31}}(z) = r^n \cdot z \tag{5.2}$$

for some $r \in \mathbb{C}^\times$ and for some affine coordinate z of C_{31} with $z(C_{31} \cap E_1) = 0$. Note here that $(1/x, z) = \mathfrak{m}_{X, E_1 \cap C_{31}}$, as C_{31} and E_1 meet transversally at the point $E_1 \cap C_{31}$. Note also that $\psi \in \mathrm{Dec}(X, E_1)$ again by the definition of ψ. Thus

$$\psi_n = f^{-4n} \circ \psi \circ f^{4n} \in \mathrm{Dec}(X, E_1)$$

as well. Note that ψ is an involution on E_1 which satisfies

$$\psi(E_1 \cap C_{21}) = E_1 \cap C_{41}, \quad \psi(E_1 \cap C_{41}) = E_1 \cap C_{21}, \quad \psi(E_1 \cap C_{31}) = E_1 \cap C_{31},$$

that is, under the affine coordinate x of E_1,

$$\psi(1) = 0, \quad \psi(0) = 1, \quad \psi(\infty) = \infty.$$

Hence, under the coordinate x, we have

$$\psi|_{E_1}(x) = 1 - x.$$

Combining this with Equation (5.2), we readily obtain that

$$\psi_n|_{E_1}(x) = -x + \frac{1}{r^n}.$$

So, to complete the proof of (2), it suffices to show that r is not a root of 1.

To show this, we will use the theory of Mordell-Weil lattice due to Shioda [Sh90]. Recall that

$$\mathrm{MW}(\Phi_{|D_1|}) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \tag{5.3}$$

by [Og89, Theorem 2.1].

We compute the Shioda's height paring value $\langle C_{12}, C_{12}\rangle$ of the section C_{12} of $\Phi_{|D_1|}$ with respect to the zero section C_{21} by using the formula [Sh90, Theorem 8.6]. To use this, first note that the reducible fibers of $\Phi_{|D_1|}$ are

$$D_1 = C_{31} + E_1 + C_{41} + F_4 + C_{42} + E_2 + C_{32} + F_3,$$

$$D_1' := C_{13} + F_1 + C_{14} + E_4 + C_{24} + F_2 + C_{23} + E_3,$$

as $\Phi_{|D_1|}$ is of Type $\mathcal{I}_1$ in [Og89, Table 2].

The zero section C_{21} meets D_1 and D_1' at E_1 and F_2 respectively, while the section C_{12} meets D_1 and D_1' at E_2 and F_1 respectively. Thus, by [Sh90, Theorem 8.6, Table 8.16], we compute that

$$\langle C_{12}, C_{12}\rangle = 2\cdot 2 + 2\cdot 0 - 2\cdot \frac{4(8-4)}{8} = 0.$$

Thus C_{12} corresponds to a torsion element of $\mathrm{MW}(\Phi_{|D_1|})$ by [Sh90, Equation 8.10]. Therefore C_{12} corresponds to the unique 2-torsion element of $\mathrm{MW}(\Phi_{|D_1|})$.

Now assume to the contrary that r is a root of 1, say the Nth root of 1. Note that $(1/x, z) = \mathfrak{m}_{X,E_1\cap C_{31}}$ as C and E_1 meets transversally at the point $E_1 \cap C_{31}$. Then, by Equation (5.2), $f^{4nN} = \mathrm{id}_X$ around the point $C_{31}\cap E_1$ and hence on X. In particular $f \in \mathrm{MW}(\Phi_{|D_1|})$ would be a torsion element, but

$$f(C_{21}) = C_{43} \neq C_{21}, C_{12},$$

which is a contradiction to the observation above. This shows that r is not a root of 1 and therefore, $\psi_n \neq \psi_m$ if $n \neq m$. This proves (2).

Let us finally show (3). Observe that ϵ (*cf.* (4.6) and (4.7)) acts on the bases B_i of $\Phi_{|D_i|}$ ($i = 1$ and 2) as an involution, it follows that

$$\tilde{f} := \epsilon^{-1}\circ f^{-1}\circ \epsilon \circ f \in \mathrm{Aut}(X/B_1),$$

$$\tilde{\psi} := \epsilon^{-1}\circ \psi^{-1}\circ \epsilon \circ \psi \in \mathrm{Aut}(X/B_2),$$

and by the shape of $\tilde{f}$ and $\tilde{\psi}$, we have $\tilde{f}^*\omega_X = \omega_X$ and $\tilde{\psi}^*\omega_X = \omega_X$. Thus the action of $\tilde{f} \in \mathrm{Aut}(X/B_1)$ on the generic fiber $F_{1,\eta}$ of $\Phi_{|D_1|}$ (resp. the action of $\tilde{\psi} \in \mathrm{Aut}(X/B_2)$ on the generic fiber $F_{2,\eta}$ of $\Phi_{|D_2|}$) is a translation. Therefore $\tilde{f} \in \mathrm{MW}(\Phi_{|D_1|})$ (resp. $\tilde{\psi} \in \mathrm{MW}(\Phi_{|D_2|})$).

Since $\sigma, \sigma' \in \mathrm{MW}(\Phi_{|D_1|})$ defined by $\sigma(C_{21}) = C_{12}$ and $\sigma'(C_{34}) = C_{43}$ are both non-trivial torsion elements for the same reason as in the proof of (2), it follows that $\sigma = \sigma'$. As $\sigma, f \in \mathrm{MW}(\Phi_{|D_1|})$ and $\mathrm{MW}(\Phi_{|D_1|})$ is abelian,

$$f(C_{12}) = f\circ\sigma(C_{21}) = \sigma\circ f(C_{21}) = \sigma(C_{43}) = C_{34}$$

and hence $f^{-1}(C_{34}) = C_{12}$. Thus, under $\tilde{f} := \epsilon^{-1}\circ f^{-1}\circ\epsilon\circ f$, we have:

$$C_{21} \mapsto C_{43} \mapsto C_{34} \mapsto C_{12} \mapsto C_{21}.$$

Since $\tilde{f} \in \mathrm{MW}(\Phi_{|D_1|})$ and C_{21} is a section of $\Phi_{|D_1|}$, it follows that $\tilde{f} = \mathrm{id}_X$, that is,

$$f\circ\epsilon = \epsilon\circ f. \tag{5.4}$$

Similarly, C_{13} is a 2-torsion element of $\mathrm{MW}(\Phi_{|D_2|})$ with respect the zero section C_{31} for the same reason as in the proof of (2) and ψ is the inversion with respect to the zero section C_{31}, it follows that

$$\psi(C_{31}) = C_{31}, \quad \psi(C_{13}) = C_{13}.$$

Thus, under $\tilde{\psi} := \epsilon^{-1} \circ \psi^{-1} \circ \epsilon \circ \psi$, we have

$$C_{31} \mapsto C_{31} \mapsto C_{13} \mapsto C_{13} \mapsto C_{31}.$$

Since $\tilde{\psi} \in \mathrm{MW}(\Phi_{|D_2|})$ and C_{31} is a section of $\Phi_{|D_2|}$, it follows that $\tilde{\psi} = \mathrm{id}_X$, that is,

$$\psi \circ \epsilon = \epsilon \circ \psi. \tag{5.5}$$

By Equations (5.4) and (5.5), $\psi_n = f^{-4n} \circ \psi \circ f^{4n}$ also commutes with ϵ. Hence $\psi_n \in \mathrm{Ine}(X, C_{31})$ descends to an element of $\mathrm{Ine}(Z, D_{31})$. Under the inclusion in Proposition 5.1 (1), it follows that $\mathcal{S} \subset \mathrm{Ine}(Z, D_{31}) = \mathrm{Aut}(Y)$ as claimed.

This completes the proof of Proposition 5.2. □

Set

$$\mathrm{Cent}(\psi) := \{g \in \mathrm{Aut}(X) \,|\, g \circ \psi = \psi \circ g\}.$$

As we remarked, the next proposition will complete the proof of Theorem 4.8.

Proposition 5.3. (1) *There is a nef and big curve $\Sigma \subset X$ such that $g(\Sigma) = \Sigma$ for all $g \in \mathrm{Cent}(\psi)$. In particular,* $\mathrm{Cent}(\psi)$ *is a finite group.*

(2) *The set of conjugacy classes of $\mathcal{S}$ in* $\mathrm{Ine}(X, C_{31})$ *is an infinite set.*

Proof. Let us show (1). We will show that X^{ψ} contains a unique irreducible smooth curve, say Σ, of general type. Note then that Σ is nef and big, as

$$(\Sigma^2)_X = 2g(\Sigma) - 2 > 0.$$

So, once we show the existence and uniqueness of Σ, it follows that $g(\Sigma) = \Sigma$ if $g \in \mathrm{Cent}(\psi)$ and, as Σ is nef and big, $\mathrm{Cent}(\psi)$ is then finite for the same reason as in [DO19, Lemma 4.4], by the Hodge index theorem and the global Torelli theorem for K3 surfaces or by [Br18, Proposition 2.25].

Let us show the existence and uniqueness of Σ. Since ψ is of finite order, the action of ψ is locally algebraically linearizable at any point of X^{ψ}, say $P \in X^{\psi}$ (See e.g., [Ka84, Lemma 1.3]). Since ψ is of order two and satisfies $\psi^*\omega_X = -\omega_X$, one can find a system of local parameters $(x_P, y_P) = \mathfrak{m}_{X,P}$ at $P \in X^{\psi}$ such that

$$\psi(x_P, y_P) = (-x_P, y_P).$$

In particular, X^{ψ} is a smooth (not necessarily irreducible) curve unless it is empty.

In our case, the irreducible components of X^{ψ} in fibers are (necessarily) smooth rational curves. Any other irreducible component, say D, of X^{ψ} has to dominate the base B_2 and D meets any smooth fiber of $\Phi_{|D_2|}$ in (some of) the two-torsion points. As before, from [Og89, Theorem 1.2],

$$\mathrm{MW}(\Phi_{|D_2|}) = \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$$

and for the same reason as before, the unique non-trivial torsion element is given by $C_{31} \mapsto C_{13}$. We take C_{31} as the zero section. Then C_{13} is the unique two-torsion section and there is no other two-torsion section. Thus, the irreducible components of X^{ψ} which dominate B_2 are the zero section C_{31}, the unique two-torsion section C_{13} and a necessarily irreducible smooth curve, say Σ. Then Σ meets each smooth fiber of $\Phi_{|D_2|}$ at the remaining

2 two-torsion points (outside C_{31} and C_{13}) and 8 singular fibers of Kodaira type I_1 at the unique singular points. Thus the projection $\Sigma \to B_2$ is a finite double cover branched at least 8 points. It follows that the smooth curve Σ is of genus at least 3.

This completes the proof of the existence and uniqueness of Σ and thus completes the proof of (1).

Now we show (2). Recall that $\psi_n \neq \psi_m$ if $n \neq m$ by Proposition 5.2 (2). So, as in [DO19, Lemma 4.5], it suffices to show that for each fixed n, there are only finitely many m such that

$$\psi_m = h^{-1} \circ \psi_n \circ h \tag{5.6}$$

for some $h \in \mathrm{Ine}(X, C_{31})$.

Since $\psi_n = f^{-4n} \circ \psi \circ f^{4n}$, we have from Equation (5.6) that

$$f^{-4m} \circ \psi \circ f^{4m} = h^{-1} \circ f^{-4n} \psi \circ f^{4n} \circ h$$

and equivalently

$$f^{4n} \circ h \circ f^{-4m} \circ \psi = \psi \circ f^{4n} \circ h \circ f^{-4m}.$$

Thus

$$f^{4n} \circ h \circ f^{-4m} \in \mathrm{Cent}(\psi).$$

Recall from Equation (5.2) and the proof of Proposition 5.1 (2) that there are an affine coordinate z of C_{31} (with $z(C_{31} \cap F_1) = \infty$) and $r \in \mathbb{C}^\times$, which is not a root of 1, such that

$$f^{4n}|_{C_{31}}(z) = r^n \cdot z$$

for all n. Since $h|_{C_{31}}(z) = z$ as $h \in \mathrm{Ine}(X, C_{31})$, it follows that

$$f^{4n} \circ h \circ f^{-4m}|_{C_{31}}(z) = r^{n-m} \cdot z.$$

Since $\mathrm{Cent}(\psi)$ is a finite set, it follows that

$$\mathcal{R}_n := \{r^{n-m} \,|\, m \in \mathbb{Z}\}$$

has to be a finite set, too. As r is not a root of 1, it follows that integers $n - m$, and hence integers m for each fixed n, are at most finite. This completes the proof of (2). ☐

By Propositions 5.2 and 5.3 (2), the set $\mathcal{S}$ satisfies the assumption made in Proposition 5.1 (3). Hence Y has infinitely many mutually different real forms by Proposition 5.1 (3).

This completes the proof of Theorem 4.8 and hence the proof of Theorem 1.1 (2).

References

[Be16] Benzerga, M., : *Real structures on rational surfaces and automorphisms acting trivially on Picard groups*, Math. Z. **282** (2016), no. 3-4, 1127–1136. 2

[Be17] Benzerga, M., : *Finiteness of real structures on klt Calabi-Yau regular smooth pairs of dimension 2*, arXiv:1702.08808. 9

[BHPV04] Barth, W., Hulek, K., Peters, C., Van de Ven, A., : *Compact complex surfaces*. Second enlarged edition. Springer Verlag, Berlin-Heidelberg (2004). 1

[Bo21] Bot, A., : *A smooth complex rational affine surface with uncountably many real forms*, preprint (2021).`arXiv:2105.08044` 2

[BS64] Borel, A., Serre, J.-P., : *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. **39** (1964) 111–164. 6, 8, 11

[Br18] Brion, M., : *Notes on automorphism groups of projective varieties*, `https://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos_final.pdf`. 8, 10, 24

[Ca99] Cantat, S., : *Dynamique des automorphismes des surfaces projectives complexes*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 901–906. 12

[CF19] Cattaneo, A., Fu, L., : *Finiteness of Klein actions and real structures on compact hyperkähler manifolds*, Math. Ann. **375** (2019) 1783–1822. 2, 6, 9, 11

[Di12] Dinh, T.-C., : *Tits alternative for automorphism groups of compact Kähler manifolds*, Acta Math. Vietnam. **37** (2012) 513–529. 2

[DIK00] Degtyarev, A., Itenberg, I., Kharlamov, V., : *Real Enriques surfaces*. Lecture Notes in Mathematics, 1746. Springer-Verlag, Berlin, 2000. 2, 6, 8

[DLOZ22] T.-C. Dinh, H.-Y. Lin, K. Oguiso and D.-Q. Zhang, : *Zero entropy automorphisms of compact Kähler manifolds and dynamical filtrations*, Geom. Funct. Anal. **32** (2022), no. 3, 568–594. 9

[DN11] T.-C. Dinh, V.-A. Nguyên, : *Comparison of dynamical degrees for semi-conjugate meromorphic maps*, Comment. Math. Helv. **86** (2011), no. 4, 817–840. 12

[DO19] Dinh, T.-C., Oguiso, K., : *A surface with discrete and nonfinitely generated automorphism group*, Duke Math. J. **168** (2019) 941–966. 1, 3, 15, 16, 24, 25

[DOY21] Dinh, T.-C., Oguiso, K., Yu, X., : *Smooth complex projective rational surfaces with infinitely many real forms*, arXiv:2106.05687. 1, 3, 15, 19

[FGIKNV] Fantechi, B., Göttsche, L., Illusie, L., Kleiman, S.-L., Nitsure, N., Vistoli, A., : *Fundamental algebraic geometry. Grothendieck's FGA explained*, Mathematical Surveys and Monographs **123** American Mathematical Society, Providence, RI (2005). 4

[HS97] Hilton, P. J., Stammbach, U., : *A Course in Homological Algebra, second ed.*, Grad. Texts Math. 4, Springer-Verlag, New York, 1997. 7

[Ka84] Katsura, T., : *The unirationality of certain elliptic surfaces in characteristic p*, Tohoku Math. J. (2) **36** (1984) 217–231. 24

[Ka97] Kawamata, Y., : *On the cone of divisors of Calabi-Yau fiber spaces*, Internat. J. Math. **8** (1997) 665–687. 2, 14

[Kh02] Kharlamov, V., : *Topology, moduli and automorphisms of real algebraic surfaces*, Milan J. Math. **70** (2002) 25–37. 1

[Ki20] Kim, J.-H., :*On the finiteness of real structures of projective manifolds*, Bull. Korean Math. Soc., **57** (2020) 109–115. 2

[KO19] Keum J.H., Oguiso, K., : *A surface birational to an Enriques surface with non-finitely generated automorphism group*, arXiv:1904.04451 (unpublished). 3

[Ko63] Kodaira, K., : *On compact analytic surfaces. II*, Ann. of Math. **77** (1963), 563–626. 22

[Le18] Lesieutre, J., : *A projective variety with discrete non-finitely generated automorphism group*, Invent. Math. **212** (2018) 189–211. 1, 3, 6

[Lo14] Looijenga, E., : *Discrete automorphism groups of convex cones of finite type*, Compos. Math. **150** (2014) 1939–1962. 10, 14

[MO67] Matsumura, H., Oort, F., : *Representability of group functors, and automorphisms of algebraic schemes*, Invent. Math. **4** (1967), 1–5. 4

[MO15] Mumford, D., Oda, T., ; *Algebraic geometry II*, Texts and Readings in Mathematics **73** Hindustan Book Agency, New Delhi (2015). 9

[Mi17] Milne, J. S., : *Algebraic groups*. Cambridge Studies in Advanced Mathematics, 170. Cambridge University Press, Cambridge, 2017. 7

[Mu10] Mukai, S., : *Numerically trivial involutions of Kummer type of an Enriques surface*, Kyoto J. Math. **50** (2010) 889–902. 3, 16, 17, 18

[Na85] Namikawa, Y., : *Periods of Enriques surfaces*, Math. Ann. **270** (1985) 201–222. 2, 14

[Og89] Oguiso, K., : *On Jacobian fibrations on the Kummer surfaces of the product of non-isogenous elliptic curves*, J. Math. Soc. Japan **41** (1989) 651–680. 16, 21, 23, 24

[Se83] Segal, D., : *Polycyclic groups*, Cambridge Tracts in Mathematics **82** Cambridge University Press, Cambridge, 1983. 8, 13

[Se02] Serre, J.-P., : *Galois cohomology*, english ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, Translated from the French by Patrick Ion and revised by the author. 3, 5, 8, 9, 11

[Sh90] Shioda, T., : *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul. **39** (1990) 211–240. 23

[Si82] Silhol, R., : *Real abelian varieties and the theory of Comessatti*, Math. Z., **181** (1982) 345–364. 7, 11

[St85] Sterk, H., : *Finiteness results for algebraic K3 surfaces*, Math. Z. **189** (1985) 507–513. 2, 14

[Tr20] Truong, T.T., : *Relative dynamical degrees of correspondences over a field of arbitrary characteristic*, J. Reine Angew. Math. **758** (2020), 139–182. 12

[Ue75] Ueno, K., : *Classification theory of algebraic varieties and compact complex spaces.* Lecture Notes in Mathematics, **439**, Springer-Verlag, 1975. 12

[Wa21] Wang, L., : *On automorphisms and the cone conjecture for Enriques surfaces in odd characteristic*, Math. Res. Lett. **28** (2021) 1263–1281. 3

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