Positivity of higher exterior powers of the tangent bundle

Cécile Gachet

Université Côte d'Azur, CNRS, LJAD, France

August 2, 2022

Abstract

We prove that a smooth projective variety X of dimension n with strictly nef third, fourth or $(n-1)$-th exterior power of the tangent bundle is a Fano variety. Moreover, in the first two cases, we provide a classification for X under the assumption that $\rho(X) \neq 1$.

1 Introduction

Positivity notions are numerous in algebraic geometry: a line bundle can be considered positive, e.g., if it is very ample, ample, strictly nef, nef, big, semiample, effective, pseudoeffective... Some of these notions relate: a very ample line bundle is ample, an ample line bundle is strictly nef and big, a strictly nef line bundle (i.e., a line bundle that has positive intersection with any curve) is nef, a nef line bundle and an effective line bundle are pseudoeffective. These positivity notions, as they tremendously matter in algebraic geometry, have been the subject of a lot of work, to which the books by Lazarsfeld [Laz04a, Laz04b] are a great introduction. Proving new relationships between these various positivity notions is however a rather naive ambition, if not under strong additional assumptions.

From this perspective, the conjecture by Campana and Peternell [CP91] is surprising: they predict that, if X is a smooth projective variety, and the anticanonical bundle $-K_X$ is strictly nef, then $-K_X$ is ample, i.e., X is a Fano manifold. Their conjecture was in fact proven in dimension 2 and 3, by Maeda and Serrano [Mae93, Ser95]. As all Fano manifolds are rationally connected [Cam92, KMM92], an interesting update on the conjecture is the recent proof by Li, Ou and Yang [LOY19, Theorem 1.2] that if X is a smooth projective variety, and the anticanonical bundle $-K_X$ is strictly nef, then X is rationally connected. Their proof uses important results on the Albanese map of varieties with nef anticanonical bundle. Such varieties have been extensively studied too [DPS94, Zha96, PS98, Dem15, CH17, Cao19, CH19].

Positivity notions extend to vector bundles [Laz04b, Definition 6.1.1] in the following fashion: a vector bundle E is stricly nef if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is strictly nef on $\mathbb{P}(E)$. Instead of asking about the positivity of the top exterior power of the tangent bundle, $-K_X = \bigwedge^{\dim(X)} T_X$, it makes sense to ask about the positivity of intermediate exterior powers $\bigwedge^r T_X$, for $1 \leq r \leq \dim(X)-1$.

For $r = 1$, it is known since Mori [Mor79] that projective spaces are the only smooth projective varieties with ample tangent bundle. They are also the only smooth projective varieties with strictly nef tangent bundle, by [LOY19, Theorem 1.4]. Varieties with nef tangent bundle are, on the other hand, governed by another conjecture of Campana and Peternell [CP91] which has received a lot of attention: see the survey [MnOSC+15], and *inter alia* [CP91, DPS94, Wat14, Kan17, Kan16, MnOSCW15, Yan, Li17, Dem18, Wat21a, KW].

For $r = 2$, it has been proven that varieties with ample second exterior power of the tangent bundle are projective spaces and quadric hypersurfaces [CS95], varieties with strictly nef second exterior power of the tangent bundle alike.

Theorem 1.1. [LOY19, Theorem 1.5] *Let X be a smooth projective variety of dimension $n \geq 2$, such that $\bigwedge^2 T_X$ is strictly nef. Then X is isomorphic to the projective space $\mathbb{P}^n$, or to a smooth quadric hypersurface Q^n.*

Partial results were obtained under the nef assumption [Wat21b, Sch].

These results lead us to the following questions.

Question. Let X be a smooth projective variety of dimension n. Suppose that $\bigwedge^r T_X$ is strictly nef for some integer $1 \leq r \leq n$. Is X a Fano variety?

Question. Let X be a smooth projective variety of dimension n. Suppose that $\bigwedge^r T_X$ is nef for some integer $1 \leq r < n$, and that X is rationally connected. Is X a Fano variety?

Note that an affirmative answer to the second question would imply an affirmative answer to the first question, by [LOY19, Theorem 1.2]. Also note that the second question is answered negatively for $r = n$, as there are smooth rationally connected threefolds with $-K_X$ nef but not semiample [Xie]. The first question is answered affirmatively for smooth toric varieties by [Sch]. In this paper, we answer the first question for arbitrary smooth projective varieties for $r = 3, 4$ and the second question for $r = n - 1$.

Theorem 1.2. *Let X be a smooth projective variety of dimension $n \geq 2$ such that the vector bundle $\bigwedge^{n-1} T_X$ is nef and X is rationally connected. Then X is a Fano variety.*

This theorem is reminiscent of [DPS94, Proposition 3.10], which states a dichotomy for varieties X with nef tangent bundle: either X is a Fano manifold, or $\chi(X, \mathcal{O}_X) = 0$. The proof similarly involves Chern classes inequalities and the Hirzebruch-Riemann-Roch formula.

Theorem 1.3. *Let X be a smooth projective variety of dimension at least 4 such that the vector bundle $\bigwedge^3 T_X$ is strictly nef. Then either $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$, or X is a Fano variety of Picard rank $\rho(X) = 1$.*

Let us briefly discuss the case when $\rho(X) = 1$ and $\bigwedge^3 T_X$ is strictly nef. We note that, if X is a cubic or a complete intersection of two quadrics in $\mathbb{P}^n$, the vector bundle $\bigwedge^3 T_X$ is ample. These are two examples of del Pezzo manifolds, i.e., Fano n-folds of Picard number 1 and of index $n-1$. But we do not know whether the other del Pezzo manifolds have ample $\bigwedge^3 T_X$, and the converse seems even harder. It is worth noting that the case $\rho(X) = 1$ and $\bigwedge^2 T_X$ strictly nef was successfully studied in Theorem 1.1 thanks to a characterization of rationally connected varieties such that $-K_X \cdot C \geq n$ for every rational curve C in X [DH17], a result that we can hardly hope for when $-K_X \cdot C \geq n - 1$. It is moreover not clear how to use the positivity of $\bigwedge^3 T_X$ beyond the inequality $-K_X \cdot C \geq n - 1$ for every rational curve C in X, *cf.* Lemma 2.1.

Theorem 1.4. *Let X be a smooth projective variety of dimension at least 5 such that the vector bundle $\bigwedge^4 T_X$ is strictly nef. Then either X is isomorphic to one of the following Fano varieties*

$$\mathbb{P}^2 \times Q^3;\ \mathbb{P}^2 \times \mathbb{P}^3;\ \mathbb{P}(T_{\mathbb{P}^3});\ \mathrm{Bl}_\ell(\mathbb{P}^5) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1));\ \mathbb{P}^3 \times \mathbb{P}^3$$

or X is a Fano variety of Picard rank $\rho(X) = 1$.

These two theorems were to our knowledge unknown even under the stronger, more classical assumption that $\bigwedge^3 T_X$ or $\bigwedge^4 T_X$ be ample. The proof of both theorems goes by classifying possible Mori contractions for X. A delicate point is that, while we know that our varieties X with $\rho(X) \geq 2$ admit one Mori contraction by the Cone Theorem, we need to construct by hand a second Mori contraction, e.g., to control higher-dimensional fibres in case of a first fibred Mori contraction. Depending on circumstances, we use unsplit covering families of deformations of rational curves, and a result by Bonavero, Casagrande and Druel [BCD07], or, if X has the right dimension, Theorem 1.2, to produce this second Mori contraction.

Acknowledgments. I am grateful to my advisor A. Höring for regular helpful discussions, to S. Tanimoto for pointing out that the complete intersection of two quadrics in a projective space should satisfy Theorem 1.3, and to J. Cao for suggesting the second question in the introduction.

Conventions. We work over the field of complex numbers $\mathbb{C}$. Varieties (and in particular curves) are always assumed irreducible and reduced. We use the expressions "smooth projective variety" and "projective manifold" interchangeably. We refer to [Deb01] for birational geometry, in particular Mori theory, [Laz04a, Laz04b] for positivity notions, [Kol96] for rational curves and their deformations. We write $c_i(X) = c_i(T_X)$ for the Chern classes of the tangent bundle of X.

2 A first lemma

We start with a simple lemma.

Lemma 2.1. *Let X be a smooth projective variety of dimension n such that $\bigwedge^r T_X$ is strictly nef, for some $1 \leq r \leq n-1$. Then any rational curve C in X satisfies*

$$-K_X \cdot C \geq n+2-r.$$

Proof. The proof goes as [LOY19, Proof of Theorem 1.5]. Let $f : \mathbb{P}^1 \to C$ be the normalization of the curve. Write

$$f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),$$

with $(a_i)_{1\leq i\leq n}$ ordered increasingly. It holds $a_n \geq 2$, as $T_{\mathbb{P}^1}$ maps non-trivially to f^*T_X, and we have $a_1 + \ldots + a_r > 0$ because $\mathcal{O}_{\mathbb{P}^1}(a_1 + \ldots + a_r)$ is a direct summand of the strictly nef vector bundle $\bigwedge^r f^*T_X$. In particular, $a_{r+1} \geq a_r \geq 1$. Hence,

$$-K_X \cdot C = \deg f^*(-K_X) = a_1 + \ldots + a_n \geq 1 + n - r - 1 + 2 = n + 2 - r.$$

$\square$

This result is all the more valuable as, by [LOY19, Theorem 1.2], if X is a smooth projective variety of dimension n such that $\bigwedge^r T_X$ is strictly nef, then it is rationally connected, in particular, it contains numerous rational curves.

We will also need the following result.

Lemma 2.2. *Let X be a smooth projective variety of dimension n such that $\bigwedge^r T_X$ is nef, for some $1 \leq r \leq n-1$. Then any rational curve C in X satisfies $-K_X \cdot C \geq 2$.*

Proof. Let $f : \mathbb{P}^1 \to C$ be the normalization of the curve. Write

$$f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n),$$

with $(a_i)_{1\leq i\leq n}$ ordered increasingly. It holds $a_n \geq 2$, as $T_{\mathbb{P}^1}$ maps non-trivially to f^*T_X, and we have $a_1 + \ldots + a_r \geq 0$ because $\mathcal{O}_{\mathbb{P}^1}(a_1 + \ldots + a_r)$ is a direct summand of the nef vector bundle $\bigwedge^r f^*T_X$. Hence, $a_{r+1} \geq a_r \geq 0$, and summing up those inequalities, we obtain the estimate

$$-K_X \cdot C = a_1 + \ldots + a_n \geq 2.$$

$\square$

3 Results on $\bigwedge^{n-1} T_X$

The following lemma is the main step in the proof of Theorem 1.2.

Lemma 3.1. *Let X be a projective n-dimensional manifold such that $\bigwedge^{n-1} T_X$ is nef and X is rationally connected. Then $-K_X$ is nef and big.*

Proof. By [Laz04b, Theorem 6.2.12(iv)], the anticanonical bundle $-K_X$ is nef. By the Hirzebruch-Riemann-Roch formula, there is a homogeneous polynomial P of degree n in $\mathbb{Q}[X_1, \ldots, X_n]$ with grading $\deg X_i = i$ such that

$$\chi(X, \mathcal{O}_X) = P(c_1(X), \ldots, c_n(X)).$$

Note that, as $\bigwedge^{n-1} T_X = \Omega^1_X \otimes \mathcal{O}_X(-K_X)$, and by [Ful98, Remark 3.2.3(b)], we have

$$c_i\left(\bigwedge^{n-1} T_X\right) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{i-j} c_j(X) c_1(-K_X)^{i-j}. \tag{$*$}$$

Let us show by induction that $c_i(X)$ is a rational polynomial in the $c_j(\bigwedge^{n-1} T_X)$, for $0 \leq j \leq i$. Indeed, $c_1(X) = \frac{1}{n} c_1(\bigwedge^{n-1} T_X)$. Assume now that for some i, for all $0 \leq j \leq i$, there is a polynomial $P_j \in \mathbb{Q}[X_1, \ldots, X_j]$ such that $c_j(X) = P_j(c_1(\bigwedge^{n-1} T_X), \ldots, c_j(\bigwedge^{n-1} T_X))$. Then, setting

$$P_{i+1}(X_1, \ldots, X_{i+1}) = (-1)^{i+1} X_{i+1} - \sum_{j=0}^{i} (-1)^{i+j+1} \binom{n-j}{i+1-j} P_j(X_1, \ldots, X_j)(P_1(X_1))^{i+1-j},$$

we have $c_{i+1}(X) = P_{i+1}(c_1(\bigwedge^{n-1} T_X), \ldots, c_{i+1}(\bigwedge^{n-1} T_X))$ by $(*)$. This perpetuates the induction.

In particular, we have

$$\chi(X, \mathcal{O}_X) = P\left(P_1\left(c_1\left(\bigwedge^{n-1} T_X\right)\right), \ldots, P_n\left(c_1\left(\bigwedge^{n-1} T_X\right), \ldots, c_n\left(\bigwedge^{n-1} T_X\right)\right)\right),$$

which is a homogeneous polynomial of degree n in $c_1(\bigwedge^{n-1} T_X), \ldots, c_n(\bigwedge^{n-1} T_X)$.

Now, if we suppose that $-K_X$ is not big, then $c_1(\bigwedge^{n-1} T_X)$ is not big. Thus, [DPS94, Corollary 2.7] implies $\chi(X, \mathcal{O}_X) = 0$. But on the other hand, X is rationally connected, so $\chi(X, \mathcal{O}_X) = 1$, contradiction. □

Remark 3.2. If $n = 4$, we cannot write $c_3(X)$ as a polynomial in

$$\begin{aligned} c_1\left(\bigwedge^{n-2} T_X\right) &= 3c_1(X), \\ c_2\left(\bigwedge^{n-2} T_X\right) &= 3c_1(X)^2 + 2c_2(X), \\ c_3\left(\bigwedge^{n-2} T_X\right) &= c_1(X)^3 + 4c_1(X)c_2(X), \end{aligned}$$

these formulas coming from [Ien, 4.5.2].

Lemma 3.3. *Let X be a projective n-dimensional manifold such that $\bigwedge^{n-1} T_X$ is nef and X is rationally connected. Then $-K_X$ is ample.*

Proof of Theorem 1.2. By Lemma 3.1, $-K_X$ is nef and big. By the base-point-free theorem [Deb01, Theorem 7.32], we dispose of an integer m such that $-mK_X$ is globally generated. Let $\varepsilon : X \to Z$ be the $|-mK_X|$-morphism.

Suppose that it is not finite. By [Kaw91, Theorem 2], any irreducible component E of the exceptional locus is covered by rational curves that are contracted by ε. Let C be one of them: we have $0 = -K_X \cdot C \geq 2$ by Lemma 2.2, contradiction. So $-K_X$ is ample. □

4 Studying Mori contractions

The strategy for proving Theorems 1.3 and 1.4 is to show that there are only few possible birational contractions for X. In the following, if R is an extremal ray of the Mori cone $\overline{NE}(X)$, its *length* denoted by $\ell(R)$ is defined to be the minimal value of $-K_X \cdot C$, for a rational curve C with class in R. A Mori contraction is said to be *of length ℓ* if it is a contraction of a ray R with $\ell(R) = \ell$.

4.1 Small contractions

Lemma 4.1. *Let $r \in [\![1, 4]\!]$. Let X be a smooth projective variety of dimension at least $r + 1$ such that $\bigwedge^r T_X$ is strictly nef. Then X has no small contraction.*

Proof. Let n be the dimension of X. Let $\varphi : X \to Y$ be a birational contraction, E be an irreducible component of the exceptional locus, F an irreducible component of the general fibre of $\varphi|_E$, and R the corresponding extremal ray. Applying Ionescu-Wiśnewski inequality [Ion86, Theorem 0.4], [Wiś91a, Theorem 1.1] together with Lemma 2.1 yields

$$\dim E + \dim F \geq n + \ell(R) - 1 \geq 2n + 1 - r.$$

Since $r \leq 4$, we have $\dim E \geq n - 1$, and thus φ is a divisorial contraction. □

4.2 Fibred Mori contractions

We move on to studying fibred Mori contractions.

4.2.1 Generalities about fibred Mori contractions

If X is a normal projective variety, and C is a rational curve in X, we may denote by $\mathcal{V}$ its family of deformations, that is an irreducible component of $\mathrm{Chow}(X)$ containing the point corresponding to C. Denoting by $\phi : \mathrm{Univ}(X) \to \mathrm{Chow}(X)$ the universal family and by $\mathrm{ev} : \mathrm{Univ}(X) \to X$ the evaluation map, we define

$$\mathrm{Locus}(\mathcal{V}) := \mathrm{ev}(\phi^{-1}(\mathcal{V})) \subset X.$$

We say that $\mathcal{V}$ is *covering* if $\mathrm{Locus}(\mathcal{V}) = X$.
We say that $\mathcal{V}$ is *unsplit* if it only parametrizes irreducible cycles.

For $x \in \mathrm{Locus}(\mathcal{V})$, we define $\mathcal{V}_x := \phi(\mathrm{ev}^{-1}(x))$ the family of deformations of C through x. We finally define

$$\mathrm{Locus}(\mathcal{V}_x) := \mathrm{ev}(\phi^{-1}(\mathcal{V}_x)) \subset X.$$

We use families of deformations of rational curves to prove the following proposition.

Proposition 4.2. *Let X be a smooth projective rationally connected variety of dimension n. Let $r \in [\![1, n-1]\!]$. Suppose that $-K_X \cdot C \geq n+2-r$ for any rational curve C in X. Suppose that there is a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$. Then the general fibre has dimension at most $r-1$.*

If equality holds, then there is a rational curve C in X, not contracted by π, whose family of deformations $\mathcal{V}$ is unsplit covering and satisfies $\dim \mathrm{Locus}(\mathcal{V}_x) = n+1-r$ for $x \in \mathrm{Locus}(\mathcal{V})$ general.

The proof relies on the following lemmas.

Lemma 4.3. *Let X be a smooth projective variety. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$, and let C be a rational curve such that $\pi(C) \neq \{\mathrm{pt}\}$ and such that its family of deformations $\mathcal{V}$ is unsplit. Then, for any $x \in \mathrm{Locus}(\mathcal{V})$,*

$$\dim \mathrm{Locus}(\mathcal{V}_x) \leq \dim Y.$$

Proof of Lemma 4.3. We claim that $\pi|_{\mathrm{Locus}(\mathcal{V}_x)}$ is finite onto its image. If it is not, it contracts a curve B to a point: for some ample divisor H on Y, we have $B \cdot \pi^* H = 0$. By [ACO09, Lemma 4.1], the numerical class of $B \subset \mathrm{Locus}(\mathcal{V}_x)$ is a multiple of $C \in N_1(X)_{\mathbb{Q}}$, whence $C \cdot \pi^* H = 0$, which is a contradiction. So $\pi|_{\mathrm{Locus}(\mathcal{V}_x)}$ is finite onto its image: this implies $\dim \mathrm{Locus}(\mathcal{V}_x) \leq \dim Y$. □

Lemma 4.4. *Let X be a smooth projective variety. Suppose that $-K_X \cdot C > 0$ for every rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$, and let C be a rational curve such that $\pi(C) \neq \{\mathrm{pt}\}$ and such that*

$$-K_X \cdot C = \min\{-K_X \cdot B \mid B \text{ rational curve in } X, \pi(B) \neq \{\mathrm{pt}\}\}.$$

Then the family of deformations of C is unsplit.

Proof of Lemma 4.4. Let $\mathcal{V}$ be the family of deformations of C. Suppose that it is splitting, i.e.,

$$C \underset{\text{num}}{\equiv} \sum_i a_i C_i,$$

with rational curves C_i and coefficients $a_i \geq 1$ such that $\sum_i a_i \geq 2$. Since $-K_X$ is positive on rational curves, we have $-K_X \cdot C_i < -K_X \cdot C$ for all i. So, by minimality of $-K_X \cdot C$, the fibration π contracts all curves C_i. Let H be an ample divisor on Y. We obtain $\sum_i a_i C_i \cdot \pi^* H = 0$, contradiction. □

Proof of Proposition 4.2. Since X is rationally connected and $-K_X$ is Cartier, we dispose of a rational curve C such that $\pi(C) \neq \{\mathrm{pt}\}$ and $-K_X \cdot C \geq n+2-r \geq 3$ is minimal with this condition. Let $\mathcal{V}$ be the corresponding family of deformations. By Lemma 4.4, it is unsplit.

Fix $x \in \mathrm{Locus}(\mathcal{V})$ general. By [Kol96, Proposition IV.2.6] and Lemma 2.1, we derive

$$\dim \mathrm{Locus}(\mathcal{V}) + \dim \mathrm{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + n - 1 \geq 2n + 1 - r.$$

So $\dim \mathrm{Locus}(\mathcal{V}_x) \geq n + 1 - r$.

Let d denote the dimension of the general fibre of π. Then, by Lemma 4.3,

$$d \leq n - \dim \mathrm{Locus}(\mathcal{V}_x) \leq r - 1.$$

As for the equality case, if $d = r-1$, then $\dim \mathrm{Locus}(\mathcal{V}_x) = n - r + 1$, and so C is such a rational curve as we claimed existed in the equality case of the proposition. □

Proposition 4.2 has an important consequence.

Corollary 4.5. *Let X be a smooth projective rationally connected variety of dimension n such that, for some $r \in [\![1, n-1]\!]$, one has $-K_X \cdot C \geq n+2-r$ for any rational curve $C \subset X$. Suppose that there is a fibred contraction $\pi : X \to Y$ with $\dim Y > 0$. Then $n \leq 2r-2$.*

If equality holds, then a general fibre of π has dimension $r-1$, and there is a rational curve C in X, not contracted by π, whose family of deformations $\mathcal{V}$ is unsplit covering and satisfies $\dim \operatorname{Locus}(\mathcal{V}_x) = n+1-r$ for $x \in \operatorname{Locus}(\mathcal{V})$ general.

Proof. Let F be a general fiber of π. By Proposition 4.2, we have $r-1 \geq \dim F$. Adding n to both sides and applying Ionescu-Wiśnewski inequality (with the exceptional locus $E = X$ of dimension n), it holds

$$n + r - 1 \geq n + \dim F \geq n + \ell(R) - 1 \geq 2n + 1 - r.$$

If there is an equality, then $\dim F = r-1$, and so we are in the equality case of Proposition 4.2. □

4.2.2 Fibred Mori contractions for certain varieties of even dimension

The set-up for this paragraph is the following. Let r be 3 or 4. Let X be a smooth projective rationally connected variety of dimension $2r-2$ such that $-K_X \cdot C \geq r$ for any rational curve $C \subset X$. Suppose that there is a fibred contraction $\pi : X \to Y$ with $\dim Y > 0$. Let us classify what happens.

Lemma 4.6. *Let r be 3 or 4. Let X be a smooth projective rationally connected variety of dimension $2r-2$ such that $-K_X \cdot C \geq r$ for any rational curve $C \subset X$. Suppose that there is a fibred contraction $\pi : X \to Y$ with $\dim Y > 0$. Then there is another equidimensional fibred Mori contraction $\varphi : X \to Z$ with $\dim Z = r-1$.*

Proof. We are in the case of equality of Corollary 4.5. In particular, the general fibre F of π has dimension $r-1$, and there is a rational curve C in X that is not contracted by π whose family of deformations $\mathcal{V}$ is unsplit covering and satisfies $\dim \operatorname{Locus}(\mathcal{V}_x) = r-1 \geq (2r-2)-3 = \dim X - 3$.

By [BCD07, Theorem 2, Proposition 1(i)], there is a fibred Mori contraction $\varphi : X \to Z$ whose fibres exactly are the $\mathcal{V}$-equivalence classes, and its general fibre has dimension $\dim \operatorname{Locus}(\mathcal{V}_x) = r-1$.

Let G be a fibre of φ. We claim that $\pi|_G$ is finite. Indeed, if it is not, then there is a curve $B \subset G$ that is contracted by π. The curve B lies in a $\mathcal{V}$-equivalence class, so by [BCD07, Remark 1], as $\mathcal{V}$ is unsplit, B is numerically equivalent to a multiple of C, so it cannot be contracted by π, contradiction! So $\pi|_G$ is finite onto its image, which is contained in Y, so $\dim G \leq \dim Y = r-1$.

So φ is indeed equidimensional. □

Proposition 4.7. *Let $r \geq 3$ be an integer. Let X be a smooth projective rationally connected variety of dimension $2r-2$ such that $-K_X \cdot C \geq r$ for any rational curve $C \subset X$. Suppose that there is an equidimensional fibred Mori contraction $\pi : X \to Y$ with $\dim Y = r-1$. Then $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.*

This proposition relies on the following lemma.

Definition 4.8. Let $\pi : X \to Y$ be a fibration whose general fibre is a projective space. Let $f : \mathbb{P}^1 \to C \subset Y$ be a rational curve whose image lies in the smooth locus of π. The fibre product π_C of π by f is the projectivization of a bundle $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_k)$, with the (a_i) ordered increasingly. A *minimal section* over C is the section $s : \mathbb{P}^1 \to X$ of π_C corresponding to a quotient $\mathcal{O}_{\mathbb{P}^1}(a_1)$.

Remark 4.9. There may be several minimal sections as soon as $a_1 = a_2$.

Lemma 4.10. *Let X be a smooth projective variety with a fibration $\pi : X \to Y$ whose general fiber is a projective space. Then for any rational curve $f : \mathbb{P}^1 \to C \subset Y^0 \subset Y$ in the smooth locus of π, for any minimal section s of it, it holds $-K_Y \cdot C \geq -K_X \cdot s(\mathbb{P}^1)$. In particular,*

$$-K_Y \cdot C \geq \min\{-K_X \cdot C' \mid C' \text{ is a rational curve in } X\}. \tag{$**$}$$

If there is an equality in ($**$), then the base change of π by f is isomorphic to $\mathbb{P}({\mathcal{O}_{\mathbb{P}^1}}^{\oplus k}) \to \mathbb{P}^1$. If there is almost an equality, i.e.,

$$-K_Y \cdot C = \min\{-K_X \cdot C' \mid C' \text{ is a rational curve in } X\} + 1,$$

then the base change of π by f is isomorphic to to $\mathbb{P}({\mathcal{O}_{\mathbb{P}^1}}^{\oplus k}) \to \mathbb{P}^1$ or to $\mathbb{P}({\mathcal{O}_{\mathbb{P}^1}}^{\oplus k-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$.

Proof. By Tsen's theorem, the base change π_C of π by f is the natural projection morphism of a projectivized vector bundle V on $\mathbb{P}^1$. We write $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_k)$, with (a_i) ordered increasingly, and consider s the section of π_C satisfying $s^*\mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{O}_{\mathbb{P}^1}(a_1)$. The degree of $\det(s^*\mathcal{O}_{\mathbb{P}(V)}(1)) \otimes V^*$ is non-positive, equals zero if and only if $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k}$, and equals one if and only if $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k-1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1+1)$.

Pulling-back the Euler exact sequence of π_C by s, we get

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to s^*\mathcal{O}_{\mathbb{P}(V)}(1) \otimes V^* \to s^*T_{X/Y} \to 0.$$

Thus, $s^*T_{X/Y}$ has non-positive degree. We also have the tangent bundle exact sequence:

$$0 \to s^*T_{X/Y} \to s^*T_X \to f^*T_Y \to 0,$$

Since $s^*T_{X/Y}$ has non-positive degree, we obtain

$$-K_Y \cdot C \geq -K_X \cdot s(C) \geq \min\{-K_X \cdot C' \mid C' \text{ is a rational curve in } X\}.$$

Moreover, if there is an equality, then we have $-K_Y \cdot C = -K_X \cdot s(C)$, and so $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k}$.

If there is almost an equality, then $-K_Y \cdot C = -K_X \cdot s(C)$ or $-K_Y \cdot C = -K_X \cdot s(C) + 1$, so $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k}$ or $V \simeq \mathcal{O}_{\mathbb{P}^1}(a_1)^{\oplus k-1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1+1)$. □

Proof of Proposition 4.7. By [HN13, Theorem 1.3], as $\pi : X \to Y$ is an equidimensional fibration with fibres of dimension $r-1$, and as it is a Mori contraction of length at least r as well, it is a $\mathbb{P}^{r-1}$-bundle. Let us show that Y is isomorphic to $\mathbb{P}^{r-1}$. Since X is smooth and a projective bundle over Y, the variety Y is smooth. By Lemma 4.10, any rational curve C in Y satisfies $-K_Y \cdot C \geq r$. Moreover, X is rationally connected, so Y is too. By [CMSB02, Cor.0.4, $1 \Leftrightarrow 10$], we get $Y \simeq \mathbb{P}^{r-1}$.

As $\mathbb{P}^{r-1}$ has trivial Brauer group, there is a vector bundle V of rank r on Y such that π identifies with the natural projection $\mathbb{P}(V) \to \mathbb{P}^{r-1}$. Without loss of generality, we can twist V by a line bundle so that $\deg_\Delta V|_\Delta \in [\![0, r-1]\!]$ for any line Δ in $\mathbb{P}^{r-1}$. Let Δ be a line in $\mathbb{P}^{r-1}$. Then $-K_{\mathbb{P}^{r-1}} \cdot \Delta = r$. By the equality case in Lemma 4.10, the restriction $V|_\Delta$ is isomorphic to $L^{\oplus r}$ for some line bundle L on Δ. Hence $\deg L = 0$, so $L = \mathcal{O}_\Delta$. By [OSS80, Theorem 3.2.1], the vector bundle V is globally trivial. Hence, $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$. □

4.2.3 Fibred Mori contractions for certain fivefolds

The goal in this section is prove the following result.

Proposition 4.11. *Let X be a smooth projective fivefold such that $\bigwedge^4 T_X$ is strictly nef. Suppose that X admits a fibred Mori contraction. Then X is isomorphic to one of the following projective manifolds*

$$\mathbb{P}^2 \times Q^3;\ \mathbb{P}^2 \times \mathbb{P}^3;\ \mathbb{P}(T_{\mathbb{P}^3});\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)).$$

We first establish this classification under the simplifying assumption that X has a $\mathbb{P}^2$-bundle structure, instead of a fibred Mori contraction.

Lemma 4.12. *Let X be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that $p : X \to Y$ is a $\mathbb{P}^2$-bundle. Then Y is a smooth projective variety, and X is isomorphic to one of the following projective manifolds*

$$\mathbb{P}^2 \times Q^3;\ \mathbb{P}^2 \times \mathbb{P}^3;\ \mathbb{P}(T_{\mathbb{P}^3});\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)).$$

Among other things, the proof uses the following lemma.

Lemma 4.13. *Let V be a vector bundle on a smooth quadric hypersurface Q^n. If V is trivial on all lines in Q^n, then V is trivial.*

Proof. Note that by [Erm15, Theorem 7], it is enough to show that for any $x, z \in Q^n$, there exists a point $y \in Q^n$ such that the lines (xy) and (yz) belong to Q^n. Intersecting with $n-2$ hyperplanes, we can reduce to $n = 2$, in which case $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is covered by two family of lines corresponding to the two rulings. Hence, the point $y = (pr_1(x), pr_2(z))$ satisfies our requirement. □

Proof of Lemma 4.12. Since X is smooth and $X \to Y$ is a projective bundle, Y is smooth as well. Since X is rationally connected, Y is rationally connected and by Lemma 4.10, one has $-K_Y \cdot C \geq 3$ for any rational curve C in Y. By [DH17, Cor.1.4], Y is a quadric hypersurface Q^3 or the projective space $\mathbb{P}^3$. In either case, Y is rational and so it has trivial Brauer group. Hence, $X = \mathbb{P}(V)$ for some vector bundle V on Y.

If Y is a quadric, then all lines $\Delta \subset Y$ satisfy $-K_Y \cdot \Delta = 3$, and thus by the equality case in Lemma 4.10, $V|_\Delta \simeq {L_\Delta}^{\oplus 3}$ for some line bundle L on Δ. Fixing a line Δ_0, we have, as $\rho(Y) = 1$,

$$\deg L_\Delta \otimes {L_{\Delta_0}}^{-1} = \frac{1}{3}(\deg V|_\Delta \otimes V^*|_{\Delta_0}) = \frac{1}{3}(\det V \cdot \Delta - \det V \cdot \Delta_0) = 0,$$

so $(V \otimes {L_{\Delta_0}}^{-1})|_\Delta = {\mathcal{O}_\Delta}^{\oplus 3}$ for any line Δ in Y. By Lemma 4.13, this twist of V is globally trivial and thus $X \simeq \mathbb{P}^2 \times Q^3$.

Suppose now that Y is a projective space. By the almost-equality case in Lemma 4.10, for every line Δ in Y,

$$V|_\Delta \simeq \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(a_{i,\Delta}),$$

with either $a_{1,\Delta} = a_{2,\Delta} = a_{3,\Delta}$ or $a_{1,\Delta} = a_{2,\Delta} = a_{3,\Delta} - 1$. Note that the sum $a_{1,\Delta} + a_{2,\Delta} + a_{3,\Delta} = \det V \cdot \Delta$ is independent of the chosen line Δ. If it is divisible by 3, then we are in the first case, else it is congruent to 1 modulo 3 and we are in the second case. In both cases, the $a_{i,\Delta}$ are thus independent of the line Δ. Fixing a line Δ_0, the restricted twisted bundle $(V \otimes \mathcal{O}_{\mathbb{P}^1}(-a_{1,\Delta_0}))|_\Delta$ therefore is a uniform bundle of type $(0,0,0)$ or $(0,0,1)$. In the first case, this twist of V is globally trivial by [OSS80], and so $X \simeq \mathbb{P}^2 \times \mathbb{P}^3$. In the second case, by [Sat76], this twist of V is either $\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ or $T_{\mathbb{P}^3}(-1)$, which concludes the classification. □

Let us now study a more general fibred Mori contraction of X.

Lemma 4.14. *Let X be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$. Then* $\dim Y \leq 3$.

Proof. If $\dim(Y) = 4$, the general fibre of π is a smooth curve C with trivial normal bundle. By assumption,

$$2 = -K_X \cdot C = \deg_C(-K_C) \geq 3,$$

absurd. □

Let us cover the case when X has a fibred Mori contraction $\pi : X \to Y$ with $1 \leq \dim(Y) \leq 2$.

Lemma 4.15. *Let X be a smooth projective rationally connected fivefold and such that, for any rational curve $C \subset X$, one has $-K_X \cdot C \geq 3$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $1 \leq \dim Y \leq 2$. Then there is a fibred Mori contraction $p : X \to Z$ that is a $\mathbb{P}^2$-bundle.*

Proof. We dispose of a rational curve C such that $\pi(C) \neq \{\text{pt}\}$ and $-K_X \cdot C \geq 3$ is minimal with this condition. Let $\mathcal{V}$ be the corresponding family of deformations. By Lemma 4.4, $\mathcal{V}$ is unsplit. Fix $x \in \operatorname{Locus}(\mathcal{V})$ general. By [Kol96, Proposition IV.2.6] and by assumption, we derive

$$\dim \operatorname{Locus}(\mathcal{V}) + \dim \operatorname{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + 5 - 1 \geq 7.$$

So $\dim \operatorname{Locus}(\mathcal{V}_x) \geq 2$. By Lemma 4.3, $\dim \operatorname{Locus}(\mathcal{V}_x) \leq \dim Y \leq 2$.

As equality holds, $\mathcal{V}$ is a covering family of rational 1-cycles with $\dim \operatorname{Locus}(\mathcal{V}_x) = 2 \geq 5 - 3$, so by [BCD07, Theorem 2, Proposition 1(i)], it admits a geometric quotient $p : X \to Z$, that is a fibred Mori contraction, with a general fibre of dimension 2. If a fibre F of p has dimension 3 or more, then since $\dim Y \leq 2$, $\pi|_F$ cannot be finite. So π contracts at least a curve B contained in F, which is numerically equivalent to a multiple of C as it lies in a $\mathcal{V}$-equivalence class [BCD07, Remark 1], contradiction.

So p is an equidimensional fibred Mori contraction with fibres of dimension 2, of length $-K_X \cdot C \geq 3$. By [HN13, Theorem 1.3], the morphism p is a $\mathbb{P}^2$-bundle. □

We are left supposing that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim(Y) = 3$ that is not a $\mathbb{P}^2$-bundle. Let us first prove a few generalities about its fibres.

Lemma 4.16. *Let X be a smooth projective n-dimensional variety with a fibred Mori contraction π of length $n-k+1$ onto a variety Y of dimension k. Then the general fibre is isomorphic to $\mathbb{P}^{n-k}$.*

Proof. The general fibre is a smooth variety F of dimension $n-k$ such that $-K_F \cdot C \geq n-k+1$ for any rational curve C in F, and $-K_F$ is ample. By [CMSB02, Keb02], [HN13, Theorem 2.1], we obtain $F \simeq \mathbb{P}^{n-k}$. □

We recall and prove a fact mentioned in [HN13, 1.C].

Lemma 4.17. *Let X be a smooth projective variety of dimension $n \geq 4$ with a fibred Mori contraction π of length $n-2$ onto a threefold Y. Suppose that π is not equidimensional. Then for any irreducible component F of a fibre of π of dimension $n-2$, the normalization $\tilde{F}$ of F is isomorphic to $\mathbb{P}^{n-2}$.*

Proof. By [HN13, Theorem 1.3], and as $\mathrm{Univ}_{n-3}(X/Y) \to \mathrm{Chow}_{n-3}(X/Y)$ is a universal family for the $(n-3)$-cycles of X over Y, there is a commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "\eta'"'] \arrow[rr, bend left, "\mu'"] \arrow[d, "\pi'"'] & \overline{X} \arrow[r, "\varepsilon'"'] \arrow[d, "\overline{\pi}"'] & X \arrow[d, "\pi"] \\
Y' \arrow[r, "\eta"] \arrow[rr, bend right, "\mu"'] & \overline{Y} \arrow[r, "\varepsilon"] & Y
\end{tikzcd}$$

where $\overline{Y}$ is the normalization of the closure of the π-equidimensional locus of Y in $\mathrm{Chow}_{n-3}(X/Y)$, $\overline{X}$ is the normalization of the universal family over it, ε' is the evaluation map, Y' is a resolution of $\overline{Y}$, X' is the corresponding normalized fibred product, π' is a $\mathbb{P}^{n-3}$ bundle. Note that since Y is $\mathbb{Q}$-factorial, the exceptional loci of μ and of ε are unions of surfaces, hence the exceptional locus of μ' is a union of $\mathbb{P}^{n-3}$-bundles on surfaces.

Let F be an irreducible component of dimension $n-2$ of a fibre of π, let $\nu : \tilde{F} \to F$ be its normalization. Let $\Sigma \subset \overline{Y}$ be one of the surfaces that ε contracts onto $\pi(F)$, chosen such that $\Gamma := \overline{\pi}^{-1}(\Sigma)$ dominates F. Let S be the strict transform of Σ by η, and let $P := \pi'^{-1}(S)$: it is a $\mathbb{P}^{n-3}$-bundle over S and it dominates Γ. By the universal property of the normalization, we have a map $f : P \to \tilde{F}$, that fits into the following commutative diagram.

$$\begin{tikzcd}
 & & \tilde{F} \arrow[d, "\nu"] \\
P \arrow[urr, bend left, "f"] \arrow[r, "\eta'"'] \arrow[rr, bend left, "\mu'"] \arrow[d, "\pi'"'] & \Gamma \arrow[r, "\varepsilon'"'] \arrow[d, "\overline{\pi}"'] & F \arrow[d, "\pi"] \\
S \arrow[r, "\eta"] \arrow[rr, bend right, "\mu"'] & \Sigma \arrow[r, "\varepsilon"] & \{\mathrm{pt}\}
\end{tikzcd}$$

Let ℓ be a line contained in a fibre of $\pi'|_P$. Let $\mathcal{V}$ be the family of deformation of $f_*\ell$ in $\tilde{F}$.

Let us show that this family satisfies the hypotheses of [HN13, Theorem 2.1]. First, note that $\nu^*(-K_X|_F)$ is ample. Since there is a line in X' numerically equivalent to ℓ that is disjoint from all exceptional divisors of μ', and since ℓ is contracted by π',

$$\nu^*(-K_X|_F) \cdot f_*\ell = -K_X \cdot \mu'_*\ell = -K_{X'} \cdot \ell = -K_{X'/Y'} \cdot \ell = -K_{\mathbb{P}^{n-3}} \cdot \ell = n-2.$$

Since for any rational curve C in $\tilde{F}$, it holds $\nu^*(-K_X|_F) \cdot C \geq n-2$ by assumption, the family $\mathcal{V}$ is unsplit. Moreover, it is a covering family, as ν is birational, μ' is surjective and the family of deformations of ℓ is covering. Hence, by [Kol96, Proposition IV.2.5], for a general point $x \in \tilde{F}$,

$$\dim \mathcal{V} = n - 2 + \dim \mathrm{Locus}(\mathcal{V}_x) + 1 - 3,$$

so we are left to show that $\dim \mathrm{Locus}\,(\mathcal{V}_x) = n-2$ to conclude.

Let us take x and y general in F. It suffices to show that the image by $\mu'|_P$ of a certain fibre $\mathbb{P}^{n-3}$ of $\pi'|_P$ contains both x and y, since then there is a line through any two points in $\mathbb{P}^{n-3}$.

Since x is general and Γ dominates F, it holds $\dim \varepsilon'^{-1}(x) = \dim \Gamma - \dim F = n-3+2-(n-2) = 1$, so there is a one-dimensional family of cycles passing through x, parametrized by a curve in Σ. As there is a finite map $\Sigma \to \mathrm{Chow}_{n-3}(F)$ (a composition of inclusions and a normalization), this is a non-trivial family of divisors. Hence, it must cover F, in particular there is one divisor passing through y and x. This divisor is dominated by a fibre of $\pi'|_P$, which concludes. $\square$

We now use the fact that π is not a $\mathbb{P}^2$-bundle (in fact, that π is not equidimensional) to construct covering families of rational curves on X. Before that, we prove a simple lemma.

Definition 4.18. Let $f : X \dashrightarrow Y$ be a rational map. We say that f is *almost holomorphic* if there is are Zariski open subsets $U \subset X$ and $V \subset Y$ such that $f|_U : U \to V$ is a proper holomorphic map.

Lemma 4.19. *Let $f : X \dashrightarrow Y$ be almost holomorphic map. If Y is a curve, then f is holomorphic.*

Proof. Let $\varepsilon : X' \to X$ be a resolution of indeterminacies for f, let $f' : X' \to Y$ be the induced holomorphic map. As f is almost holomorphic, no component of the exceptional locus of ε is dominant onto Y. As Y is curve, this means that the exceptional locus of ε is sent onto finitely many points in Y. So f' factors through ε, i.e., f is holomorphic. $\square$

Lemma 4.20. *Let X be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y = 3$. If π is not a $\mathbb{P}^2$-bundle, then any rational curve $C \subset X$ such that $\pi(C) \neq \{\mathrm{pt}\}$, and which deforms in an unsplit family, deforms in a family covering X.*

Proof. Note that if π is equidimensional, by [HN13, Theorem 1.3] it is a $\mathbb{P}^2$-bundle. Hence, we assume that a variety F of dimension 3 is contained in a fibre of π. By contradiction, we consider a rational curve $C \subset X$ such that $\pi(C) \neq \{\mathrm{pt}\}$, and the family $\mathcal{V}$ of deformations of C is unsplit and not covering X.

Fix $x \in \mathrm{Locus}(\mathcal{V})$ general. By Lemma 4.3, $\dim \mathrm{Locus}(\mathcal{V}_x) \leq \dim Y \leq 3$. Since the family $\mathcal{V}$ is unsplit,

$$\dim \mathrm{Locus}(\mathcal{V}) + \dim \mathrm{Locus}(\mathcal{V}_x) \geq -K_X \cdot C + 5 - 1 \geq 7,$$

in particular as $\mathcal{V}$ is not covering, $\dim \mathrm{Locus}(\mathcal{V}) = 4$ and $\dim \mathrm{Locus}(\mathcal{V}_x) = 3$.

Let $n : \tilde{D} \to D$ denote the normalization of $D = \mathrm{Locus}(\mathcal{V})$, and let $\tilde{\mathcal{V}}$ be the covering family on $\tilde{D}$. Note that π induces a fibration of $\tilde{D}$ onto a variety of smaller dimension that is not a point, in particular $\rho(\tilde{D}) \geq 2$. Thus, by [ACO09, Corollary 4.4], $\tilde{D}$ cannot be $\tilde{\mathcal{V}}$-chain-connected.

Considering the dominant almost holomorphic map $r : \tilde{D} \dashrightarrow Z$ whose general fibre is a $\tilde{\mathcal{V}}$-equivalence class [BCD07, Section 2], the variety Z is thus not a point. Since $\dim \mathrm{Locus}(\tilde{\mathcal{V}}_x) = 3$ for a general $x \in \mathrm{Locus}(\tilde{\mathcal{V}})$, the variety Z must be a curve, in particular, by Lemma 4.19, the map r is holomorphic.

Note that, as D is a relatively ample Cartier divisor with respect to π, it intersects the three-dimensional variety F along a surface S. Since $\dim n^{-1}(S) = 2 > \dim Z = 1$, the restriction $r|_{n^{-1}(S)} : n^{-1}(S) \to Z$ cannot be finite. So it contracts a curve B. Its image $n(B)$ is in a $\mathcal{V}$-equivalence class, so as $\mathcal{V}$ is unsplit, it is numerically equivalent to a multiple of C. But $n(B) \subset F$, so this curve is contracted by π, contradiction. $\square$

Definition 4.21. Let $f : X \to Y$ be a finite surejctive map. We say that f is quasiétale if it is étale in codimension 1.

Remark 4.22. Note that if $f : X \to Y$ is quasiétale and Y is smooth, then by Zariski purity of the branch locus, f is étale.

Lemma 4.23. *Let X be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim Y > 0$. If X is not a $\mathbb{P}^2$-bundle over any smooth projective base, then $Y \simeq \mathbb{P}^3$. Moreover, $\rho(X) = 2$, and if C is a line in the smooth locus $Y^0 \subset Y$ of π and s a minimal section over C in X, the class of $s(\mathbb{P}^1)$ generates the other extremal ray in $\overline{NE}(X)$, induces a fibred Mori contraction to a positive dimensional variety too, and satisfies $-K_X \cdot s(\mathbb{P}^1) = 3$.*

Proof. Note that $\dim(Y) = 3$, by Lemmas 4.14, 4.15. By [DP], let C be a minimal free rational curve in the smooth locus $Y^0 \subset Y$ of π. Let s be a minimal section over C. Lemma 4.10 yields

$$4 \geq -K_Y \cdot C \geq -K_X \cdot s(\mathbb{P}^1).$$

The family $\mathcal{V}$ of deformations of $s(\mathbb{P}^1)$ is unsplit. Indeed, suppose by contradiction that it is splitting, i.e. that there is a cycle

$$\sum_i a_i C_i \underset{\text{num}}{\equiv} s(\mathbb{P}^1),$$

with C_i rational curves, $a_i \geq 1$ integers, and $\sum_i a_i \geq 2$. Then, intersecting with $-K_X$ yields $4 \geq -K_X \cdot s(\mathbb{P}^1) \geq 6$, contradiction.

By Lemma 4.20, $\mathcal{V}$ therefore is a covering family. By [Kol96, Proposition IV.2.6], it moreover holds

$$\dim \operatorname{Locus}(\mathcal{V}_x) \geq -K_X \cdot s(\mathbb{P}^1) - 1 \geq 2 = 5 - 3,$$

so by [BCD07, Theorem 2, Proposition 1(i)], there is a geometric quotient $p : X \to Z$, that is a fibred Mori contraction, with general fibre of dimension at least $-K_X \cdot s(\mathbb{P}^1) - 1$. By Lemma 4.14, we have $\dim Z \leq 3$ and by Lemma 4.15, we have $\dim(Z) = 3$, or X is a $\mathbb{P}^2$-bundle over some three-dimensional base. So $\dim Z = 3$, hence $-K_X \cdot s(\mathbb{P}^1) = 3$. It also follows that $s(\mathbb{P}^1)$ is an extremal class in the Mori cone, as wished.

Again, X not being a $\mathbb{P}^2$-bundle over any smooth base, p is not equidimensional by [HN13, Theorem 1.3], so a variety F of dimension 3 is contained in a fibre of p. By Lemma 4.17, the normalization $n : \tilde{F} \to F$ satisfies $\tilde{F} \simeq \mathbb{P}^3$.

Since π and p are distinct Mori contractions, they contract no common numerical class of curve, in particular $\pi|_F : F \to Y$ is finite onto its image, hence finite surjective for dimensional reasons. There is an effective ramification divisor $R \in \operatorname{Pic}(\mathbb{P}^3)$ such that $-K_{\mathbb{P}^3} = n^*\pi|_F{}^*(-K_Y) - R$. As F is an irreducible component of a $\mathcal{V}$-equivalence class, and as $\mathcal{V}$ is unsplit, F contains a deformation of $s(\mathbb{P}^1)$. Let $\tilde{C}$ be the lift to $\tilde{F}$ of a deformation of $s(\mathbb{P}^1)$ that is contained in F. Then $-K_{\mathbb{P}^3} \cdot \tilde{C} \geq 4$, and $n^*\pi|_F{}^*(-K_Y) \cdot \tilde{C} = -K_Y \cdot C \leq 4$. So $R \cdot \tilde{C} \leq 0$, but $R \in \operatorname{Pic}(\mathbb{P}^3)$ is effective, thus ample or trivial, so R is trivial. The finite map $\pi|_F \circ n : \mathbb{P}^3 \to Y$ is thus quasiétale. So, its base change $\mathbb{P}^3 \underset{Y}{\times} X \to X$ is also quasiétale, as $\pi : X \to Y$ contracts no divisor. But X is rationally connected, hence simply-connected, and smooth, so $\mathbb{P}^3 \underset{Y}{\times} X \to X$ is an isomorphism. Hence $\pi|_F \circ n : \mathbb{P}^3 \to Y$ is an isomorphism too.

Since $\rho(Y) = 1$, we have $\rho(X) = 2$. Since $Y \simeq \mathbb{P}^3$ and $4 \geq -K_Y \cdot C$, the curve C is a line. □

Lemma 4.24. *Let X be a smooth projective rationally connected fivefold, such that $-K_X \cdot C \geq 3$ for any rational curve $C \subset X$. Suppose that X has a fibred Mori contraction $\pi : X \to Y$ with $\dim(Y) > 0$. If X is not a $\mathbb{P}^2$-bundle over any smooth projective base, then $\rho(X) = 2$ and X has two distinct fibred Mori contractions onto $\mathbb{P}^3$, with corresponding extremal rays generated by the minimal sections $s(\mathbb{P}^1), \sigma(\mathbb{P}^1)$ above lines that lie in each $\mathbb{P}^3$ in the smooth locus of the fibration. Moreover,*

$$-K_X \cdot s(\mathbb{P}^1) = -K_X \cdot \sigma(\mathbb{P}^1) = 3.$$

Proof. Apply Lemma 4.23 twice. □

Proof of Proposition 4.11. If X has a $\mathbb{P}^2$-bundle structure, then Lemma 4.12 concludes. Suppose that X is not a $\mathbb{P}^2$-bundle. By Lemma 4.24, X admits exactly two fibred Mori contractions π and p, both onto $\mathbb{P}^3$. Given the intersection number of $-K_X$ with both extremal rays, and as $\pi_* s(\mathbb{P}^1)$ is a line in $\mathbb{P}^3$ and as $p_* s(\mathbb{P}^1) = 0$, we have

$$-K_X \cdot s(\mathbb{P}^1) = 3 = \pi^*\mathcal{O}_{\mathbb{P}^3}(3) \cdot s(\mathbb{P}^1) = (\pi^*\mathcal{O}_{\mathbb{P}^3}(3) \otimes p^*\mathcal{O}_{\mathbb{P}^3}(3)) \cdot s(\mathbb{P}^1),$$

and similarly

$$-K_X \cdot \sigma(\mathbb{P}^1) = (\pi^*\mathcal{O}_{\mathbb{P}^3}(3) \otimes p^*\mathcal{O}_{\mathbb{P}^3}(3)) \cdot \sigma(\mathbb{P}^1).$$

Hence, as $\rho(X) = 2$, and $s(\mathbb{P}^1)$ and $\sigma(\mathbb{P}^1)$ are independent,

$$\omega_X{}^* = \pi^*\mathcal{O}_{\mathbb{P}^3}(3) \otimes p^*\mathcal{O}_{\mathbb{P}^3}(3).$$

By Theorem 1.2, $-K_X$ is ample. So X is a Fano fivefold, and we just showed that it has index 3. By the classification in [Wiś91b], X must then be a $\mathbb{P}^2$-bundle, which is a contradiction. □

4.3 Divisorial contractions

Let us classify divisorial Mori contraction of large length.

Proposition 4.25. *Let X be a smooth projective rationally connected variety of dimension n such that $-K_X \cdot C \geq 3$ for every rational curve C. Then X admits no divisorial Mori contraction of length greater or equal to $n-1$.*

Remark 4.26. In particular, the assumptions are fulfilled if there is $1 \leq r \leq n-1$ such that $\bigwedge^r T_X$ is strictly nef, by [LOY19, Theorem 1.2] and Lemma 2.1.

The proof uses the following lemma, that excludes some special contractions of length $n-1$.

Lemma 4.27. *Let X be a smooth projective rationally connected variety of dimension n such that $-K_X \cdot C \geq 3$ for every rational curve C. Then there is no morphism $X \to Y$ that is a blow-up of a smooth point in a smooth variety.*

Proof of Lemma 4.27. By contradiction, consider such a smooth blow-up:

$$f : E \subset X \to p \in Y$$

Note that since X is rationally connected, so Y is too. Let C be a rational curve through p.

Since $-f^*K_Y = -K_X + (n-1)E$ and since no curve is contained in the blown-up locus p, the anticanonical divisor $-K_Y$ is stricly nef. By bend-and-break [Deb01, Proposition 3.2] on the smooth variety Y, one can thus assume $-K_Y \cdot C \leq n+1$. The strict transform $C' \subset X$ of C satisfies $E \cdot C' > 0$. Since $K_X = f^*K_Y + (n-1)E$, we have

$$3 \leq -K_X \cdot C' \leq -K_Y \cdot C - (n-1) \leq 2,$$

contradiction! □

Proof of Proposition 4.25. By Ionescu-Wiśnewski inequality, if X admits a divisorial Mori contraction of length $\ell \geq n-1$, the exceptional divisor E and the general fibre $F \subset E$ satisfy:

$$\dim E + \dim F \geq n + \ell - 1 \geq 2n - 2,$$

i.e., $\ell = n-1$ and $E = F$ is contracted onto a point. So [AO02, Theorem 5.2] applies and shows that this divisorial Mori contraction of X correponds to a blow-up of a smooth point in a smooth variety, which contradicts Lemma 4.27. □

We now consider divisorial Mori contractions of length $n-2$.

Proposition 4.28. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $-K_X \cdot C \geq n-2$ for any rational curve $C \subset X$. Then X has no divisorial Mori contraction contracting the exceptional divisor to a point.*

Remark 4.29. These assumptions are fulfilled if $\bigwedge^4 T_X$ is strictly nef, by [LOY19, Theorem 1.2] and Lemma 2.1.

Proof. Assume that $\varepsilon : X \to Y$ is a divisorial Mori contraction contracting the exceptional divisor E to a point. Note that as X is rationally connected, there exists a rational curve C that intersects E without being contained in E. In particular, $E \cdot C > 0$. Among all such curves, let actually C be one such that $-K_X \cdot C$ is minimal. Then we claim that the family $\mathcal{V}$ of deformations of C is unsplit. Indeed, suppose by contradiction that it is splitting, i.e.,

$$C \underset{\text{num}}{\equiv} \sum_i a_i C_i,$$

with rational curves C_i and coefficients $a_i \geq 1$ such that $\sum a_i \geq 2$. Then $E \cdot C > 0$, so without loss of generality, $E \cdot C_1 > 0$. In particular, C_1 intersects E and is not contracted by ε, hence not contained in E. Since $-K_X$ has positive degree on all rational curves in X, we have $-K_X \cdot C_1 < -K_X \cdot C$, which contradicts the minimality of $-K_X \cdot C$.

By [Kol96, Proposition IV.2.6.1], for a general $x \in \mathrm{Locus}(\mathcal{V})$,

$$\dim \mathrm{Locus}(\mathcal{V}) + \dim \mathrm{Locus}(\mathcal{V}_x) \geq n + n - 2 - 1.$$

In particular, $\dim \mathrm{Locus}(\mathcal{V}_x) \geq n-3$, and as X is smooth, E is Cartier, hence intersects $\mathrm{Locus}(\mathcal{V}_x)$ along a subscheme of dimension at least $n-4 \geq 1$. Let B be a curve in this intersection. It is contained in E, hence contracted by ε, hence satisfies $E \cdot B < 0$. On the other hand, it is contained in $\mathrm{Locus}(\mathcal{V}_x)$, hence is numerically equivalent to a multiple of C by [ACO09, Lemma 4.1]. It has to be a positive multiple, as one sees when intersecting with any ample divisor. But $E \cdot C > 0$, contradiction. □

Corollary 4.30. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $-K_X \cdot C \geq n-2$ for any rational curve $C \subset X$. Suppose that $\varepsilon : X \to Y$ is a divisorial Mori contraction. Then Y is smooth and ε is the blow-up of a smooth curve in Y.*

Proof. Recall [Deb01, Proposition 6.10(b)] that the divisorial Mori contraction ε has a unique exceptional divisor E as its exceptional locus. By [KM98, Lemma 2.62], a ray $\mathbb{R}_+[C]$ associated to ε satisfies $E \cdot C < 0$, so such C has negative intersection with at least one effective divisor. Moreover, ε is a Mori contraction of length $n-2$. So [AO02, Theorem 5.3] applies, showing that ε either contracts a divisor to a point, or is a blow-up of a smooth curve in a smooth variety Y. By Proposition 4.28, only the latter can occur. □

Let us finally describe more precisely what happens in the occurrence of Corollary 4.30.

Lemma 4.31. *Let X be a smooth projective variety of dimension $n \geq 3$, that is rationally connected and such that for some $1 \leq r \leq n-1$, for any rational curve $C \subset X$, it holds $-K_X \cdot C \geq n+2-r$. If there is a morphism $\varepsilon : X \to Y$ that is a blow-up of a smooth curve in the smooth variety Y, then $r = n-1$.*

Proof. Consider such a smooth blow-up:

$$f : E \subset X \to \ell \subset Y$$

As X is rationally connected, so is Y. Fix H an ample divisor on Y. Let $C \subset Y$ be a rational curve other than ℓ passing through a point $p \in \ell$, with $H \cdot C$ minimal among the degrees of all rational curves intersecting ℓ other than ℓ. Fix another point $q \in C \setminus C \cap \ell$. By bend-and-break [Deb01, Proposition 7.3], as Y is smooth, if $-K_Y \cdot C \geq n+2$, then there is a connected non-integral 1-cycle that is a deformation of C passing through p and q. In particular,

$$\sum_{i=1}^{k} a_i C_i \underset{\text{num}}{\equiv} C,$$

with rational curves C_i such that $p \in C_1$, $q \in C_{i_0}$ for some i_0, coefficients $a_i \geq 1$, and $\sum_{i=1}^{k} a_i \geq 2$. As $q \notin \ell$, we have that $C_{i_0} \neq \ell$, so either $C_1 \neq \ell$, or $C_1 = \ell$ and $k \geq 2$. Intersecting with H, we see that $H \cdot C_i < H \cdot C$ for all i, in particular for C_1. If $C_1 \neq \ell$, then $H \cdot C_1$ contradicts the minimality of $H \cdot C$. If $C_1 = \ell$, then $k \geq 2$ and by connectedness of the rational cycle, there is a curve $C_{i_1} \neq \ell$ that intersects $C_1 = \ell$. So $C_{i_1} \neq \ell$ intersects ℓ and contradicts the minimality, as $H \cdot C_{i_1} < H \cdot C$ again. So $-K_Y \cdot C \leq n+1$.

The strict transform $C' \subset X$ of C satisfies $E \cdot C' > 0$. Since $K_X = f^* K_Y + (n-2)E$, and by assumption,

$$n+2-r \leq -K_X \cdot C' \leq -K_Y \cdot C - (n-2) \leq 3,$$

so $r = n-1$. □

Proposition 4.32. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $\bigwedge^4 T_X$ is strictly nef. If there is a morphism $\varepsilon : X \to Y$ that is a blow-up of a smooth curve in the smooth variety Y, then X is a fivefold and there is a fibred Mori contraction $\pi : X \to Z$ with $\dim(Z) > 0$.*

Proof. By Lemma 4.31, we have $n = 5$. So by Theorem 1.2, $-K_X$ is ample. The Mori cone $NE(X)$ is closed, generated by finitely many classes of rational curves. Let E be the exceptional divisor of ε. Note that there exists an extremal ray $R = \mathbb{R}_+[C]$ of $NE(X)$ on which $E \cdot C > 0$. Indeed, if there were not such a ray, then E would be non-positive on all curves in X, which is absurd for an effective divisor. So, let $R = \mathbb{R}_+[C]$ be an extremal ray on which $E \cdot C > 0$.

Denote the associated Mori contraction by $\pi : X \to Z$. Since X already had a non-trivial Mori contraction ε, we have $\dim(Z) > 0$. Let us prove that π is a fibred Mori contraction.

By Lemma 4.1, π cannot be a small contraction. Assume by contradiction that it is a divisorial contraction. By Corollary 4.30, the variety Z is smooth and π is a blow-up along a smooth curve of Z. Let E' be the π-exceptional divisor. Let ℓ, respectively ℓ', be the image of E, respectively E', in Y, respectively Z. Let F' be a general fibre of $\pi|_{E'}$. It has dimension $n-2$. Note that F' and E intersect, since $E \cdot C > 0$. Hence, $E \cap F'$ is a subscheme of X of dimension at least $n-3$. Since ε and π are distinct Mori contractions, the restriction $\varepsilon|_{E\cap F'}$ must be finite onto its image, which is contained in ℓ. So $n-3 \leq 1$, contradiction!

So π is a fibred Mori contraction. $\square$

Proposition 4.33. *Let X be a smooth projective variety of dimension $n \geq 5$, that is rationally connected and such that $\bigwedge^4 T_X$ is strictly nef. If there is a morphism $\varepsilon : X \to Y$ that is a blow-up of a smooth curve, then $Y \simeq \mathbb{P}^5$ and ε is the blow-up of a line.*

Proof. By Proposition 4.32, X is a fivefold and admits a fibred Mori contraction onto a positive dimensional base. So Proposition 4.11 applies, showing that X belongs to a list of certain varieties of Picard number two. Only one of them has a divisorial Mori contraction, namely $\mathrm{Bl}_\ell(\mathbb{P}^5) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$. $\square$

5 Results on $\wedge^3 T_X$

Proof of Theorem 1.3. Note that $-K_X$ is nef, and non-trivial (as it is positive on rational curves by Lemma 2.1, and X is rationally connected by [LOY19, Theorem 1.2]). If $\rho(X) = 1$, $-K_X$ is ample and X is thus a Fano variety. If $\rho(X) \geq 2$, by the Cone Theorem, X admits a Mori contraction, which by Lemma 4.1 and Proposition 4.25 is a fibred Mori contraction. Corollary 4.5 implies that X is a fourfold. By Lemma 4.6, X has an equidimensional fibred Mori contraction to a surface, so by Proposition 4.7, we have $X \simeq \mathbb{P}^2 \times \mathbb{P}^2$. $\square$

Remark 5.1. It is easy to check that $\bigwedge^3 T_{\mathbb{P}^2\times\mathbb{P}^2}$ is ample.

Example 5.2. Let X be a cubic in $\mathbb{P}^n$ with $n \geq 5$. From the tangent exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to \mathcal{O}_X(3) \to 0,$$

we can use [Har77, II.Ex.5.16(d)] to derive the existence of a surjection

$$0 \to F_4 \to \bigwedge^4 T_{\mathbb{P}^n}|_X \to \bigwedge^3 T_X \otimes \mathcal{O}_X(3) \to 0.$$

As $T_{\mathbb{P}^n}|_X \otimes \mathcal{O}_X(-1)$ is nef, the quotient of its fourth exterior power $\bigwedge^3 T_X \otimes \mathcal{O}_X(-1)$ is also nef, and thus $\bigwedge^3 T_X$ is ample.

Example 5.3. Let X be the complete intersection of two quadrics in $\mathbb{P}^n$ with $n \geq 6$. From the tangent exact sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to \mathcal{O}_X(2) \oplus \mathcal{O}_X(2) \to 0,$$

we can use [Har77, II.Ex.5.16(d)] to derive the existence of a surjection

$$0 \to F_4 \to \bigwedge^5 T_{\mathbb{P}^n}|_X \to \bigwedge^3 T_X \otimes \mathcal{O}_X(4) \to 0.$$

As $T_{\mathbb{P}^n}|_X \otimes \mathcal{O}_X(-1)$ is nef, the quotient of its fifth exterior power $\bigwedge^3 T_X \otimes \mathcal{O}_X(-1)$ is also nef, and thus $\bigwedge^3 T_X$ is ample.

6 Results on $\wedge^4 T_X$

6.1 Examples

Lemma 6.1. *Let X be the fivefold $\mathbb{P}(T_{\mathbb{P}^3})$. Then $\bigwedge^4 T_X$ is ample.*

Proof. Denote the natural projection by $p : X \to \mathbb{P}^3$, the tautological line bundle on X by $\mathcal{O}_X(1)$. By [Har77, II.Ex.5.16(d)], there is an exact sequence

$$0 \to \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3} \to \bigwedge^4 T_X \to T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3}) \to 0.$$

Let us prove that $E_1 = T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3})$ is ample. We have the relative Euler sequence

$$0 \to \mathcal{O}_X \to p^*\Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_X(1) \to T_{X/\mathbb{P}^3} \to 0.$$

The bundle E_1 is a quotient of $p^*\Omega^1_{\mathbb{P}^3}(4) \otimes \mathcal{O}_X(1)$. But as $T_{\mathbb{P}^3}$ is ample, $\mathcal{O}_X(1)$ is ample. Moreover, $\Omega^1_{\mathbb{P}^3}(4) \simeq \bigwedge^2 T_{\mathbb{P}^3}$ is ample too, which concludes by [Laz04b, 6.1.16].

Let us prove that $E_2 = \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3}$ is ample. This would settle the ampleness of $\bigwedge^4 T_X$ by [Laz04b, 6.1.13(ii)]. From [Har77, II.Ex.5.16(d)] and the relative Euler sequence, we derive

$$0 \to T_{X/\mathbb{P}^3} \to p^*T_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_X(2) \to \bigwedge^2 T_{X/\mathbb{P}^3} \to 0.$$

Since E_2 is a quotient of $p^*(T_{\mathbb{P}^3}(-4) \otimes \bigwedge^2 T_{\mathbb{P}^3}) \otimes \mathcal{O}_X(2)$, we are left proving that the latter is ample. Notice that $T_{\mathbb{P}^3}(-1)$ is globally generated and thus nef. So the bundle $T_{\mathbb{P}^3}(-3) \otimes \bigwedge^2 T_{\mathbb{P}^3} = T_{\mathbb{P}^3}(-1) \otimes \bigwedge^2 T_{\mathbb{P}^3}(-1)$ is nef as well. Finally, $\mathcal{O}_X(1)$ is ample, and we see that $\mathcal{O}_X(1) \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-1)$ is a quotient of $p^*T_{\mathbb{P}^3}(-1)$ (dualizing the relative Euler exact sequence and twisting by $\mathcal{O}_X(1)$), hence it is nef. We conclude by [Laz04b, 6.2.12(iv)]. □

Lemma 6.2. *Let X be the fivefold $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$. Then $\bigwedge^4 T_X$ is ample.*

Remark 6.3. Note that $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$ is isomorphic to the blow-up of line in $\mathbb{P}^5$ [EH16, Section 9.3.2].

Proof. Denote the natural projection by $p : X \to \mathbb{P}^3$, the tautological line bundle on X by $\mathcal{O}_X(1)$. By [Har77, II.Ex.5.16(d)], there is an exact sequence

$$0 \to \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3} \to \bigwedge^4 T_X \to T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3}) \to 0.$$

Let us prove that $E_1 = T_{X/\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-K_{\mathbb{P}^3})$ is ample. We have the relative Euler sequence

$$0 \to \mathcal{O}_X \to p^*(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1)) \otimes \mathcal{O}_X(1) \to T_{X/\mathbb{P}^3} \to 0.$$

The bundle E_1 is a quotient of $p^*(\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(4) \oplus \mathcal{O}_{\mathbb{P}^3}(4)) \otimes \mathcal{O}_X(1)$. Since $\mathcal{O}_{\mathbb{P}^3}(3) \oplus \mathcal{O}_{\mathbb{P}^3}(4) \oplus \mathcal{O}_{\mathbb{P}^3}(4)$ is ample and $\mathcal{O}_X(1)$ is nef and p-ample, the bundle E_1 is thus ample.

Let us prove that $E_2 = \bigwedge^2 T_{X/\mathbb{P}^3} \otimes p^* \bigwedge^2 T_{\mathbb{P}^3}$ is ample. From [Har77, II.Ex.5.16(d)] and the relative Euler sequence, we derive

$$0 \to T_{X/\mathbb{P}^3} \to p^*(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}) \otimes \mathcal{O}_X(2) \to \bigwedge^2 T_{X/\mathbb{P}^3} \to 0.$$

It is thus enough to prove that $p^* \bigwedge^2 T_{\mathbb{P}^3} \otimes p^*\mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_X(2)$ is ample, which is clear since $\bigwedge^2 T_{\mathbb{P}^3}(-1) = (\bigwedge^2 T_{\mathbb{P}^3})(-2)$ is globally generated and thus nef, and since $p^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_X(2)$ is ample. □

Remark 6.4. It is easy check to that $\bigwedge^4 T_{\mathbb{P}^2 \times \mathbb{P}^3}$, $\bigwedge^4 T_{\mathbb{P}^2 \times Q^3}$, $\bigwedge^4 T_{\mathbb{P}^3 \times \mathbb{P}^3}$ are ample.

6.2 Proof of Theorem 1.4

Proof of Theorem 1.4. Note that $-K_X$ is nef, and non-trivial (as it is positive on rational curves by Lemma 2.1, and X is rationally connected by [LOY19, Theorem 1.2]). If $\rho(X) = 1$, $-K_X$ is ample and X is thus a Fano variety. If $\rho(X) \geq 2$, by the Cone Theorem, X admits a Mori contraction. By Lemma 4.1, it cannot be a small contraction.

Suppose that it is a divisorial contraction. By Corollary 4.30, it is a smooth blow-up of a smooth curve, so by Proposition 4.33, $X \simeq \mathrm{Bl}_\ell \mathbb{P}^5$.

Suppose that X has no divisorial contraction. Then it has a fibred Mori contraction onto a positive dimensional variety. Corollary 4.5 implies that X is a fivefold or a sixfold. If X is a sixfold, by Lemma 4.6, X has an equidimensional fibred Mori contraction to a threefold, so by Proposition 4.7, we have $X \simeq \mathbb{P}^3 \times \mathbb{P}^3$. Else, X is a fivefold with a fibred Mori contraction to a positive dimensional variety. Proposition 4.11 concludes. □

References

[ACO09] M. Andreatta, E. Chierici, and G. Occhetta. Generalized Mukai conjecture for special Fano varieties. *Central Europen Journal of Mathematics*, 2(2):272–293, 2009.

[AO02] M. Andreatta and G. Occhetta. Special rays in the Mori cone of a projective variety. *Nagoya Mathematical Journal*, 168:127–137, 2002.

[BCD07] L. Bonavero, C. Casagrande, and S. Druel. On covering and quasi-covering families of curves. *Journal of the European Mathmatical Society*, 9:45–57, 2007.

[Cam92] F. Campana. Connexité rationnelle des variétés de Fano. *Ann. Sci. École Norm. Sup. (4)*, 25(5):539–545, 1992.

[Cao19] J. Cao. Albanese maps of projective manifolds with nef anticanonical bundles. *Ann. Sci. Éc. Norm. Supér. (4)*, 52(5):1137–1154, 2019.

[CH17] J. Cao and A. Höring. Manifolds with nef anticanonical bundle. *J. Reine Angew. Math.*, 724:203–244, 2017.

[CH19] J. Cao and A. Höring. A decomposition theorem for projective manifolds with nef anticanonical bundle. *J. Algebraic Geom.*, 28(3):567–597, 2019.

[CMSB02] K. Cho, Y. Miyaoka, and N.I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. In *Higher dimensional birational geometry*, volume 35 of *Higher dimensional birational geometry*, pages 1–88. Advanced Studies in Pure Mathematics, Tokyo, 2002.

[CP91] F. Campana and T. Peternell. Projective manifolds whose tangent bundles are numerically effective. *Mathematische Annalen*, 289(1):1–33, 1991.

[CS95] K. Cho and E. Sato. Smooth projective varieties with the ample vector bundle $\bigwedge^2 T_X$ in any characteristic. *J. Math. Kyoto Univ.*, 35(1):1–33, 1995.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[Dem15] J.-P. Demailly. Structure theorems for compact Kähler manifolds with nef anticanonical bundles. In *Complex analysis and geometry*, volume 144 of *Springer Proc. Math. Stat.*, pages 119–133. Springer, Tokyo, 2015.

[Dem18] J.-P. Demailly. Fano manifolds with nef tangent bundles are weakly almost Kähler-Einstein. *Asian J. Math.*, 22(2):285–290, 2018.

[DH17] T. Dedieu and A. Höring. Numerical characterisation of quadrics. *Algebraic Geometry*, 4(1):120–135, 2017.

[DP] S. Druel and M. Paris. Erratum to "characterizations of projective spaces and hyperquadrics". `https://druel.perso.math.cnrs.fr/textes/epq2_erratum.pdf`.

[DPS94] J.-P. Demailly, T. Peternell, and M. Schneider. Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.*, 3(2):295–345, 1994.

[EH16] D. Eisenbud and J. Harris. *3264 and all that—a second course in algebraic geometry.* Cambridge University Press, Cambridge, 2016.

[Erm15] S. Ermakova. Vector bundles of finite rank on complete intersections of finite codimension in ind-Grassmannians. *Complex Manifolds*, 2(1):78–88, 2015.

[Ful98] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, second edition, 1998.

[Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics.* Springer, 1977.

[HN13] A. Höring and C. Novelli. Mori contractions of maximal length. *Publ. Res. Inst. Math. Sci.*, 49(1):215–228, 2013.

[Ien] O. Iena. On symbolic compoutations with Chern classes: Remarks on the library `chern.lib` for `singular`. `https://orbilu.uni.lu/bitstream/10993/21949/2/ChernLib.pdf`.

[Ion86] P. Ionescu. Generalized adjunction and applications. *Mathematical Proceedings of the Cambridge Philosophical Society*, 99(3):457–472, 1986.

[Kan16] A. Kanemitsu. Fano n–folds with nef tangent bundle and Picard number greater than $n-5$. *Math. Z.*, 284:195–208, 2016.

[Kan17] A. Kanemitsu. Fano 5–folds with nef tangent bundles. *Math. Res. Lett.*, 24(5):1453–1475, 2017.

[Kaw91] Y. Kawamata. On the length of an extremal rational curve. *Invent. Math.*, 105(3):609–611, 1991.

[Keb02] S. Kebekus. Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron. In *Complex geometry (Göttingen, 2000)*, pages 147–155. Springer, Berlin, 2002.

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.*, 36(3):765–779, 1992.

[Kol96] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1996.

[KW] A. Kanemitsu and K. Watanabe. Projective varieties with nef tangent bundles in positive characteristics. arXiv:2012.09419.

[Laz04a] R. Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Laz04b] R. Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Li17] Q. Li. Smooth projective horospherical varieties with nef tangent bundles. *Indiana Univ. Math. J.*, 66(6):1879–1893, 2017.

[LOY19] D. Li, W. Ou, and X. Yang. On projective varieties with strictly nef tangent bundles. *J. Math. Pures Appl. (9)*, 128:140–151, 2019.

[Mae93] H. Maeda. A criterion for a smooth surface to be Del Pezzo. *Math. Proc. Cambridge Philos. Soc.*, 113(1):1–3, 1993.

[MnOSC$^+$15] R. Muñoz, G. Occhetta, L. E. Solá Conde, K. Watanabe, and J. A. Wiśniewski. A survey on the Campana-Peternell conjecture. *EUT Edizioni Università di Trieste*, 47:127–185, 2015.

[MnOSCW15] R. Muñoz, G. Occhetta, L. E. Solá Conde, and K. Watanabe. Rational curves, Dynkin diagrams and Fano manifolds with nef tangent bundle. *Math. Ann.*, 361(3):583–609, 2015.

[Mor79] S. Mori. Projective manifolds with ample tangent bundle. *Annals of Mathematics*, 110:593–606, 1979.

[OSS80] C. Okonek, M. Schneider, and H. Spindler. *Vector Bundles on Complex Projective Spaces*. Progress in Mathematics. Birkhäuser, 1980.

[PS98] T. Peternell and F. Serrano. Threefolds with nef anticanonical bundles. volume 49, pages 465–517. 1998. Dedicated to the memory of Fernando Serrano.

[Sat76] E. Sato. Uniform vector bundles on a projective space. *J. Math. Soc. Japan*, 28(1):123–132, 1976.

[Sch] D. Schmitz. On exterior powers of the tangent bundle on toric varieties. arXiv:1811.02603.

[Ser95] F. Serrano. Strictly nef divisors and Fano threefolds. *Journal für die reine und angewandte Mathematik*, 464:187–206, 1995.

[Wat14] K. Watanabe. Fano 5–folds with nef tangent bundles and Picard numbers greater than one. *Math. Z.*, 276:39–49, 2014.

[Wat21a] K. Watanabe. Fano manifolds of coindex three admitting nef tangent bundle. *Geom. Dedicata*, 210:165–178, 2021.

[Wat21b] K. Watanabe. Positivity of the second exterior power of the tangent bundles. *Adv. Math.*, 385:Paper No. 107757, 27, 2021.

[Wiś91a] J.A. Wiśniewski. On contractions of extremal rays of Fano manifolds. *Journal für die reine und angewandte Mathematik*, 417:141–157, 1991.

[Wiś91b] J.A. Wiśniewski. On Fano manifolds of large index. *Manuscripta Math.*, 70(2):145–152, 1991.

[Xie] Z. Xie. Rationally connected threefolds with nef and bad anticanonical divisor. Preprint arXiv:2006.08234.

[Yan] Q. Yang. Toric Fano manifolds with nef tangent bundles. arXiv:1506.05565.

[Zha96] Q. Zhang. On projective manifolds with nef anticanonical bundles. *J. Reine Angew. Math.*, 478:57–60, 1996.