NEF CONES OF FIBER PRODUCTS AND AN APPLICATION TO THE CONE CONJECTURE

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Abstract. We prove a decomposition theorem for the nef cone of smooth fiber products over curves, subject to the necessary condition that their Néron–Severi space decomposes. We apply it to describe the nef cone of so-called Schoen varieties, which are the higher dimensional analogues of the Calabi–Yau threefolds constructed by Schoen. Schoen varieties give rise to Calabi–Yau pairs, and in each dimension at least three, there exist Schoen varieties with non-polyhedral nef cone. We prove the Kawamata–Morrison–Totaro Cone Conjecture for the nef cones of Schoen varieties, which generalizes the work by Grassi and Morrison.

1. Introduction

1.1. **Cone Conjecture.** To understand the geometry of a smooth projective variety X, studying the Mori cone of curves $\overline{\mathrm{NE}}(X)$ and its dual, the nef cone $\mathrm{Nef}(X)$, is central, especially from the viewpoint of the minimal model program (MMP).

An important part of the relationship between the Mori cone and the MMP is captured by the Cone Theorem and the Contraction Theorem. These theorems assert that the K_X-negative part of the Mori cone of a smooth projective variety X is rational polyhedral away from the K_X-trivial hyperplane, and the extremal rays of the K_X-negative part correspond to some morphisms from X, involved in the MMP. In particular, when X is a Fano variety (namely, $-K_X$ is ample), the cone $\mathrm{Nef}(X)$ is a rational polyhedral cone, and its extremal rays are generated by semiample classes. In general, however, it is difficult to describe the whole Mori cone, or dually the whole nef cone, even under the slightly weaker assumption that $-K_X$ is semiample. For instance, if X is the blowup of $\mathbb{P}^2$ at the base points of a general pencil of cubic curves in $\mathbb{P}^2$, then $-K_X$ is semiample but $\mathrm{Nef}(X)$ is not rational polyhedral.

When X is K-trivial, we expect nevertheless that some essential parts of the nef cone of X are rational polyhedral, up to the action of $\mathrm{Aut}(X)$. A precise statement, known as the Cone Conjecture, was first formulated by Morrison [26] and Kawamata [16]. It was later generalized by Totaro [37] to klt Calabi–Yau pairs (X, Δ) (see Section 2.2), thus including much more examples, already in dimension 2.

In this work, we study the Cone Conjecture for the nef cones of certain Calabi–Yau pairs. Let us recall the statement of the Cone Conjecture for nef cones formulated by Totaro in [37, Conjecture 2.1] (in the absolute situation). For a pair

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(X, Δ), we define

$$\mathrm{Aut}(X, \Delta) := \{ \, f \in \mathrm{Aut}(X) \mid f(\mathrm{supp}(\Delta)) = \mathrm{supp}(\Delta) \, \} \, .$$

We also define the *nef effective cone* $\mathrm{Nef}^e(X)$ as

$$\mathrm{Nef}^e(X) := \mathrm{Nef}(X) \cap \mathrm{Eff}(X),$$

where $\mathrm{Eff}(X)$ is the effective cone of X.

Conjecture 1.1 (Kawamata–Morrison–Totaro Cone Conjecture)**.** *Let (X, Δ) be a klt Calabi–Yau pair. There exists a rational polyhedral cone Π in $\mathrm{Nef}^e(X)$ which is a fundamental domain for the action of $\mathrm{Aut}(X, \Delta)$ on $\mathrm{Nef}^e(X)$, in the sense that*

$$\mathrm{Nef}^e(X) = \bigcup_{g \in \mathrm{Aut}(X, \Delta)} g^*\Pi,$$

and $\Pi^\circ \cap (g^*\Pi)^\circ = \varnothing$ unless $g^* = \mathrm{id}$.

An important prediction of the Cone Conjecture to the Minimal Model Program is that the number of $\mathrm{Aut}(X, \Delta)$-equivalence classes of faces of the nef effective cone $\mathrm{Nef}^e(X)$ corresponding to birational contractions or fiber space structures is finite (see e.g. [37, p.243]).

There is also a birational version of Conjecture 1.1, involving the action of pseudo-automorphisms on the movable cone (see e.g. [37, Conjecture 2.1.(2)]), which we will not study here.

Thanks to the fundamental work of Looijenga [24], it is natural and well-known to divide Conjecture 1.1 into two parts as follows (see Corollary 2.6). Let $\mathrm{Nef}^+(X)$ denote the convex hull of

$$\mathrm{Nef}(X) \cap N^1(X)_{\mathbb{Q}},$$

where $N^1(X)_{\mathbb{Q}}$ is the rational Néron–Severi space of X.

Conjecture 1.2. *Let (X, Δ) be a klt Calabi–Yau pair.*

1. *There exists a rational polyhedral cone in $\mathrm{Nef}^+(X)$ which is a fundamental domain for the action of $\mathrm{Aut}(X, \Delta)$ on $\mathrm{Nef}^+(X)$.*
2. *We have*
$$\mathrm{Nef}^+(X) = \mathrm{Nef}^e(X).$$

Let us note that in Conjecture 1.2.(2), the inclusion $\mathrm{Nef}^e(X) \subset \mathrm{Nef}^+(X)$ is known in general (see [23, Lemma 5.1]), while the reverse is still wide open even in dimension 3.

1.2. **Nef cones of fiber products.** The starting point of this work is a decomposition theorem for the nef cone of a fiber product over a curve.

It begins with the following general question. Let W_1 and W_2 be smooth projective varieties and let $\phi_1 : W_1 \to B$ and $\phi_2 : W_2 \to B$ be surjective morphisms with connected fibers over a smooth base B. Assume that the fiber product $W := W_1 \times_B W_2$ is smooth.

Question 1.3. *Let $p_i : W \to W_i$ be the projection. When do we have*

$$p_1^*\mathrm{Nef}(W_1) + p_2^*\mathrm{Nef}(W_2) = \mathrm{Nef}(W)? \tag{1.1}$$

As the nef cone $\mathrm{Nef}(X)$ of a smooth projective variety X spans the whole space $N^1(X)_{\mathbb{R}}$ of numerical classes of $\mathbb{R}$-divisors, such a decomposition exists only if

$$p_1^*N^1(W_1)_{\mathbb{R}} + p_2^*N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}}. \tag{1.2}$$

We may then ask which fiber products satisfying the decomposition (1.2) also have the decomposition (1.1).

When B is a point, it follows from the projection formula that (1.2) implies (1.1). When B is $\mathbb{P}^1$ and the varieties W_i are certain rational elliptic surfaces, the decomposition (1.1) was proven in [12, Proposition 3.1]. We show that the implication (1.2) $\Rightarrow$ (1.1) continues to hold for an arbitrary fiber product over a curve.

Theorem 1.4. *For $i = 1, 2$, let $\phi_i : W_i \to B$ be a surjective morphism with connected fibers from a smooth projective variety to a smooth projective curve B. Assume that*

(1) *the variety $W = W_1 \times_B W_2$ is smooth;*
(2) *we have*
$$p_1^*N^1(W_1)_{\mathbb{R}} + p_2^*N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}}.$$

Then
$$p_1^*\mathrm{Nef}(W_1) + p_2^*\mathrm{Nef}(W_2) = \mathrm{Nef}(W).$$
As a consequence, we also have $p_1^*\mathrm{Amp}(W_1) + p_2^*\mathrm{Amp}(W_2) = \mathrm{Amp}(W)$.

In Examples 3.5 and 3.6, we construct explicit examples of fiber products over bases of dimension at least 2 that fail the implication (1.2) $\Rightarrow$ (1.1).

Theorem 1.4 has the following corollary.

Corollary 1.5. *In the setting of Theorem 1.4, $E \in \mathrm{Nef}(W_i)$ is extremal if and only if $p_i^*E \in \mathrm{Nef}(W)$ is extremal. As a consequence, $\mathrm{Nef}(W)$ is rational polyhedral if and only if both $\mathrm{Nef}(W_1)$ and $\mathrm{Nef}(W_2)$ are rational polyhedral.*

It provides a way of constructing fiber products (over curves) whose nef cones are not rational polyhedral.

1.3. **Cone Conjecture for Schoen varieties.** Among the strict Calabi–Yau manifolds (see Definition 2.2) whose nef cones are known to be non rational polyhedral, to our knowledge, the Cone Conjecture is known so far for only two special cases. One of them is the desingularized Horrocks–Mumford quintics, studied by Borcea in [4] (see also [11]); the other is the fiber product of two general rational elliptic surfaces with sections over $\mathbb{P}^1$ constructed by Schoen in [34], and investigated by Namikawa and Grassi–Morrison [28, 12]. Both examples are of dimension three.

The main goal of this paper is to prove the Cone Conjecture for higher dimensional generalizations of Schoen's Calabi–Yau threefolds. These are a certain type of fiber products over $\mathbb{P}^1$ as in Theorem 1.4, which we call *Schoen varieties.*

Let us first summarize the construction of Schoen varieties; we refer to Subsections 4.1 and 4.2 for more details. Let Z_1 and Z_2 be Fano manifolds of dimension at least two. For $i = 1, 2$, let D_i be an ample and globally generated divisor on Z_i such that $-(K_{Z_i} + D_i)$ is globally generated. Let $W_i \subset \mathbb{P}^1 \times Z_i$ be a general member in the linear system $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_i}(D_i)|$. We have a fibration $\phi_i : W_i \to \mathbb{P}^1$. Consider the fiber product over $\mathbb{P}^1$:

$$\phi : X := W_1 \times_{\mathbb{P}^1} W_2 \to \mathbb{P}^1.$$

Such a smooth projective variety X is called a *Schoen variety* under an extra assumption (see the second paragraph of Subsection 4.2). It follows from the construction that $-K_X$ is globally generated, so X has many effective $\mathbb{Q}$-divisors Δ which are $\mathbb{Q}$-linearly equivalent to $-K_X$. Any such Δ forms a Calabi–Yau pair (X, Δ), that we call a *Schoen pair*.

We prove the following result.

Theorem 1.6. *Let (X, Δ) be a Schoen pair. Then there exists a rational polyhedral fundamental domain for the action of* $\mathrm{Aut}(X, \Delta)$ *on* $\mathrm{Nef}^e(X) = \mathrm{Nef}^+(X) = \mathrm{Nef}(X)$.

Note that, by Corollary 1.5, the cone $\mathrm{Nef}(X)$ is not rational polyhedral as long as one of $\mathrm{Nef}(W_1)$ and $\mathrm{Nef}(W_2)$ is not. This is the case when there exists i such that $Z_i = \mathbb{P}^2$ and $D_i = -K_{Z_i}$ (in which case W_i is a rational elliptic surface). In particular, our construction provides the first series of strict Calabi–Yau manifolds, and also Calabi–Yau pairs in arbitrary dimension, for which the Cone Conjecture holds and whose nef cones are not rational polyhedral (see Example 5.6). We also note that X is a complete intersection of two hypersurfaces, which are nef but not ample, in the Fano manifold $\mathbb{P}^1 \times Z_1 \times Z_2$. That the cone $\mathrm{Nef}(X)$ may admit infinitely many faces resonates with Theorem 1.7 below.

As well-known corollaries of the Cone Conjecture, we also obtain the finite presentation of the group of components $\pi_0\mathrm{Aut}(X)$ and the finiteness of real structures on X up to equivalence; see Corollary 5.7.

1.4. **Relation to other work.**

1.4.1. *Cone Conjecture.* We refer to [22] and the references therein for a survey of the Cone Conjecture without the boundary (namely with $\Delta = 0$). As for the Cone Conjecture for Calabi–Yau pairs, the 2-dimensional case was proven by Totaro [37]. Kopper proved the Cone Conjecture for Calabi–Yau pairs arising from Hilbert schemes of points on certain rational elliptic surfaces in [20]. In this case, the nef cone may admit infinitely many faces, while the dimensions of these varieties are always even. See also [10, 23] for some recent results.

1.4.2. *Cone Conjectures for varieties with rational polyhedral nef cones.* One way of proving the Cone Conjecture for a smooth projective variety X is to show that $\mathrm{Nef}(X)$ is a rational polyhedral cone and that $\mathrm{Nef}(X) = \mathrm{Nef}^e(X)$ (see e.g. [21, Proposition 6.5]). This is the case when X is a smooth anticanonical hypersurface in a Fano manifold Y with $\dim Y \geq 4$ [1, Proposition 3.5] (based on [3] due to Kollár).

Theorem 1.7. *Let D be a smooth anticanonical hypersurface in a Fano manifold Y of dimension at least* 4. *Then the natural restriction map* $\mathrm{Nef}(D) \to \mathrm{Nef}(Y)$ *is an isomorphism. In particular,* $\mathrm{Nef}(D)$ *is a rational polyhedral cone which is generated by classes of semi-ample divisors.*

See [30, 7, 8], due to Coskun and Prendergast-Smith, for other examples of varieties X whose nef cones are rational polyhedral with $\mathrm{Nef}(X) = \mathrm{Nef}^e(X)$.

1.4.3. *Fiber product constructions.* Constructing Calabi–Yau threefolds as fiber products of two general rational elliptic surfaces with sections over $\mathbb{P}^1$ was first considered and investigated by Schoen [34]. It recently came back to light as Suzuki considered a certain higher-dimensional generalization of Schoen's construction and studied its arithmetic properties in [35]. Similar ideas are also involved in Sano's constructions of non-Kähler Calabi–Yau manifolds with arbitrarily large second Betti number in [32].

1.4.4. *Cone conjecture for movable cones.* We have already mentioned the Cone Conjecture for movable cones [37, Conjecture 2.1.(2)]. In particular, it predicts that a Calabi–Yau variety has only finitely many minimal models up to isomorphisms; see [27, 16, 37, 22] for more details. This conjecture was verified for some cases. In [6], Cantat and Oguiso produced the first series of strict Calabi–Yau manifolds in arbitrary dimension whose movable cones are not rational polyhedral and for which the Cone Conjecture for movable cones holds. We refer to [38] and references therein for more examples.

In [28] Namikawa showed that for a certain Schoen threefold (which is a Calabi–Yau threefold), the number of its minimal models up to isomorphism is finite. It would be interesting to investigate a similar problem in arbitrary dimension.

1.5. **Structure of the paper.** Section 2 is devoted to some preliminary and fundamental results. We will prove Theorem 1.4 in Section 3. After constructing Schoen varieties and Schoen pairs in Section 4, we will prove Theorem 1.6 in Section 5.

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2. Preliminaries

We work over the field $\mathbb{C}$ of complex numbers throughout this paper. For basics of birational geometry, we refer to [19].

2.1. **Notation.** We start with some notations. Let X be a normal projective variety. We write $N^1(X)$ for the free abelian group generated by the classes of Cartier divisors modulo numerical equivalence.

Inside the vector space $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$, we denote by $\mathrm{Nef}(X)$ the *nef cone*, i.e., the closure of the ample cone $\mathrm{Amp}(X)$, and by $\mathrm{Eff}(X)$ the *effective cone*. The *nef effective cone* $\mathrm{Nef}^e(X)$ is defined as

$$\mathrm{Nef}^e(X) := \mathrm{Nef}(X) \cap \mathrm{Eff}(X).$$

Let $\mathrm{Nef}^+(X)$ denote the convex hull of

$$\mathrm{Nef}(X) \cap N^1(X)_{\mathbb{Q}},$$

where $N^1(X)_{\mathbb{Q}} := N^1(X) \otimes \mathbb{Q}$. We denote by $N_1(X)$ the group of 1-cycles modulo numerical equivalence. The intersection product defines a perfect pairing between two vector spaces $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$. Under this pairing, the nef cone $\mathrm{Nef}(X)$ is dual to the *Mori cone* $\overline{\mathrm{NE}}(X)$, which is the closure of the convex cone of effective 1-cycles in $N_1(X)_{\mathbb{R}}$.

The group of automorphisms of X is denoted by $\mathrm{Aut}(X)$, and acts on $N^1(X)$ by pullback. This action

$$\rho : \mathrm{Aut}(X) \to \mathrm{GL}(N^1(X))$$

linearly extends to $N^1(X)_{\mathbb{R}}$, preserving the cones $\mathrm{Nef}^e(X)$ and $\mathrm{Nef}^+(X)$. The connected component of the identity in $\mathrm{Aut}(X)$ is a normal subgroup $\mathrm{Aut}^0(X)$, which acts trivially on $N^1(X)$ [5, Lemma 2.8].

2.2. **Klt Calabi–Yau pairs.** A *pair* is the data (X, Δ) of a normal projective variety X together with an effective $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

Definition 2.1. Following [37], we say that a pair (X, Δ) is *Calabi–Yau* if X is $\mathbb{Q}$-factorial and $K_X + \Delta$ is numerically trivial.

Let us briefly recall the definition of a klt pair. For any pair (X, Δ) and any birational morphism $\mu : \widetilde{X} \to X$, there exists a unique $\mathbb{R}$-divisor $\widetilde{\Delta}$ on $\widetilde{X}$ such that

$$K_{\widetilde{X}} + \widetilde{\Delta} = \mu^*(K_X + \Delta) \quad \text{and} \quad \mu_* \widetilde{\Delta} = \Delta.$$

A pair (X, Δ) is called *klt* (short for *Kawamata log terminal*), if for any birational morphism $\mu : (\widetilde{X}, \widetilde{\Delta}) \to (X, \Delta)$ as above, each irreducible component of $\widetilde{\Delta}$ has coefficient less than one. It suffices to check this property for one resolution $\widetilde{X}$ of X where $\widetilde{\Delta}$ has simple normal crossings.

Definition 2.2. Let X be a smooth projective variety. We say that X is a *Calabi–Yau manifold* if the canonical line bundle K_X is trivial and $h^i(X, \mathcal{O}_X) = 0$ for any $0 < i < \dim X$. If in addition, X is simply-connected, it is called a *strict Calabi–Yau manifold*.

2.3. **Looijenga's result.** We will use the following crucial result in this paper.

Proposition 2.3. *Let X be a normal projective variety and let $H \leq \mathrm{Aut}(X)$ be a subgroup. Assume that there is a rational polyhedral cone $\Pi \subset \mathrm{Nef}^+(X)$ such that $\mathrm{Amp}(X) \subset H \cdot \Pi$. Then*

1. *$H \cdot \Pi = \mathrm{Nef}^+(X)$, and the H-action on $\mathrm{Nef}^+(X)$ has a rational polyhedral fundamental domain.*
2. *The group $\rho(H)$ is finitely presented.*

Such a result and related statements are well-known to experts. We include a proof for the sake of completeness. It relies on the fundamental results due to Looijenga [24, Proposition 4.1, Application 4.14, and Corollary 4.15], which we extract and formulate here as Lemma 2.4. Recall that a cone $C \subset N_{\mathbb{R}}$ in a finite dimensional $\mathbb{R}$-vector space $N_{\mathbb{R}}$ is called *strict* if its closure $\overline{C} \subset N_{\mathbb{R}}$ contains no line.

Lemma 2.4. *Let N be a finitely generated free $\mathbb{Z}$-module, and let C be a strict convex open cone in the $\mathbb{R}$-vector space $N_{\mathbb{R}} := N \otimes \mathbb{R}$. Let C^+ be the convex hull of $\overline{C} \cap N_{\mathbb{Q}}$. Let $(C^\vee)^\circ \subset N_{\mathbb{R}}^\vee$ be the interior of the dual cone of C. Let Γ be a subgroup of $\mathrm{GL}(N)$ which preserves the cone C. Suppose that:*

- *there is a rational polyhedral cone $\Pi \subset C^+$ such that $C \subset \Gamma \cdot \Pi$;*
- *there exists an element $\xi \in (C^\vee)^\circ \cap N_{\mathbb{Q}}^\vee$ whose stabilizer in Γ (with respect to the dual action $\Gamma \circlearrowleft N_{\mathbb{Q}}^\vee$) is trivial.*

Then $\Gamma \cdot \Pi = C^+$ and the Γ-action on C^+ has a rational polyhedral fundamental domain. Moreover, the group Γ is finitely presented.

Lemma 2.5. *There exists an ample class $\eta \in N^1(X)$ such that for every $g \in \mathrm{Aut}(X)$, we have $g^*\eta = \eta$ if and only if g^* acts trivially on $N^1(X)$.*

Proof. Our proof is inspired by the argument of [21, Proposition 6.5].

Let $\Gamma := \rho(\mathrm{Aut}(X)) \subset \mathrm{GL}(N^1(X))$ and for every $\theta \in N^1(X)$, let Γ_θ be the stabilizer of θ of the Γ-action on $N^1(X)$. It suffices to find an ample class $\eta \in N^1(X)$ such that Γ_η is trivial.

By Fujiki–Liebermann's theorem [5, Theorem 2.10], the action of Γ on $C \cap N$ has finite stabilizers. Take an element $\theta \in C \cap N_{\mathbb{Q}}$ such that the order of the stabilizer Γ_θ is minimal. Since the Γ-action on $N_{\mathbb{R}}$ preserves N, we can find an open neighborhood $U \subset C$ of θ, such that $\gamma U \cap U = \varnothing$ for every $\gamma \notin \Gamma_\theta$. Thus, for every $\theta' \in U \cap N_{\mathbb{Q}}$, we have $\Gamma_{\theta'} \subset \Gamma_\theta$, which then implies $\Gamma_{\theta'} = \Gamma_\theta$ by the minimality of Γ_θ. It follows that every $\gamma \in \Gamma_\theta$ satisfies $\gamma|_{U \cap N_{\mathbb{Q}}} = \mathrm{id}_{U \cap N_{\mathbb{Q}}}$, and since γ acts linearly, necessarily $\gamma = \mathrm{id}$. This proves that $\theta \in C \cap N_{\mathbb{Q}}$ has trivial stabilizer, and so do some positive multiple $\eta \in C \cap N$ of θ. $\square$

Proof of Proposition 2.3. In Lemma 2.4, now set $N = N^1(X)$, $C = \mathrm{Amp}(X)$, and $\Gamma = \rho(H)$. By Lemma 2.4, it suffices to construct an element $\xi \in (C^\vee)^\circ \cap N_{\mathbb{Q}}^\vee$ with trivial stabilizer with respect to the induced Γ-action.

For every $\theta \in N$, let Γ_θ be the stabilizer of θ of the Γ-action on N. Choose any $\xi \in (C^\vee)^\circ$. Since $\xi(x) > 0$ for any $x \in \overline{C} \backslash \{0\}$, the subset

$$\left\{ x \in \overline{C} \;\middle|\; \xi(x) \leq r \right\} \subset V$$

is bounded, so compact for any $r > 0$. Since $C \cap N$ is discrete, among

$$\Sigma := \{ \eta \in C \cap N \mid \Gamma_\eta \text{ is trivial} \},$$

which is nonempty by Lemma 2.5, there are only finitely many $\eta \in \Sigma$ minimizing $\xi|_\Sigma$. Again, as $C \cap N$ is discrete, we can perturb ξ and obtain $\xi_0 \in (C^\vee)^\circ \cap N_{\mathbb{Q}}^\vee$ such that there is a unique $\eta \in \Sigma$ minimizing $\xi_0|_\Sigma$. As Σ is Γ-invariant, we have

$$(\gamma\xi_0)(\eta) = \xi_0(\gamma\eta) > \xi_0(\eta)$$

for every $\gamma \notin \Gamma_\eta$. Since $\eta \in \Sigma$, the stabilizer Γ_η is trivial, so the stabilizer of ξ_0 in Γ is trivial as well. $\square$

Corollary 2.6. *Conjecture 1.1 and Conjecture 1.2 are equivalent.*

Proof. It is clear that Conjecture 1.2 implies Conjecture 1.1. Now assume Conjecture 1.1. Let Π be a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(X, \Delta)$ on $\mathrm{Nef}^e(X)$. Then $\Pi \subset \mathrm{Nef}^+(X)$. By Proposition 2.3.(1),

$$\mathrm{Nef}^e(X) = \mathrm{Aut}(X, \Delta) \cdot \Pi = \mathrm{Nef}^+(X).$$

So Conjecture 1.2 holds. $\square$

3. The nef cone of a fiber product over a curve

We now prove Theorem 1.4 about the decomposition of the nef cone.

For $i = 1, 2$, recall that $\phi_i : W_i \to B$ is a surjective morphism with connected fibers from a smooth projective variety to a smooth projective curve B. We consider the fiber product

$$\begin{tikzcd} & W = W_1 \times_B W_2 \arrow[dl, "p_1"'] \arrow[dd, "p"] \arrow[dr, "p_2"] & \\ W_1 \arrow[dr, "\phi_1"'] & & W_2 \arrow[dl, "\phi_2"] \\ & B & \end{tikzcd}.$$

and work under the following assumptions:

(1) the variety $W = W_1 \times_B W_2$ is smooth;
(2) for every $D \in N^1(W)_{\mathbb{R}}$, there exist $D_1 \in N^1(W_1)_{\mathbb{R}}$ and $D_2 \in N^1(W_2)_{\mathbb{R}}$ such that
$$D = p_1^* D_1 + p_2^* D_2.$$

Proof of Theorem 1.4. Let $D \in \mathrm{Nef}(W)$ and let
$$D = p_1^* D_1 + p_2^* D_2 \in N^1(W)_{\mathbb{R}}$$
be a decomposition as in (2).

First, note the following simple fact.

Lemma 3.1. *Let $C_i \subset W_i$ be an irreducible curve. If $\phi_i(C_i)$ is a point, then $D_i \cdot C_i \geq 0$.*

Proof. We may only consider $i = 1$. Choose any point $s \in \phi_2^{-1}(\phi_1(C_1))$ and let $\widetilde{C_1} := C_1 \times_B \{s\} \subset W$. We have
$$0 \leq D \cdot \widetilde{C_1} = (p_1^* D_1 + p_2^* D_2) \cdot \widetilde{C_1} = D_1 \cdot p_{1*}\widetilde{C_1} + D_2 \cdot p_{2*}\widetilde{C_1} = D_1 \cdot C_1.$$
This proves the assertion. $\square$

We use this fact to prove the following two lemmas.

Lemma 3.2. *Either D_1 or D_2 is nef.*

Proof. Assume by contradiction that both D_1 and D_2 are not nef. Then for each i, there exists an irreducible curve $C_i \subset W_i$ such that $D_i \cdot C_i < 0$. By Lemma 3.1, we have $\phi_i(C_i) = B$, so $\widetilde{C} := C_1 \times_B C_2$ is a curve. Let $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$ be such that $p_{i*}\widetilde{C} = \beta_i C_i$. Then
$$0 > \beta_1 D_1 \cdot C_1 + \beta_2 D_2 \cdot C_2 = (p_1^* D_1 + p_2^* D_2) \cdot \widetilde{C} = D \cdot \widetilde{C} \geq 0,$$
which is a contradiction. $\square$

Now we fix a point $b \in B$.

Lemma 3.3. *For $i = 1, 2$, there exists $N_i \in \mathbb{R}$ such that the divisor $D_i + n\phi_i^* b$ is nef if $n \geq N_i$.*

Proof. We may only consider the case when $i = 2$.
Let $C_1 \subset W_1$ be an irreducible curve such that $\phi_1(C_1) = B$. Define
$$D_1' := D_1 - N_2 \phi_1^* b \quad \text{and} \quad D_2' := D_2 + N_2 \phi_2^* b$$
where
$$N_2 := \frac{D_1 \cdot C_1}{\deg(C_1 \xrightarrow{\phi_1} B)}.$$

By construction, we have

$$D_1' \cdot C_1 = 0 \text{ and } D = p_1^* D_1' + p_2^* D_2'.$$

Let us show that D_2' is nef. Let $C_2 \subset W_2$ be an irreducible curve. If $\phi_2(C_2)$ is a point, then $D_2' \cdot C_2 \geq 0$ by Lemma 3.1. Suppose now that $\phi_2(C_2) = B$. Set $\widetilde{C} := C_1 \times_B C_2$ and define $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$ such that $p_{i*}\widetilde{C} = \beta_i C_i$. We have

$$\begin{aligned}\beta_2 D_2' \cdot C_2 &= \beta_1 D_1' \cdot C_1 + \beta_2 D_2' \cdot C_2 \\ &= (p_1^* D_1' + p_2^* D_2') \cdot \widetilde{C} \\ &= D \cdot \widetilde{C} \geq 0.\end{aligned}$$

This shows that D_2' is nef. Hence, for $n \geq N_2$, the divisor

$$D_2 + n\phi_2^* b = D_2' + (n - N_2)\phi_2^* b$$

is nef. □

We can now resume the proof of Theorem 1.4. For any $t \in \mathbb{R}$, let

$$D_1(t) := D_1 - t\phi_1^* b \quad \text{and} \quad D_2(t) := D_2 + t\phi_2^* b.$$

By Lemma 3.3, there exist

$$I_1 =]-\infty, -N_{1,\min}] \quad \text{and} \quad I_2 = [N_{2,\min}, +\infty[$$

such that $D_i(t)$ is nef if and only if $t \in I_i$. Since we have

$$D = p_1^* D_1(t) + p_2^* D_2(t),$$

Lemma 3.2 shows that either $D_1(t)$ or $D_2(t)$ is nef, namely, $I_1 \cup I_2 = \mathbb{R}$. Thus, $I_1 \cap I_2$ is non-empty. As both $D_1(t)$ and $D_2(t)$ are nef whenever $t \in I_1 \cap I_2$, this gives a desired decomposition.

The last statement about the decomposition of the ample cone follows from [31, Corollary 6.6.2]. □

Remark 3.4. In the setup of Theorem 1.4, we also have the decomposition of the relative nef cone

$$\mathrm{Nef}(W/B) = p_1^* \mathrm{Nef}(W_1/B) + p_2^* \mathrm{Nef}(W_2/B)$$

by the projection formula – this is exactly Lemma 3.1.

Now we prove Corollary 1.5.

Proof of Corollary 1.5. We may assume $i = 1$.

First assume that $p_1^* E$ is extremal. Let $E = F + F'$ be a decomposition with $F, F' \in \mathrm{Nef}(W_1)$. Then $p_1^* E = p_1^* F + p_1^* F'$ with $p_1^* F, p_1^* F' \in \mathrm{Nef}(W)$, and thus, $p_1^* F$ and $p_1^* F'$ are proportional by assumption. Since $p_1^* : N^1(W_1)_{\mathbb{R}} \to N^1(W)_{\mathbb{R}}$ is injective, F and F' are proportional as well. This shows that E is extremal.

Next assume that $E \in \mathrm{Nef}(W_1)$ is extremal. Let $p_1^* E = D + D'$ be a decomposition with $D, D' \in \mathrm{Nef}(W)$. Up to adding terms to D', we can assume that D is extremal. By Theorem 1.4, we can write

$$D = p_1^* D_1 + p_2^* D_2, \quad \text{and} \quad D' = p_1^* D_1' + p_2^* D_2'$$

with $D_i, D_i' \in \mathrm{Nef}(W_i)$. As D is extremal, the divisors D, $p_1^* D_1$ and $p_2^* D_2$ are proportional. Moreover $p_1^*(E - D_1 - D_1') = p_2^*(D_2 + D_2') \in \mathrm{Nef}(W)$. Hence, by the projection formula, $E - D_1 - D_1'$ is nef. But E is extremal in the cone $\mathrm{Nef}(W_1)$, so

E, D_1, and D_1' are proportional. In particular, $p_1^*E, p_1^*D_1, p_1^*D_1'$ and $p_2^*D_2$ are all proportional, which concludes the proof. □

Now we construct fiber products showing that Theorem 1.4 fails in general when $\dim B \geq 2$. First we construct such examples of fiber products over a surface.

Example 3.5. Take $S := \mathbb{P}^2$, and take four points P_1, P_2, P_3, P_4 in S so that no three of them lie on a line. Let ℓ_1 be the line through P_1, P_2, and let ℓ_2 be the line through P_3, P_4. Take

$$W_1 := \mathrm{Bl}_{P_1,P_2}(S) \quad \text{and} \quad W_2 := \mathrm{Bl}_{P_3,P_4}(S).$$

As the blown-up points are distinct, $W := W_1 \times_S W_2$ is isomorphic to $\mathrm{Bl}_{P_1,P_2,P_3,P_4}(S)$, which is smooth. Moreover, the decomposition of the Picard group

$$\mathrm{Pic}(W) = p_1^*\mathrm{Pic}(W_1) + p_2^*\mathrm{Pic}(W_2)$$

clearly holds.

Denote by ℓ_1' and ℓ_2' the strict transforms of ℓ_1 and ℓ_2 in W_1 and W_2 respectively. Then ℓ_i' is an effective non-nef divisor on W_i as $(\ell_i')^2 = -1$. Let

$$D := p_1^*\ell_1' + p_2^*\ell_2'.$$

We show that D is nef; this also shows that Lemma 3.2 fails when $\dim B \geq 2$. As D is effective, it is enough to check that its intersections with its components are all non-negative. By symmetry, it is enough to compute

$$D \cdot p_1^*\ell_1' = (\ell_1')^2 + \ell_2' \cdot \phi_2^*\ell_1 = -1 + 1 = 0.$$

So D is nef, and has vanishing intersection with the curves $p_1^*\ell_1'$ and $p_2^*\ell_2'$.

Now assume by contradiction that D has another decomposition $D = p_1^*D_1 + p_2^*D_2$ with $D_i \in \mathrm{Nef}(W_i)$. Then we have

$$p_1^*(\ell_1' - D_1) = p_2^*(D_2 - \ell_2').$$

As $p_1^*N^1(W_1)_{\mathbb{R}} \cap p_2^*N^1(W_2)_{\mathbb{R}}$ clearly has dimension one, it equals $\mathbb{R}[p^*\ell]$, where $p : W \to S$ is the blow up, and ℓ is a line passing through none of P_1, P_2, P_3, P_4 in S. It follows that

$$p_1^*(\ell_1' - D_1) = p_2^*(D_2 - \ell_2') = cp^*\ell$$

for some $c \in \mathbb{R}$.

Since

$$p_1^*D_1 \cdot p_i^*\ell_i' + p_2^*D_2 \cdot p_i^*\ell_i' = D \cdot p_i^*\ell_i' = 0,$$

and both $p_1^*D_1$ and $p_2^*D_2$ are nef, we have $p_i^*D_i \cdot p_i^*\ell_i' = 0$. Thus

$$-1 = p_1^*\ell_1' \cdot p_1^*(\ell_1' - D_1) = cp_1^*\ell_1' \cdot p^*\ell = c$$

and similarly,

$$1 = p_2^*\ell_2' \cdot p_2^*(D_2 - \ell_2') = cp_2^*\ell_2' \cdot p^*\ell = c,$$

which is a contradiction.

Example 3.6. As for examples of fiber products over a base of higher dimension, we continue with the notations of Example 3.5, and introduce

$$W \times T = (W_1 \times T) \times_{(S \times T)} (W_2 \times T)$$

where T is an arbitrary smooth projective variety. As in Example 3.5, W, W_1 and W_2 are rationally connected, hence have trivial irregularity, so that

$$N^1(Z \times T)_{\mathbb{R}} = p_Z^*N^1(Z)_{\mathbb{R}} \oplus p_T^*N^1(T)_{\mathbb{R}},$$

for $Z = W, W_1$ or W_2. This implies that

$$N^1(W \times T)_{\mathbb{R}} = (p_1 \times \mathrm{id}_T)^* N^1(W_1 \times T)_{\mathbb{R}} + (p_2 \times \mathrm{id}_T)^* N^1(W_2 \times T)_{\mathbb{R}}.$$

Note that by the projection formula,

$$\mathrm{Nef}(Z \times T) = p_Z^* \mathrm{Nef}(Z) \oplus p_T^* \mathrm{Nef}(T),$$

for $Z = W, W_1$ or W_2. So, if we assume by contradiction that

$$\mathrm{Nef}(W \times T) = (p_1 \times \mathrm{id}_T)^* \mathrm{Nef}(W_1 \times T) + (p_2 \times \mathrm{id}_T)^* \mathrm{Nef}(W_2 \times T),$$

we get $\mathrm{Nef}(W) = p_1^* \mathrm{Nef}(W_1) + p_2^* \mathrm{Nef}(W_2)$, which contradicts Example 3.5.

For a morphism $\pi : X \to Y$, we define

$$\mathrm{Aut}(X/Y) = \{g \in \mathrm{Aut}(X) \mid \pi \circ g = \pi\}.$$

We have the following corollary of Theorem 1.4.

Corollary 3.7. *For $i = 1, 2$, let $\phi_i : W_i \to B$ be a surjective morphism with connected fibers from a smooth projective variety to a smooth projective curve B. Assume that*

(1) *the variety $W = W_1 \times_B W_2$ is smooth;*
(2) *it holds*
$$p_1^* N^1(W_1)_{\mathbb{R}} + p_2^* N^1(W_2)_{\mathbb{R}} = N^1(W)_{\mathbb{R}},$$
where p_i denotes the projection from W onto W_i.

For $i = 1, 2$, let $H_i \leq \mathrm{Aut}(W_i/B)$ be a subgroup. Let $H \leq \mathrm{Aut}(W)$ be a subgroup containing $H_1 \times H_2$. Assume that there exists a rational polyhedral cone $\Pi_i \subset \mathrm{Nef}^+(W_i)$ such that $H_i \cdot \Pi_i \supset \mathrm{Amp}(W_i)$. Then $\mathrm{Nef}^+(W)$ admits a rational polyhedral fundamental domain for the H-action.

Proof. Let Π be the convex hull of $p_1^* \Pi_1 + p_2^* \Pi_2$. Then Π is a rational polyhedral cone contained in $\mathrm{Nef}^+(W)$. Moreover,

$$\mathrm{Amp}(W) \subset (H_1 \times H_2) \cdot \Pi \subset H \cdot \Pi$$

as $p_1^* \mathrm{Amp}(W_1) + p_2^* \mathrm{Amp}(W_2) = \mathrm{Amp}(W)$ by Theorem 1.4. The existence of a rational polyhedral fundamental domain then follows from Proposition 2.3.(1). □

4. Construction of Schoen varieties

Schoen varieties will be constructed as a fiber product of two fibrations over $\mathbb{P}^1$. Let us first construct these fibrations.

4.1. **The factor W with a fibration over $\mathbb{P}^1$.** The construction relies on a pencil of ample hypersurfaces in a Fano manifold.

Let Z be a Fano manifold of dimension at least 2, and let D be an ample divisor in Z such that both $\mathcal{O}_Z(D)$ and $\mathcal{O}_Z(-K_Z - D)$ are globally generated. Note that $\mathcal{O}_Z(-K_Z)$ is then globally generated as well.

Example 4.1. Take any toric Fano manifold Z of dimension at least 2. Since nef line bundles on a projective toric manifold are globally generated, any decomposition $-K_Z = D + D'$ as the sum of an ample divisor D and a nef divisor D' yields a pair (Z, D) satisfying the above condition.

Let $W \subset \mathbb{P}^1 \times Z$ be a general member of the ample and basepoint-free linear system $|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Z(D)|$. We have a fibration $\phi : W \to \mathbb{P}^1$ via the first projection, and the second projection $\varepsilon : W \to Z$ is the blow-up of Z along the smooth subvariety Y of codimension two cut out by the members of the pencil in $|D|$ defined by W. Since Z is Fano, W is rationally connected. By construction, the rational curve $\varepsilon^{-1}(y) \simeq \mathbb{P}^1$ for any $y \in Y$ is a section of $\phi : W \to \mathbb{P}^1$.

Note that

$$\mathcal{O}_W(-K_W) = (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Z(-K_Z - D))\,|_W \tag{4.1}$$

by the adjunction formula. So $\mathcal{O}_W(-K_W)$ is globally generated; in particular, it is nef and effective.

The following lemma describes W when $\dim W = 2$. Recall that a smooth projective surface S is called *weak del Pezzo* if its anticanonical divisor $-K_S$ is nef and big.

Lemma 4.2. *If* $\dim W = 2$*, then either* $D \in |-K_Z|$ *and* $W \xrightarrow{\phi} \mathbb{P}^1$ *is a rational elliptic surface with* $-K_W$ *globally generated, or* W *is a weak del Pezzo surface.*

Proof. Since W is rationally connected and $\dim W = 2$, we know that W is rational.

If $D \in |-K_Z|$, then $\mathcal{O}_W(-K_W) = \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$. So $-K_W$ is globally generated and W is a rational elliptic surface.

Suppose that $D \notin |-K_Z|$. As $-K_Z - D$ is effective and $-K_Z$ and D are ample, we have $-K_Z(-K_Z - D) > 0$ and $D(-K_Z - D) > 0$, and thus,

$$K_Z^2 > -K_Z \cdot D > D^2.$$

As W is the blowup of Z at (D^2) points, we have $K_W^2 = K_Z^2 - D^2 > 0$. Since $-K_W$ is nef, W is a weak del Pezzo surface. ☐

The nef cone of W constructed as above has the following properties.

Proposition 4.3. *We have*

$$\mathrm{Nef}^e(W) = \mathrm{Nef}^+(W) = \mathrm{Nef}(W).$$

Moreover, if $\dim W \geq 3$*, or if* W *is a weak del Pezzo surface, then the cone* $\mathrm{Nef}(W)$ *is rational polyhedral, spanned by classes of semiample divisors.*

Proof. We start with the last statement, which is a corollary of some known results. If W is a weak del Pezzo surface, then W is log Fano (see e.g. [25, Proposition 2.6]). Hence by the Cone Theorem [19, Theorem 3.7], its nef cone is a rational polyhedral cone spanned by classes of semiample divisors. Assume that $\dim W \geq 3$. Since $\mathbb{P}^1 \times Z$ is a smooth Fano variety of dimension ≥ 4, and $W \subset \mathbb{P}^1 \times Z$ is a smooth ample divisor such that

$$\mathcal{O}_{\mathbb{P}^1 \times Z}(-K_{\mathbb{P}^1 \times Z} - W) = \mathcal{O}_Z(-K_Z - D) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$$

is nef, by [1, Proposition 3.5] (based on [3, Appendix]) we have

$$j_* : \overline{\mathrm{NE}}(W) \xrightarrow{\sim} \overline{\mathrm{NE}}(\mathbb{P}^1 \times Z)$$

where $j : W \hookrightarrow \mathbb{P}^1 \times Z$ is the inclusion. As the nef cone is dual to the Mori cone, we have

$$j^* : \mathrm{Nef}(\mathbb{P}^1 \times Z) \xrightarrow{\sim} \mathrm{Nef}(W).$$

Since $\mathrm{Nef}(\mathbb{P}^1 \times Z)$ is rational polyhedral and spanned by classes of semiample divisors, so is $\mathrm{Nef}(W)$.

Now we prove the first statement. Since it holds in particular if $\mathrm{Nef}(W)$ is rational polyhedral spanned by classes of semiample divisors, by the last statement of Proposition 4.3 and Lemma 4.2 it remains to study the case where W is a rational elliptic surface. Clearly $\mathrm{Nef}^e(W)$ and $\mathrm{Nef}^+(W)$ are subcones of $\mathrm{Nef}(W)$. Moreover, $\mathrm{Nef}^+(W) \subset \mathrm{Nef}^e(W)$ by [37, Lemma 4.2], so we only need to show that $\mathrm{Nef}(W) = \mathrm{Nef}^+(W)$. This follows from [29, Corollary 3.3.(c)] that as a cone, $\overline{\mathrm{NE}}(W)$ is generated by curve classes, and that $\mathrm{Nef}(W)$ is dual to $\overline{\mathrm{NE}}(W)$. □

Finally, note that if $D \in |-K_Z|$, then by (4.1), a general fiber of $\phi : W \to \mathbb{P}^1$ is a smooth K-trivial variety. If W has dimension 2, it must be an elliptic curve. In general, we can say the following.

Lemma 4.4. *If $D \in |-K_Z|$, then a general fiber F of $\phi : W \to \mathbb{P}^1$ is a Calabi–Yau manifold, that is, $\omega_F \simeq \mathcal{O}_F$ and $h^i(F, \mathcal{O}_F) = 0$ for $0 < i < \dim F$.*

Proof. Since $D \in |-K_Z|$, we have $\mathcal{O}_W(F) \simeq \mathcal{O}_W(-K_W)$ by (4.1). So by adjunction, $\omega_F \simeq \mathcal{O}_F$, and also we have the exact sequence

$$0 \to \omega_W \to \mathcal{O}_W \to \mathcal{O}_F \to 0.$$

Since W is rationally connected, we have

$$h^{\dim W - i}(W, \omega_W) = h^i(W, \mathcal{O}_W) = 0$$

for $i \geq 1$. Hence $h^i(F, \mathcal{O}_F) = 0$ whenever $1 \leq i \leq \dim W - 2 = \dim F - 1$. □

4.2. **The fiber product $X = W_1 \times_{\mathbb{P}^1} W_2$.** We are ready to generalize Schoen's construction and obtain Calabi–Yau pairs in arbitrary dimension. For $i = 1, 2$, let Z_i, D_i, W_i be as in §4.1. We denote by $\phi_i : W_i \to \mathbb{P}^1$ the associated fibration, and recall that it has a section.

Denoting by S_i the images of the singular fibers of ϕ_i in $\mathbb{P}^1$, we assume $S_1 \cap S_2 = \varnothing$. Moreover, if $\phi_1 : W_1 \to \mathbb{P}^1$ and $\phi_2 : W_2 \to \mathbb{P}^1$ are two rational elliptic surfaces with sections, we require that the elliptic curves $\phi_1^{-1}(t)$ and $\phi_2^{-1}(t)$ are non-isogenous for a general point $t \in \mathbb{P}^1$.

We consider the fiber product over $\mathbb{P}^1$

$$\begin{tikzcd} & X = W_1 \times_{\mathbb{P}^1} W_2 \arrow[dl, "p_1"'] \arrow[dd, "\phi"] \arrow[dr, "p_2"] & \\ W_1 \arrow[dr, "\phi_1"'] & & W_2 \arrow[dl, "\phi_2"] \\ & \mathbb{P}^1 & \end{tikzcd}.$$

As $S_1 \cap S_2 = \varnothing$, the variety X is smooth.

One can also regard X as a complete intersection in $\mathbb{P}^1 \times Z_1 \times Z_2$ of two hypersurfaces in the linear systems

$$|\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_1}(D_1) \boxtimes \mathcal{O}_{Z_2}| \quad \text{and} \quad |\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{Z_1} \boxtimes \mathcal{O}_{Z_2}(D_2)|,$$

respectively. In particular,

$$\mathcal{O}_X(-K_X) = (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{Z_1}(-K_{Z_1} - D_1) \boxtimes \mathcal{O}_{Z_2}(-K_{Z_2} - D_2))\,|_X \tag{4.2}$$

is globally generated. In particular, $-K_X$ is effective.

Definition 4.5. The smooth projective variety X constructed above is called a *Schoen variety*. A pair (X, Δ) is called a *Schoen pair* if Δ is an effective $\mathbb{Q}$-divisor such that $K_X + \Delta \sim_{\mathbb{Q}} 0$.

As $-K_X$ is effective, any Schoen variety X underlies a Schoen pair (X, Δ). By definition, any Schoen pair (X, Δ) is Calabi–Yau in the sense of Definition 2.1. Moreover, there exists a positive integer m such that

$$\Delta = \Delta_{m,X} = \frac{1}{m}\Delta'_{m,X}, \quad \text{where } \Delta'_{m,X} \in |-mK_X|. \tag{4.3}$$

Note that the pair (X, Δ) is klt if $m \geq 2$ and $\Delta'_{m,X} \in |-mK_X|$ is general.

Lemma 4.6. *Any Schoen variety X is simply connected.*

Proof. The proof is similar to [33, Lemma 1] and [35, Lemma 2.1].

Let $U \subset \mathbb{P}^1$ be the open subset over which the morphism $\phi : X \to \mathbb{P}^1$ is smooth and set $V := \phi^{-1}(U)$. The natural map $\phi|_V : V \to U$ is topologically locally trivial with a fiber, say F. Since both ϕ_1 and ϕ_2 have sections, $\phi : X \to \mathbb{P}^1$ also admits a section $\sigma : \mathbb{P}^1 \to X$. Consider the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(F) \arrow[r] & \pi_1(V) \arrow[r, two heads, shift right] \arrow[d] & \pi_1(U) \arrow[r] \arrow[d] \arrow[l, bend right, "\sigma_{U*}"'] & 1 \\ & & \pi_1(X) \arrow[r, two heads, shift right] & \pi_1(\mathbb{P}^1). \arrow[l, bend right, "\sigma_*"'] & \end{tikzcd}$$

Here the first row is exact by the homotopy long exact sequence. By a diagram chase and the fact that $\pi_1(\mathbb{P}^1)$ is trivial, it is enough to check that the image of $\pi_1(F)$ in $\pi_1(X)$ is trivial. Write $F = F_1 \times F_2$, where F_i is a general fiber of $\phi_i : W_i \to \mathbb{P}^1$ for $i = 1, 2$. Since $\pi_1(F) = \pi_1(F_1) \times \pi_1(F_2)$, it is enough to show that the image of $\pi_1(F_i)$ in $\pi_1(X)$ is trivial, which we prove for $i = 1$.

A section of $\phi_2 : W_2 \to \mathbb{P}^1$ gives rise to a section s of $p_1 : X \to W_1$. By construction, the homomorphism $\pi_1(F_1) \to \pi_1(X)$ is induced by $F_1 \hookrightarrow W_1 \xrightarrow{s} X$, thus factors through $\pi_1(W_1)$. Since it is rationally connected, W_1 is simply-connected, and hence the image of $\pi_1(F_1)$ in $\pi_1(X)$ is trivial. □

Proposition 4.7. *Suppose that $D_i \in |-K_{Z_i}|$ for both $i = 1, 2$. Then the Schoen variety X is a strict Calabi–Yau manifold (see Definition 2.2).*

Proof. First of all, (4.2) shows that K_X is trivial. Since X is simply-connected by Lemma 4.6, it remains to show that $h^p(X, \mathcal{O}_X) = 0$ for $0 < p < \dim X$.

Lemma 4.8. *Let $g : \mathcal{X} \to \mathcal{Y}$ be a surjective morphism between smooth projective varieties. Assume that a general fiber F of g is a Calabi–Yau manifold and that $\omega_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}$. Then for every integer $i > 0$, we have*

$$R^i g_* \mathcal{O}_{\mathcal{X}} = \begin{cases} \omega_{\mathcal{Y}}, & \text{if } i = \dim \mathcal{X} - \dim \mathcal{Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Set $r := \dim \mathcal{X} - \dim \mathcal{Y}$.

Since $R^q g_* \omega_{\mathcal{X}} = R^q g_* \mathcal{O}_{\mathcal{X}}$ is reflexive by [17, Theorem 2.1.(i)] and [18, Corollary 3.9], and since $H^q(F, \mathcal{O}_F) = 0$ for all $0 < q < r$ and $\dim H^q(F, \mathcal{O}_F) = 1$ for $q = 0$ or r, we have

$$R^q g_* \mathcal{O}_{\mathcal{X}} = \begin{cases} \text{an invertible sheaf}, & \text{if } q = 0 \text{ or } r, \\ 0, & \text{otherwise.} \end{cases}$$

By Grothendieck–Verdier duality [14, Theorem 3.34], we have

$$Rg_*\omega_{\mathfrak{X}} \simeq R\mathcal{H}om(Rg_*\mathcal{O}_{\mathfrak{X}}, \omega_{\mathfrak{Y}}[-r]).$$

The Grothendieck spectral sequence then gives

$$E_2^{p,-q} := \mathcal{E}xt^p(R^q g_*\mathcal{O}_{\mathfrak{X}}, \omega_{\mathfrak{Y}}) \Rightarrow R^{p-q+r} g_*\omega_{\mathfrak{X}}.$$

(See e.g. [14, Example 2.70.ii)].) So $E_2^{p,-q} \neq 0$ only if $(p,q) = (0,0)$ or $(0,r)$, and Lemma 4.8 follows. □

Let $w_i := \dim W_i$. Since a general fiber of p_2, i.e. of ϕ_1, is a Calabi–Yau manifold by Lemma 4.4, we can apply Lemma 4.8 to $p_2 : X \to W_2$ and obtain

$$R^j p_{2*}\omega_X = \begin{cases} \mathcal{O}_{W_2}, & \text{if } j = 0, \\ \omega_{W_2}, & \text{if } j = \dim w_1 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from [18, Corollary 3.2] that

$$h^p(X, \omega_X) = h^p(W_2, \mathcal{O}_{W_2}) + h^{p-w_1+1}(W_2, \omega_{W_2})$$

for all $0 \le p \le \dim X$. Since W_2 is rationally connected, this is zero unless $p = 0$ or $w_1 + w_2 - 1 = \dim X$. □

5. Application to the Cone Conjecture

In this section, we prove Theorem 1.6.

We have defined Schoen pairs $(X, \Delta_{m,X})$ in Section 4, arising from fiber products

$$\begin{tikzcd} & X = W_1 \times_{\mathbb{P}^1} W_2 \arrow[dl, "p_1"'] \arrow[dd, "\phi"] \arrow[dr, "p_2"] & \\ W_1 \arrow[dr, "\phi_1"'] & & W_2 \arrow[dl, "\phi_2"] \\ & \mathbb{P}^1 & \end{tikzcd} .$$

Lemma 5.1. *Any line bundle L on a Schoen variety X can be written $L = p_1^*L_1 \otimes p_2^*L_2$, where L_i is a line bundle on W_i.*

Proof. Let $p \in \mathbb{P}^1$ be a general point and let $F_i := \phi_i^{-1}(p) \subset W_i$.

Claim 5.2. *The map*

$$\Psi : \mathrm{Pic}(F_1) \times \mathrm{Pic}(F_2) \to \mathrm{Pic}(F_1 \times F_2)$$

defined by $\Psi(L, M) = L \boxtimes M$ is an isomorphism.

Proof. First suppose that either W_1 or W_2 is not a rational elliptic surface. Since $H^1(F_i, \mathcal{O}_{F_i}) = 0$ for at least one $i \in \{1, 2\}$, Claim 5.2 follows from [13, Exercise III.12.6].

Assume now that W_1 and W_2 are rational elliptic surfaces. Then F_1 and F_2 are elliptic curves, and we have a short exact sequence of abelian groups [2, Theorem 11.5.1]

$$0 \to \mathrm{Pic}(F_1) \times \mathrm{Pic}(F_2) \xrightarrow{\Psi} \mathrm{Pic}(F_1 \times F_2) \to \mathrm{Hom}(F_1, F_2) \to 0$$

where $\mathrm{Hom}(F_1, F_2)$ is the group of homomorphisms of group varieties $F_1 \to F_2$. Since $p \in \mathbb{P}^1$ is general, the elliptic curves F_1 and F_2 are non-isogenous by our definition of Schoen varieties. Thus $\mathrm{Hom}(F_1, F_2) = 0$, which proves Claim 5.2. □

Let L be a line bundle on X. Claim 5.2 implies that

$$L_{|\phi^{-1}(p)} \simeq L_{|F_1\times\{u\}} \boxtimes L_{|\{v\}\times F_2},$$

for any points $u \in F_2$ and $v \in F_1$.

For each $i = 1, 2$, we choose a section $s_i : \mathbb{P}^1 \to W_i$ and let $\sigma_i : W_i \to X$ be the induced section:

$$\sigma_1(w_1) := (w_1, s_2(\phi_1(w_1))) \in W_1 \times_{\mathbb{P}^1} W_2,$$

and similarly for σ_2. We have

$$\begin{aligned} L_{|\phi^{-1}(p)} &\simeq L_{|F_1\times\{s_1(p)\}} \boxtimes L_{|\{s_2(p)\}\times F_2} \\ &\simeq (\sigma_1^*L)_{|F_1} \boxtimes (\sigma_2^*L)_{|F_2} \\ &\simeq (p_1^*\sigma_1^*L \otimes p_2^*\sigma_2^*L)_{|\phi^{-1}(p)}. \end{aligned}$$

Since $p \in \mathbb{P}^1$ is general, by [13, Exercise III.12.4]

$$L \simeq p_1^*\sigma_1^*L \otimes p_2^*\sigma_2^*L \otimes \mathcal{O}_X(D)$$

for some divisor D whose support is contained in a finite union of fibers of $\phi : X \to \mathbb{P}^1$. Since the subsets S_1, S_2 parametrizing singular fibers of ϕ_1 and ϕ_2 respectively are disjoint, the subsets paramatrizing reducible fibers are disjoint as well. Hence, an irreducible component R of a fiber of ϕ is of the form p_i^*R' where R' is a multiple of an irreducible component of a fiber of $\phi_i : W_i \to \mathbb{P}^1$. Applied to the irreducible components of D, that yields that

$$\mathrm{Pic}(W_1) \times \mathrm{Pic}(W_2) \xrightarrow{p_1^*\otimes p_2^*} \mathrm{Pic}(X)$$

is surjective. $\square$

Lemma 5.3. *For every $D \in \mathrm{Nef}(X)$, one can write $D = p_1^*D_1 + p_2^*D_2$, where $D_i \in \mathrm{Nef}(W_i)$.*

Proof. Lemma 5.3 follows from Lemma 5.1, which by $\mathbb{R}$-linearity, yields the decomposition at the level of $N^1(W)_{\mathbb{R}}$, and Theorem 1.4. $\square$

Theorem 5.4 (= Theorem 1.6)**.** *Let (X, Δ) be a Schoen pair. Then*

$$\mathrm{Nef}(X) = \mathrm{Nef}^+(X) = \mathrm{Nef}^e(X),$$

and moreover, there exists a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(X, \Delta)$ *on* $\mathrm{Nef}^e(X)$.

Proof. Since $\mathrm{Nef}(W_i) = \mathrm{Nef}^+(W_i) = \mathrm{Nef}^e(W_i)$ by Proposition 4.3, we have, by Theorem 1.4 and Lemma 5.3, $\mathrm{Nef}(X) = p_1^*\mathrm{Nef}^+(W_1)+p_2^*\mathrm{Nef}^+(W_2) \subset \mathrm{Nef}^+(X)$, so $\mathrm{Nef}(X) = \mathrm{Nef}^+(X)$. Similarly, we have $\mathrm{Nef}(X) = \mathrm{Nef}^e(X)$. This proves the first assertion.

Define the subgroups $H_i \leq \mathrm{Aut}(W_i)$ by

$$H_i = \begin{cases} \mathrm{Aut}(W_i/\mathbb{P}^1), & \text{if } W_i \text{ is a rational elliptic surface,} \\ \{\mathrm{id}_{W_i}\}, & \text{otherwise.} \end{cases}$$

Then there exists a rational polyhedral cone $\Pi_i \subset \mathrm{Nef}^+(W_i)$ such that $H_i \cdot \Pi_i$ contains $\mathrm{Amp}(W_i)$. Indeed, the case where W_i is a rational elliptic surface with $-K_{W_i}$ semiample follows from [36, Theorem 8.2], and the other cases follow from Proposition 4.3.

We want to show that $H_1 \times H_2 \leq \mathrm{Aut}(X, \Delta)$. Note that there exists a positive integer m such that

$$\Delta = \Delta_{m,X} = \frac{1}{m}\Delta'_{m,X}$$

for some $\Delta'_{m,X} \in |-mK_X|$.

We now claim that $H_1 \times H_2 \leq \mathrm{Aut}(X, \Delta'_{m,X})$. Indeed, if neither W_1 nor W_2 is a rational elliptic surface, then $H_1 \times H_2$ is trivial by definition. If both W_1 and W_2 are rational elliptic surfaces, then $\Delta'_{m,X} = 0$ and clearly, $H_1 \times H_2 \leq \mathrm{Aut}(X)$. Finally, if one of the W_i, say W_1, is a rational elliptic surface, and the other, say W_2, is not, then $\mathcal{O}_X(-K_X) \simeq p_2^*\mathcal{O}_{W_2}(-K_{Z_2} - D_2)$. Since p_2 is proper surjective with connected fibers, the pullback p_2^* induces an isomorphism

$$H^0(X, p_2^*\mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))) \simeq H^0(W_2, \mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))).$$

So $\Delta'_{m,X} = p_2^*\Delta'_{m,W_2}$, for some divisor $\Delta'_{m,W_2} \in |\mathcal{O}_{W_2}(-m(K_{Z_2} + D_2))|$. Since $H_2 = \{\mathrm{id}_{W_2}\}$ in this case, it follows that $\Delta'_{m,X}$ is invariant under $H_1 \times H_2$. This proves the claim.

It then follows from Corollary 3.7 that $\mathrm{Nef}^e(X) = \mathrm{Nef}^+(X)$ has a rational polyhedral fundamental domain Π for the $\mathrm{Aut}(X, \Delta)$-action. $\square$

Remark 5.5. In [12], the authors verified the Cone Conjecture for a strict Calabi–Yau threefold $X = W_1 \times_{\mathbb{P}^1} W_2$, where both W_i are rational elliptic surfaces with section, each of whose singular fibers is an irreducible rational curve with a node, and two generic fibers are non-isogenous.

Our proof bypasses the identification shown by Namikawa [28, Proposition 2.2 and Corollary 2.3]

$$\mathrm{Aut}(X) \cong \mathrm{Aut}(W_1) \times \mathrm{Aut}(W_2),$$

an identification that was crucial in [12] due to the lack of Looijenga's result (Lemma 2.4) at that time.

Example 5.6. Assume that $\dim Z_1 = 2$ and W_1 is a general rational elliptic surface obtained by a pencil of cubic curves in $\mathbb{P}^2$. Then $\mathrm{Nef}(W_1)$ admits infinitely many faces, and so does $\mathrm{Nef}(X)$ by Lemma 5.1 and Corollary 1.5. If in addition $D_2 \in |-K_{Z_2}|$, then the Schoen variety X is a strict Calabi–Yau manifold by Proposition 4.7.

Corollary 5.7. *Let X be a Schoen variety. Then $\pi_0\mathrm{Aut}(X)$ is finitely presented and there are at most finitely many real structures on X up to equivalence.*

Proof. The linear action $\rho : \mathrm{Aut}(X) \to \mathrm{GL}(N^1(X))$ induces and factorizes through an action

$$\overline{\rho} : \pi_0\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)).$$

We let $\mathrm{Aut}^*(X) = \rho(\mathrm{Aut}(X)) = \overline{\rho}(\pi_0\mathrm{Aut}(X))$.

Choose an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is a Schoen pair. By Theorem 1.6, there exists a rational polyhedral cone $\Pi \subset \mathrm{Nef}^+(X)$ such that

$$\mathrm{Amp}(X) \subset \mathrm{Aut}(X, \Delta) \cdot \Pi \subset \mathrm{Aut}^*(X) \cdot \Pi.$$

It follows from Proposition 2.3 that there is a rational polyhedral fundamental domain for the $\mathrm{Aut}^*(X)$-action on $\mathrm{Nef}^+(X)$ and the group $\mathrm{Aut}^*(X)$ is finitely presented.

Since $\mathrm{Ker}(\overline{\rho})$ is finite by Fujiki–Liebermann's theorem [5, Corollary 2.11], the first claim follows from [15, Corollary 10.2]. The second statement follows from Theorem 5.8 below. □

Theorem 5.8 ([9, Theorem 1.5]). *Let V be a smooth complex projective variety. Assume that there exists a rational polyhedral fundamental domain for the action of* $\mathrm{Aut}(V)$ *on* $\mathrm{Nef}^+(V)$*. Then the set of non-isomorphic real structures of V is finite.*

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