

Finite quotients of abelian varieties with a Calabi-Yau resolution

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Abstract

Let A be an abelian variety, and $G \subset \text{Aut}(A)$ a finite group acting freely in codimension two. We discuss whether the singular quotient A/G admits a resolution that is a Calabi-Yau manifold. While Oguiso constructed two examples in dimension 3 [32], we show that there are none in dimension 4. We also classify up to isogeny the possible abelian varieties A in arbitrary dimension.

1 Introduction

Since singularities are a byproduct of the Minimal Model Program, studying singular varieties with trivial canonical class, or singular K -trivial varieties, is an important question in the birational classification of complex algebraic varieties. From this point of view, the recent generalization of the Beauville-Bogomolov decomposition theorem for smooth K -trivial varieties ([4]) to klt K -trivial varieties ([14, 13, 16, 3]) is highly relevant. It indeed establishes that, after a finite quasiétale cover, any klt K -trivial variety is a product of a smooth abelian variety, some irreducible holomorphic symplectic varieties with canonical singularities, also called hyperkähler varieties, and some Calabi-Yau varieties with canonical singularities. These three main families of K -trivial varieties are the subject of large, mostly disjoint realms of the literature, ranging from the well-known theory of abelian varieties (exposed notably in the reference books [6, 39]), through the thriving study of hyperkähler varieties (see [11, 1, 19] for surveys), to the unruly “Zoo of Calabi-Yau varieties”, populated by a huge amount of examples ([24, 25] for K3 surfaces and Calabi-Yau threefolds embedded as hypersurfaces in toric varieties only), and whose boundedness is yet not established (see [44, 45, 8, 12, 5] for recent breakthroughs).

A new feature appearing in the context of singular K -trivial varieties is that birational morphisms may change the type of the Beauville-Bogomolov decomposition. For example, Kummer surfaces are K3 surfaces, but arise as minimal resolutions of finite quasiétale quotients of abelian surfaces. Similar examples of dimension 3 are numerous, as in [32, 31, 30], and even less well understood in higher dimensions, *cf.* [9, 10, 35, 2, 7]. In arbitrary dimension, it is known that a crepant resolution or terminalization only changes the type of a klt K -trivial variety if its decomposition entails an abelian factor ([13, Prop.4.10]).¹

This paper aims at describing changes of the type of a K -trivial variety through a birational morphism in the simplest case of higher dimension, i.e., when a singular variety with Beauville-Bogomolov decomposition of purely abelian type is resolved by a Calabi-Yau manifold. We work in the following set-up: By a Calabi-Yau manifold, we mean a smooth simply-connected complex projective variety of dimension n with trivial canonical bundle, without any global holomorphic differential form of degree

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¹It reflects the more general fact that the Beauville-Bogomolov decomposition type of a klt K -trivial variety X with non-trivial fundamental group $\pi_1(X_{\text{reg}})$ is not captured by its algebra of global holomorphic differential forms $H^0(X, \Omega_X^{[1]})$. Many examples supporting this fact are exposed in [14, Sec.14], and most notably, smooth K -trivial threefolds with Beauville-Bogomolov decomposition of pure abelian type and algebra of global differential forms generated by the volume form (as for a Calabi-Yau threefold) are classified in [34].

$i \in \llbracket 1, n-1 \rrbracket$. Extending the terminology of [29], we define n -dimensional Calabi-Yau manifolds of type n_0 as follows.

Theorem 1.1. [38],[26, Rem.1.5] *Let X be a Calabi-Yau manifold of dimension n . The following are equivalent:*

- (i) *There is a nef and big divisor D on X such that $c_2(X) \cdot D^{n-2} = 0$.*
- (ii) *There is an abelian variety A and a finite group G acting freely in codimension 2 on A such that X is a crepant resolution of A/G .*

If it satisfies these conditions, X is called a Calabi-Yau manifold of type n_0 .

Calabi-Yau threefolds of type III_0 appear naturally when classifying extremal contractions of Calabi-Yau threefolds [29], and fit in a more general circle of ideas on how the cubic intersection form and the second Chern class determine the birational geometry of a Calabi-Yau threefold (see, e.g., the work of Wilson [43], Oguiso and Peternell [33]). Calabi-Yau threefolds of type III_0 were classified by Oguiso, as we now recall.

Theorem 1.2. [32] *There are exactly two Calabi-Yau threefolds X_3, X_7 of type III_0 . They are the unique crepant resolution of E_j^3 quotiented by the group generated by jid_3 , and of $E_{u_7}^3$ quotiented by the group generated by:*

$$\begin{pmatrix} 0 & -8 & 7 - 10u_7 \\ 1 & -6 - 2u_7 & 11 - u_7 \\ 0 & -1 - 2u_7 & 6 + 3u_7 \end{pmatrix}.$$

where $j = e^{2i\pi/3}$, $\zeta_7 = e^{2i\pi/7}$, $u_7 = \zeta_7 + \zeta_7^2 + \zeta_7^4 = \frac{-1+i\sqrt{7}}{2}$, and for any complex number $z \in \mathbb{C} \setminus \mathbb{R}$, we denote by E_z the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus z\mathbb{Z})$.

Our first theorem restricts the isogeny type of A in arbitrary dimension.

Theorem 1.3. *Let A be an abelian variety of dimension n and G be a finite group acting freely in codimension 2 on A . If A/G has a crepant resolution that is a Calabi-Yau manifold, then A is isogenous to E_j^n or to $E_{u_7}^n$ and G is generated by its elements that admit fixed points in A .*

Moreover, the local geometry of A/G is generally quite similar to the 3-dimensional model (see Theorem 1.6 below). Two important consequences of this are the following results.

Theorem 1.4. *Let A be an abelian variety and G be a finite group acting freely in codimension 3 on A . Then A/G has no simply-connected crepant resolution.*

Theorem 1.5. *Let A be an abelian variety and G be a finite group acting freely in codimension 2 on A . If A/G has a simply-connected crepant resolution, then $\dim(A) \neq 4$.*

We know no example in dimension $n \geq 5$ either, and the discussion at the end of the introduction leads us to conjecture that there are none.

On the one hand, the Calabi-Yau assumption is crucial in Theorem 1.3, as it rules out products of the 3-dimensional examples of Oguiso, e.g., $X_3 \times X_7$, which is a resolution of a finite quotient of $E_j^3 \times E_{u_7}^3$. On the other hand, Theorem 1.5 merely requires the simply-connectedness of a crepant resolution. Let us explain why. Note that, if A is an abelian variety and G is a finite group acting freely in codimension 2 on A , then A/G cannot have a holomorphic symplectic resolution X . Indeed, a holomorphic symplectic resolution provides $(A/G)_{\text{reg}}$ with a holomorphic symplectic form. By [28, Thm, Cor.1] then, since A/G is smooth in codimension 2, it is terminal. As it is \mathbb{Q} -factorial as well, it thus admits no crepant resolution. By the Beauville-Bogomolov decomposition theorem, a smooth simply-connected K -trivial fourfold which is not holomorphic symplectic is a Calabi-Yau fourfold, whence the weaker assumption of Theorem 1.6.

The structure of the paper is as follows. Sections 2 to 10 build up to the proof of the main technical result.

Theorem 1.6. *Let A be an abelian variety of dimension n and G be a finite group acting freely in codimension 2 on A . If A/G has a crepant resolution that is a Calabi-Yau manifold, then*

- (1) *A is isogenous to E_j^n or to $E_{u_7}^n$, and G is generated by its elements that admit fixed points in A .*
- (2) *For every translated abelian subvariety $W \subset A$, there is $k \in \mathbb{N}$ such that the pointwise stabilizer*

$$\text{PStab}(W) := \{g \in G \mid \forall w \in W, g(w) = w\}$$

is isomorphic to \mathbb{Z}_3^k if A is isogenous to E_j^n , or to \mathbb{Z}_7^k if A is isogenous to $E_{u_7}^n$.

- (3) *For every translated abelian subvariety $W \subset A$, if $\text{PStab}(W)$ is isomorphic to*
 - \mathbb{Z}_3^k , *then there are k generators of it such that their matrices are similar to $\text{diag}(\mathbf{1}_{n-3}, j, j, j)$, and the j -eigenspaces of these matrices are in direct sum.*
 - \mathbb{Z}_7^k , *then there are k generators of it such that their matrices are similar to $\text{diag}(\mathbf{1}_{n-3}, \zeta_7, \zeta_7^2, \zeta_7^4)$, and all eigenspaces of these matrices with eigenvalues other than 1 are in direct sum.*

Our starting point in Section 2 is a necessary condition for a local quotient singularity to have a crepant resolution. The result is the following (Proposition 2.7): If $H \subset \text{GL}_n(\mathbb{C})$ is a finite group, and $0 \in U \subset \mathbb{C}^n$ is an H -stable analytic open set such that U/H admits a crepant resolution, then H is generated by its so-called *junior* elements, i.e., elements M with eigenvalues $(e^{2i\pi a_k/d})_{1 \leq k \leq n}$ satisfying $0 \leq a_k \leq d-1$ and $\sum a_k = d$.

Matrices inducing actions on abelian varieties satisfy a rationality requirement [6, 1.2.3], which translates into arithmetic constraints on their characteristic polynomial. These constraints allow us to classify matrices of junior elements g acting on n -dimensional abelian varieties up to similarity: In Section 3, we prove that if a junior element g acts on an abelian variety in a way that the generated group $\langle g \rangle$ acts freely in codimension 2, then the matrix of g is of one of twelve possible types (see Proposition 3.1). In particular, the order of g and the number of non-trivial eigenvalues of g are bounded independently of the dimension n .

The next step is to show that ten out of the twelve types of junior elements can not belong to G , for a mix of local and global reasons. The proof spreads throughout Sections 4, 5, 7 and 8. Let us sketch the idea of the argument in the simplest case, namely if g is a junior element of composite order other than 6, with at most four non-trivial eigenvalues. If such a junior element g belongs to G , then some non-trivial power g^α is not junior, and has a larger fixed locus in A . Fix an irreducible component W of that larger fixed locus that is not in the fixed locus of g : the pointwise stabilizer $\text{PStab}(W) \subset G$ does not contain g , but the power g^α . Now, as W has codimension less than 4, Section 4 shows that $\text{PStab}(W)$ is cyclic generated by one junior element h , and thus, up to possibly replacing h by another junior generator of $\text{Fix}(W)$, one has $g^\alpha = h^\alpha$. For well-chosen α , this is enough to yield $g = h$, and a contradiction.

This idea excludes seven out of the twelve types of junior elements (see Subsection 5.A). The three types of junior elements of order 6 are excluded by technical variations in the next sections. Ruling them out works along with classifying pointwise stabilizers in higher codimension: In codimension 4, Section 4 establishes cyclicity of the pointwise stabilizers and Section 5 deduces that junior elements with four non-trivial eigenvalues do not exist; in codimension 5 (Section 7), we first prove that junior elements with five non-trivial eigenvalues do not exist (Subsection 7.A), then deduce cyclicity of the pointwise stabilizers (Subsection 7.B). In codimension 6 (Section 8), we first classify pointwise stabilizers which do not contain junior elements of type $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega)$: they are isomorphic to $\mathbb{Z}_3, \mathbb{Z}_7, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_7 \times \mathbb{Z}_7$, or $\text{SL}_2(\mathbb{F}_3)$ (Subsection 8.A). We use this partial classification to rule out junior elements with six non-trivial eigenvalues (Subsection 8.B), and we then finally refine the study of pointwise stabilizers in codimension 6 by ruling out $\text{SL}_2(\mathbb{F}_3)$ (Subsection 8.C).

There finally remain two types of possible junior elements, which are those already appearing in dimension 3 in [32]: $\text{diag}(\mathbf{1}_{n-3}, j, j, j)$ and $\text{diag}(\mathbf{1}_{n-3}, \zeta_7, \zeta_7^2, \zeta_7^4)$.

This description of pointwise stabilizers in codimension up to 6 implies that any two junior elements admitting a common fixed point commute. Together with a simple argument about the isogeny type of A (see Section 6), it concludes the proof of Theorem 1.6. In fact, the idea that the existence of certain automorphisms on an abelian variety determines the isomorphism type of some special abelian subvarieties is general ([39]), and it applies crucially throughout this paper, starting in Section 4. From there, it is not so surprising that we are able to determine the isogeny type of A , interpreting the fact that A/G admits a Calabi-Yau resolution as an irreducibility property of the G -equivariant Poincaré decomposition of A .

Under the additional assumption that the group G is abelian, Theorem 1.6 and the results of Section 6 suffice to generalize Theorem 1.5 to higher dimensions, i.e., to the statement that, if A is an abelian variety of dimension n and G is a finite group acting freely in codimension 2 on A such that A/G admits a Calabi-Yau resolution X , then $n = 3$ and X is X_3 or X_7 .

Also note that G is abelian if and only if any two junior elements g, h of G commute, which by our results can be checked *via* their matrices acting on a vector space V of dimension 3, 4, 5, or 6. Standard finite group theory allows us to explicitly bound the order of $\langle g, h \rangle$ depending on this dimension and the isogeny type of A . If the dimension is 3 or 4, the bounds are reasonable enough to launch a computer-assisted search through all possible abstract groups $\langle g, h \rangle$. Among these, the only groups which, in a faithful 3 or 4-dimensional representation, are generated by two junior elements of the same type, are \mathbb{Z}_3 , \mathbb{Z}_7 , and the finite simple group $\text{SL}_3(\mathbb{F}_2)$ of order 168. But a geometric argument on fixed loci excludes $\text{SL}_3(\mathbb{F}_2)$, whence the wished contradiction. This reproves the classification of [32] in dimension 3, and settles Theorem 1.5.

When V has dimension 5 or 6, we could also bound the order of $\langle g, h \rangle$ explicitly. For example, we could consider the image of the faithful representation $M \oplus \overline{M}$ in $\text{SL}_{2 \dim(V)}(\mathbb{Q})$, and use the classification of irreducible maximal finite integral matrix groups in dimension less than 12 by V. Felsch, G. Nebe, W. Plesken, and B. Souvignier to obtain a bound on the order of $\langle g, h \rangle$. But the bounds obtained in this way are too large for the `SmallGroup` library. One needs to better understand the arising matrix groups of larger order, and build a reasonably smaller finite list of possibilities for the abstract group $\langle g, h \rangle$. It will then remain to figure out geometric ways for ruling out those potential groups in the list other than \mathbb{Z}_3 , \mathbb{Z}_7 , $\mathbb{Z}_3 \times \mathbb{Z}_3$, and $\mathbb{Z}_7 \times \mathbb{Z}_7$.

Some of our proofs resort to computer-searches among all finite groups of certain fixed orders (relying on the `SmallGroup` library of `GAP`). The computer-assisted results used in Subsection 4.C were actually originally proven by hand using elementary representation theory and Sylow theory. Such arguments being standard in finite group theory, we chose to keep their exposition concise for the sake of readability, and preferred invoking computer-checked facts as black boxes when needed. This approach also has the advantage of better separating abstract group-theoretic arguments on G from properties of the particular representation $G \hookrightarrow \text{GL}(H^0(T_A))$. All programs used are available in the Appendix.

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2 Some results in McKay correspondence

Definition 2.1. Let g be a matrix in $GL_n(\mathbb{C})$. Assume that it has finite order d . Since $g^d = \text{id}$, g is diagonalizable and has eigenvalues of the form $e^{2i\pi a_k/d}$, for integers $a_k \in \llbracket 0, d-1 \rrbracket$ satisfying:

$$a_1 \wedge \dots \wedge a_n \wedge d = 1.$$

Ordering the integers a_k increasingly, we define the *ranked vector of eigenvalues* of g as the tuple $(e^{2i\pi a_k/d})_{1 \leq k \leq n}$.

The *age* of g is set to be the number $\frac{a_1 + \dots + a_n}{d}$. If it equals 1, we say that g is *junior*.

Definition 2.2. If A is an abelian variety of dimension n and $g \in \text{Aut}(A)$ has finite order, then g can be written as:

$$g : [z] \in A \mapsto [M(g)z + T(g)] \in A,$$

where $M(g)$ is a matrix of finite order in $GL_n(\mathbb{C})$, $T(g)$ a vector in \mathbb{C}^n . If g fixes any point a of A , it can be represented locally in a neighborhood of a by its matrix $M(g)$. Hence, it makes sense to say that the automorphism g is *junior* if g fixes at least one point in A and the matrix $M(g)$ is junior.

Remark 2.3. Note that if $g \in \text{Aut}(A)$ admits a fixed point, then $\langle g \rangle$ contains no translation, so g and its matrix $M(g)$ have the same order.

Junior elements play a key role in the study of finite quotient singularities, as the following theorem emphasizes.

Theorem 2.4. [18] *Let \mathbb{C}^n/G be a finite Gorenstein quotient singularity, and suppose that it has a minimal model X . Then there is a natural one-to-one correspondence between conjugacy classes of junior elements in G and prime exceptional divisors in X .*

Remark 2.5. Note that such a minimal model X always exists as a relative minimal model of a resolution $\tilde{X} \rightarrow \mathbb{C}^n/G$, by [22, 1.30.6].

Quotient singularities are \mathbb{Q} -factorial, so they can not be resolved by small birational morphisms. This yields a simple corollary of the theorem.

Corollary 2.6. [18] *Let \mathbb{C}^n/G be a finite Gorenstein quotient singularity, with G acting freely in codimension 1. If the singularity \mathbb{C}^n/G admits a crepant resolution, then there is a junior element $g \in G$.*

In fact, [18, Par.4.5] conjectures that under the same hypotheses, if the singularity \mathbb{C}^n/G admits a crepant resolution, then any maximal cyclic subgroup of G contains a junior element. A counterexample to this conjecture is however presented in Remark 8.15. In this section, we prove a weak version of that conjecture. We phrase it in an analytic set-up, as our later applications call for that, but the proof works in the affine set-up just as well.

Proposition 2.7. *Let $G \subset GL_n(\mathbb{C})$ be a finite group acting freely in codimension 1 on \mathbb{C}^n , and let $U \subset \mathbb{C}^n$ be a G -stable simply-connected analytic neighborhood of $0 \in \mathbb{C}^n$. If the singularity U/G admits a crepant resolution X_G , then the group G is generated by junior elements.*

Note that a singularity admitting a crepant resolution is Gorenstein. By [20][42], the existence of a crepant resolution X_G thus implies that $G \subset SL_n(\mathbb{C})$.

In order to prove the proposition, we need some background in valuation theory.

2.A Introduction to valuation theory for singularities

Recall that an *integral valuation* v on a field K is a function $\nu : K \rightarrow \mathbb{Z} \cup \{+\infty\}$ that satisfies, for all $a, b \in K$,

- $v(a) = +\infty$ if and only if $a = 0$;

- $v(a + b) \geq \min(v(a), v(b))$;
- $v(ab) = v(a) + v(b)$.

A *discrete valuation* is an integral valuation which is surjective onto $\mathbb{Z} \cup \{+\infty\}$.

Example 2.8. We say that E is a divisor *over* a normal complex analytic variety X if there is a *partial resolution* of X , i.e., a normal complex analytic variety \tilde{X} with a proper birational morphism $\varphi : \tilde{X} \rightarrow X$, such that E is a φ -exceptional prime divisor. We say that the partial resolution φ *realizes* E . To such a divisor we associate a discrete valuation on the function field of X :

$$v_E : f \in k(X) \mapsto \text{ord}_E(f \circ \varphi) \in \mathbb{Z} \cup \{+\infty\},$$

which does not depend on the partial resolution φ chosen. A valuation of this form is called a *divisorial valuation*.

Example 2.9. [18] Let $g \in \text{SL}_n(\mathbb{C})$ be a matrix of finite order d . We can take coordinates x_1, \dots, x_n on \mathbb{C}^n that diagonalize g , so that for any $k \in \llbracket 1, n \rrbracket$, $g^*x_k = e^{2i\pi a_k/d}x_k$, with $a_k \in \llbracket 0, d-1 \rrbracket$. We define the integral valuation:

$$v_g : x_k \in k(\mathbb{C}^n) \mapsto a_k \in \mathbb{Z} \cup \{+\infty\}.$$

If g is junior, it holds $a_1 \wedge \dots \wedge a_n = 1$, and thus v_g is then a discrete valuation.

Remark 2.10. Note that the correspondence in Theorem 2.4 is just the identification of the set of divisorial valuations v_E , when E is a crepant divisor over \mathbb{C}^n/G , and the set of valuations v_g , when g is a junior element in G .

Definition 2.11. Let X, Y be normal complex analytic varieties, and $p : X \rightarrow Y$ be a finite Galois morphism of group G . Let v, w be discrete valuations on the function fields $k(X)$ and $k(Y)$. Note that $k(Y)$ identifies with the invariant subfield $k(X)^G$ of $k(X)$. The *ramification index* $\text{Ram}(v, k(Y))$ of v over $k(Y)$ is the unique non-negative integer such that:

$$v(k(Y)^*) = \text{Ram}(v, k(Y))\mathbb{Z}.$$

We say that v is an *extension* of w to $k(X)$ if:

$$w = \frac{1}{\text{Ram}(v, k(Y))} v|_{k(Y)}.$$

If v is an extension of w , then by [46, Ch.VI, Par.12], the set of all extensions of w is exactly $\{v \circ g \mid g \in G\}$. In particular, all extensions of w have the same ramification index.

Remark 2.12. When considering divisorial valuations, ramification indices and extension properties carry a geometrical meaning. Let X, Y be normal complex analytic varieties endowed with their sheaves of holomorphic functions \mathcal{H}_X and \mathcal{H}_Y . Let $p : X \rightarrow Y$ be a finite Galois morphism of group G , and let E, F be prime divisors in X, Y . The local rings $\mathcal{H}_{Y,E}$ and $\mathcal{H}_{X,F}$ are discrete valuation rings for the valuations v_E and v_F .

If we assume that F dominates E , then $p : X \rightarrow Y$ induces an injective morphism of local rings $p^\sharp : \mathcal{H}_{Y,E} \rightarrow \mathcal{H}_{X,F}$ by [40, Lem.29.8.6]. The maximal ideals $m_E \subset \mathcal{H}_{Y,E}$ and $m_F \subset \mathcal{H}_{X,F}$ relate by $p^\sharp(m_E) = m_F^r$, where r is the ramification index of F over E . Hence $v_F|_{\mathcal{H}_{Y,E}} = r v_E$, i.e., v_F is an extension of v_E to $k(X)$ with ramification index $\text{Ram}(v_F, k(Y)) = r$.

Conversely, if we assume that v_F is an extension of v_E to $k(X)$, then the structure sheaf map $p^\sharp : \mathcal{H}_Y \rightarrow p_*\mathcal{H}_X$ sends the ideal sheaf \mathcal{I}_E to a subsheaf of $p_*\mathcal{I}_F$, so F dominates E .

Another important concept when considering ramification of valuations over subfields is the following.

Definition 2.13. Let X, Y be normal complex analytic varieties, and $p : X \rightarrow Y$ be a finite Galois morphism of group G . Let v be a discrete valuation on $k(X)$. Let $R_v \subset k(X)$ be the valuation ring, and $m_v \subset R_v$ be the unique maximal ideal. We define the *inertia group*

$$G_T(v) := \{g \in G \mid \forall x \in R_v, gx - x \in m_v\}.$$

Proposition 2.14. [46, p.77, Cor.] *If the residue field R_v/m_v has characteristic zero, then the inertia group G_T is cyclic of order $\text{Ram}(v, k(Y))$.*

Proposition 2.15. [18, Cor.2.7 and p.11, Par.1] *Suppose that U is an open simply-connected subset of \mathbb{C}^n , G is a finite subgroup of $\text{GL}_n(\mathbb{C})$ stabilizing U , and $p : U \rightarrow U/G = Y$ is the quotient map. Let $h \in \text{GL}_n(\mathbb{C})$ be a junior element. Then:*

$$G_T(v_h) = G \cap \langle h \rangle.$$

2.B Proof of Proposition 2.7

Let G be a finite subgroup of $\text{GL}_n(\mathbb{C})$ acting freely in codimension 1 on \mathbb{C}^n , and U be a G -stable simply-connected neighborhood of $0 \in \mathbb{C}^n$. Suppose that U/G has a crepant resolution X_G .

Set G_0 to be the subgroup of G generated by all junior elements. We have the following commutative diagram, constructed from the lower row up:

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{p}} & X_0 & \xrightarrow{\tilde{q}} & X_G \\ \varepsilon \downarrow & & \varepsilon_0 \downarrow & & \varepsilon_G \downarrow \\ U & \xrightarrow{p} & U/G_0 & \xrightarrow{q} & U/G \end{array}$$

The commutative squares containing the normal complex analytic varieties X_0, X are obtained by taking normalized fibered products. Since quotient singularities are locally \mathbb{Q} -factorial, all birational morphisms considered here are divisorial. The morphisms $p, q, \tilde{p}, \tilde{q}$ are finite, and $\varepsilon, \varepsilon_0, \varepsilon_G$ are proper birational.

The key fact is the following.

Lemma 2.16. *The prime exceptional divisors of ε_0 are crepant.*

Proof. Let E_0 be a prime exceptional divisor of ε_0 , and denote by E its image in X_G . Since E is an exceptional divisor of ε_G , it is crepant. Let F be a prime exceptional divisor of ε dominating E_0 . By Theorem 2.4 and Remark 2.10, there is a junior element $f \in G$ such that $v_F = v_f$. We can compute the following ramification index.

$$|\text{Ram}(E_0/E)| = \frac{|\text{Ram}(F/E)|}{|\text{Ram}(F/E_0)|} = \frac{|\text{Ram}(v_f, k(U/G))|}{|\text{Ram}(v_f; k(U/G_0))|} = \frac{|\langle f \rangle \cap G|}{|\langle f \rangle \cap G_0|} = 1$$

so E_0 is generically étale over E , hence crepant [23, Prop.5.20]. \square

By this lemma, the finite proper morphism $\tilde{q} : X_0 \rightarrow X_G$ has no ramification divisor. By Zariski purity of the branch locus, since X_G is smooth, the morphism \tilde{q} is unramified, hence étale by [15, Ex.III.10.3, Ex.III.10.9].

On the other hand, X_G is locally simply-connected by [21, Thm.7.5.2]: There is a contractible neighborhood V of $0 \in U/G$, such that $\varepsilon_G^{-1}(V)$ is simply-connected. Hence the following commutative diagram.

$$\begin{array}{ccc} \varepsilon_0^{-1}(q^{-1}(V)) & \xrightarrow{\tilde{q}} & \varepsilon_G^{-1}(V) \\ \varepsilon_0 \downarrow & & \varepsilon_G \downarrow \\ q^{-1}(V) & \xrightarrow{q} & V \end{array}$$

As \tilde{q} is étale, the pre-image $\varepsilon_0^{-1}(q^{-1}(V))$ is a disjoint union of $\deg(\tilde{q})$ copies of $\varepsilon_G^{-1}(V)$. Nevertheless, the morphism ε_0 has connected fibers and the base $q^{-1}(V)$ is itself connected, hence $\varepsilon_0^{-1}(q^{-1}(V))$ is connected, and

$$\deg(\tilde{q}) = \frac{|G|}{|G_0|} = 1,$$

so $G_0 = G$ and the proof of Proposition 2.7 is settled.

2.C Global result along the same lines

We close this section with a global result along the same lines as Proposition 2.7.

Lemma 2.17. *Let G be a finite group acting freely in codimension 1 on an abelian variety A . Suppose that A/G has a crepant resolution X_G that is simply-connected. Then G is generated by its elements admitting fixed points in A .*

Proof. Let $G_0 \triangleleft G$ be the normal subgroup of G generated by elements admitting fixed points. We want to prove that $G_0 = G$. We have a commutative diagram:

$$\begin{array}{ccc} X_0 & \xrightarrow{\tilde{q}} & X_G \\ \downarrow \varepsilon_0 & & \downarrow \varepsilon_G \\ A/G_0 & \xrightarrow{q} & A/G \end{array}$$

By definition of G_0 , for every $a \in A$, the stabilizers of a in G and G_0 coincide. Hence, q is étale, and \tilde{q} is étale too by base change. But X_G is simply-connected and X_0 is connected, so $\deg(\tilde{q}) = 1$ and $G_0 = G$. \square

Remark 2.18. If G is a finite group acting freely in codimension 1 on an abelian variety A so that A/G has a simply-connected crepant resolution, then G may still contain elements that admit no fixed point. Without loss of generality, we can assume that G contains no translation, up to replacing A by an isogenous abelian variety, but that is the best we can do.

3 The twelve types of junior elements on an abelian variety

Section 2 just shows that, if we want a finite singular quotient of an abelian variety A/G to have a crepant resolution, the group G must contain some junior elements. The fact that in our set-up, G must also act freely in codimension 2 on A is restrictive enough that there are only twelve possibilities for the ranked vector of eigenvalues of a junior element $g \in G$.

Proposition 3.1. *Let A be an abelian variety of dimension n , and $g \in \text{Aut}(A)$ be a junior element such that $\langle g \rangle$ acts freely in codimension 2. Then the order d of g and the ranked vector of eigenvalues of g are in one of the twelve columns of Table 1.*

d	3	4	6
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-3}, j, j, j)$	$(\mathbf{1}_{n-4}, i, i, i, i)$	$(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$
d	6	6	7
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$	$(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$	$(\mathbf{1}_{n-3}, \zeta_7, \zeta_7^2, \zeta_7^4)$
d	8	12	15
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-4}, \zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3)$	$(\mathbf{1}_{n-4}, \zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5)$	$(\mathbf{1}_{n-4}, \zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$
d	16	20	24
$(e^{2i\pi a_k/d})$	$(\mathbf{1}_{n-4}, \zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7)$	$(\mathbf{1}_{n-4}, \zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9)$	$(\mathbf{1}_{n-4}, \zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11})$

Table 1: Possible ranked vectors of eigenvalues for junior elements in G
For $d \in \mathbb{N}$, we denote $\zeta_d = e^{2i\pi/d}$, and in particular $j = e^{2i\pi/3}$ and $\omega = e^{2i\pi/6}$. For $k \in \mathbb{N}$, $\mathbf{1}_k$ refers to a sequence of k times the symbol 1 in a row.

The proof goes by elementary arithmetic and meticulous case disjunctions. The following terminology should simplify the exposition.

Definition 3.2. A *multiset* \mathbf{A} is the data of a set A and a function $m : A \rightarrow \mathbb{Z}_{>0}$, called the *multiplicity function*. Intuitively, a multiset is like a set where elements are allowed to appear more than once.

If a multiset $\mathbf{A} = (A, m)$ is *finite*, i.e., its underlying set $A = \{a_1, \dots, a_k\}$ is finite, we may write A in the following form:

$$\{\{\underbrace{a_1, \dots, a_1}_{m(a_1) \text{ times}}, \dots, \underbrace{a_k, \dots, a_k}_{m(a_k) \text{ times}}\}\}.$$

Double-braces are used to avoid confusion between the multiset and the underlying set.

Let $\mathbf{A} = (A, m)$ be a finite multiset.

If $\alpha \in \mathbb{Z}_{>0}$ and \mathbf{A} , we denote by $\mathbf{A}^{*\alpha}$ the multiset $(A, \alpha m)$.

If A a subset of \mathbb{Q} , and p, q are rational numbers, with $q \neq 0$, we denote by $p + q\mathbf{A}$ the multiset $(p + qA, m)$.

The *cardinal* of \mathbf{A} is:

$$|\mathbf{A}| := \sum_{a \in \mathbf{A}} m(a).$$

More generally, if $f : A \rightarrow \mathbb{Q}$ is a function, we define:

$$\sum_{a \in \mathbf{A}} f(a) := \sum_{a \in A} m(a)f(a).$$

If $\mathbf{A} = (A, m)$ and $\mathbf{B} = (B, n)$ are two multisets, we define their *union*:

$$\mathbf{A} \cup \mathbf{B} := (A \cup B, \mathbf{1}_A m + \mathbf{1}_B n),$$

where $\mathbf{1}_A, \mathbf{1}_B$ are the indicator functions of A and B .

Notation 3.3. For $d \in \mathbb{N}$, we denote by Φ_d the d -th cyclotomic polynomial, and by $\phi(d)$ the degree of Φ_d . In other terms, ϕ is the Euler indicator function.

For integers a, b , the greatest common divisor of a and b is denoted $a \wedge b$.

We establish a sequence of three useful lemmas.

Lemma 3.4. *Let u be a positive integer strictly greater than 2. Then we have:*

$$\frac{\phi(u)^2}{u} \leq 8 \text{ or } \left(2 \mid u \text{ and } \frac{\phi(u)^2}{u} \leq 4 \right)$$

$$\Leftrightarrow u \in \llbracket 3, 10 \rrbracket \cup \{12, 14, 15, 16, 18, 20, 21, 24, 30, 36, 42\}.$$

Proof. Write $u = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < \dots < p_k$ are prime numbers, and $\alpha_1, \dots, \alpha_k$ positive integers, so that:

$$\frac{\phi(u)^2}{u} = \prod_{i=1}^k (p_i - 1)^2 p_i^{\alpha_i - 2}.$$

Each of the k factors of this product is greater or equal to 1, unless $p_1^{\alpha_1} = 2$ in which case the first factor is $\frac{1}{2}$.

Hence, if u satisfies:

$$\frac{\phi(u)^2}{u} \leq 8 \text{ or } \left(2 \mid u \text{ and } \frac{\phi(u)^2}{u} \leq 4 \right),$$

then each factor satisfies:

$$(p_i - 1)^2 p_i^{\alpha_i - 2} \leq 8, \quad (1)$$

which yields $p_i \in \{2, 3, 5, 7\}$. Writing $u = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $\alpha, \beta, \gamma, \delta \geq 0$ and using Inequality (1) again bounds $\alpha \leq 4$, $\beta \leq 2$, $\gamma \leq 1$, $\delta \leq 1$. Among the finitely many possibilities left, it is easy to check that the solutions exactly are $u \in \llbracket 3, 10 \rrbracket \cup \{12, 14, 15, 16, 18, 20, 21, 24, 30, 36, 42\}$. \square

Lemma 3.5. *Let $u \geq 2$ and $d \geq 3$ be integers, such that u divides d . Suppose that there are a positive integer α and a multiset \mathbf{A} such that:*

$$\mathbf{A} \cup (d - \mathbf{A}) = \left\{ \left\{ a \in \llbracket 1, d - 1 \rrbracket \mid u = \frac{d}{d \wedge a} \right\} \right\}^{*\alpha},$$

and such that the quantity:

$$S_{\mathbf{A}, d}(u) := \sum_{a \in \mathbf{A}} \frac{a}{u(a \wedge d)}$$

satisfies $S_{\mathbf{A}, d}(u) \leq 1$. Then u , $\frac{1}{d}\mathbf{A}$, α , $S_{\mathbf{A}, d}(u)$ are classified in Table 2.

Proof. We consider the following function.

$$f : a \in A \cup (d - A) \mapsto \frac{a}{a \wedge d} = \frac{ua}{d}.$$

Clearly, f is an increasing function, and takes values in $\{\ell \in \llbracket 1, u - 1 \rrbracket \mid \ell \wedge u = 1\}$. It is in fact a bijection, with converse

$$g : \ell \in \{\ell \in \llbracket 1, u - 1 \rrbracket \mid \ell \wedge u = 1\} \mapsto \frac{d\ell}{u}.$$

So $|A| \geq \frac{\phi(u)}{2}$. The restriction $f|_A$ is injective, hence takes at least $\frac{\phi(u)}{2}$ distinct values in its image set, so that:

$$1 \geq S_{\mathbf{A}, d}(u) = \frac{1}{u} \sum_{a \in \mathbf{A}} f(a) \geq \frac{\alpha}{u} \left(\sum_{\substack{1 \leq \ell \leq u/2 \\ \ell \wedge u = 1}} \ell \right). \quad (2)$$

Let us denote by $\Sigma(u)$ the sum $\sum_{\substack{1 \leq \ell \leq u/2 \\ \ell \wedge u = 1}} \ell$. We have the following coarse estimates:

$$u \geq \Sigma(u) \geq \sum_{\ell=1}^{\phi(u)/2} \ell \geq \frac{\phi(u)^2}{8}, \text{ and, if } u \text{ is even, } u \geq \Sigma(u) \geq \sum_{\ell=1}^{\phi(u)/2} 2\ell - 1 \geq \frac{\phi(u)^2}{4}.$$

Applying Lemma 3.4, these coarse estimates yield finitely many possibilities for u . Computing explicitly $\frac{1}{u}\Sigma(u)$ for the possible values and applying Inequality 2 again, we exclude a few of them, finally obtaining that:

$$u \in \llbracket 2, 10 \rrbracket \cup \{12, 14, 15, 16, 18, 20, 24\}.$$

For each u , we then list by hand the finitely many possibilities for the multiplicity α and the multiset $\frac{1}{d}\mathbf{A}$, and this is how we construct Table 2. \square

u	α	$\frac{1}{d}A$	$S_{A,d}(u) \leq 1$
2	1	$\{\frac{1}{2}\}$	$\frac{1}{2}$
	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1
3	1	$\{\frac{1}{3}\}, \{\frac{2}{3}\}$	$\frac{1}{3}, \frac{2}{3}$
	2	$\{\frac{1}{3}, \frac{1}{3}\}, \{\frac{1}{3}, \frac{2}{3}\}$	$\frac{2}{3}, 1$
	3	$\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$	1
4	1	$\{\frac{1}{4}\}, \{\frac{1}{4}\}$	$\frac{1}{4}, \frac{3}{4}$
	2	$\{\frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{4}, \frac{3}{4}\}$	$\frac{1}{2}, 1$
	3	$\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$	$\frac{3}{4}$
	4	$\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$	1
5	1	$\{\frac{1}{5}, \frac{2}{5}\}, \{\frac{1}{5}, \frac{3}{5}\}$	$\frac{3}{5}, \frac{4}{5}$
6	1	$\{\frac{1}{6}\}, \{\frac{5}{6}\}$	$\frac{1}{6}, \frac{5}{6}$
	2	$\{\frac{1}{6}, \frac{1}{6}\}, \{\frac{1}{6}, \frac{5}{6}\}$	$\frac{1}{3}, 1$
	3	$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	$\frac{1}{2}$
	4	$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	$\frac{2}{3}$
	5	$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	$\frac{5}{6}$
	6	$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	1
7	1	$\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}\}, \{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$	$\frac{6}{7}, 1$
8	1	$\{\frac{1}{8}, \frac{3}{8}\}, \{\frac{1}{8}, \frac{5}{8}\}$	$\frac{1}{2}, \frac{3}{4}$
	2	$\{\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}\}$	1
9	1	$\{\frac{1}{9}, \frac{2}{9}, \frac{4}{9}\}, \{\frac{1}{9}, \frac{2}{9}, \frac{5}{9}\}$	$\frac{7}{9}, \frac{8}{9}$
10	1	$\{\frac{1}{10}, \frac{3}{10}\}, \{\frac{1}{10}, \frac{7}{10}\}$	$\frac{2}{5}, \frac{4}{5}$
	2	$\{\frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}\}$	$\frac{4}{5}$
12	1	$\{\frac{1}{12}, \frac{5}{12}\}, \{\frac{1}{12}, \frac{7}{12}\}$	$\frac{1}{2}, \frac{2}{3}$
	2	$\{\frac{1}{12}, \frac{1}{12}, \frac{5}{12}, \frac{5}{12}\}$	1
14	1	$\{\frac{1}{14}, \frac{3}{14}, \frac{5}{14}\}, \{\frac{1}{14}, \frac{3}{14}, \frac{9}{14}\}$	$\frac{9}{14}, \frac{13}{14}$
15	1	$\{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{7}{15}\}, \{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}\}$	$\frac{14}{15}, 1$
16	1	$\{\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}\}$	1
18	1	$\{\frac{1}{18}, \frac{5}{18}, \frac{7}{18}\}, \{\frac{1}{18}, \frac{5}{18}, \frac{11}{18}\}$	$\frac{13}{18}, \frac{17}{18}$
20	1	$\{\frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{9}{20}\}$	1
24	1	$\{\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}\}$	1

Table 2: Possibilities for $u, \frac{1}{d}A, \alpha, S_{A,d}(u)$ such that $S_{A,d}(u) \leq 1$

Lemma 3.6. *Let $k \in \mathbb{N}$. For each $m \in \llbracket 1, k \rrbracket$, let $u_m \geq 2$ and $d_m \geq 3$ be integers, such that u_m divides d_m , and suppose that there are a positive integer α_m and a multiset \mathbf{A}_m such that:*

$$\mathbf{A}_m \cup (d_m - \mathbf{A}_m) = \left\{ \left\{ a \in \llbracket 1, d_m - 1 \rrbracket \mid u_m = \frac{d_m}{d_m \wedge a} \right\} \right\}^{*\alpha_m}.$$

Suppose additionally that:

$$\sum_{m=1}^k S_{\mathbf{A}_m, d_m}(u_m) = 1.$$

Then the data of k and of all $u_m, \alpha_m, \frac{1}{d_m} \mathbf{A}_m$ is classified in Table 3.

Proof. It is easily derived by hand from Table 2. □

u_1, \dots, u_k	$\alpha_1, \dots, \alpha_k$	$\frac{1}{d_1} A_1, \dots, \frac{1}{d_k} A_k$	freeness in codimension 2
2	2	$\{\frac{1}{2}, \frac{1}{2}\}$	✗
2, 3, 6	1, 1, 1	$\{\frac{1}{2}\}, \{\frac{1}{3}\}, \{\frac{1}{6}\}$	✗
2, 4	1, 2	$\{\frac{1}{2}\}, \{\frac{1}{4}, \frac{1}{4}\}$	✗
2, 6	1, 3	$\{\frac{1}{2}\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	✓
2, 8	1, 1	$\{\frac{1}{2}\}, \{\frac{1}{8}, \frac{3}{8}\}$	✗
2, 12	1, 1	$\{\frac{1}{2}\}, \{\frac{1}{12}, \frac{5}{12}\}$	✗
3	2	$\{\frac{1}{3}, \frac{2}{3}\}$	✗
	3	$\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$	✓
3, 4, 6	1, 2, 1	$\{\frac{1}{3}\}, \{\frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{6}\}$	✗
3, 6	1, 2	$\{\frac{2}{3}\}, \{\frac{1}{6}, \frac{1}{6}\}$	✗
	1, 4	$\{\frac{1}{3}\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	✓
	2, 2	$\{\frac{1}{3}, \frac{1}{3}\}, \{\frac{1}{6}, \frac{1}{6}\}$	✗
3, 12	1, 1	$\{\frac{1}{3}\}, \{\frac{1}{12}, \frac{7}{12}\}$	✗
4	2	$\{\frac{1}{4}, \frac{3}{4}\}$	✗
	4	$\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$	✓
4, 6	2, 3	$\{\frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	✗
4, 8	1, 1	$\{\frac{1}{4}\}, \{\frac{1}{8}, \frac{5}{8}\}$	✗
	2, 1	$\{\frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{8}, \frac{3}{8}\}$	✗
4, 12	2, 1	$\{\frac{1}{4}, \frac{1}{4}\}, \{\frac{1}{12}, \frac{5}{12}\}$	✗
5, 10	1, 1	$\{\frac{1}{5}, \frac{2}{5}\}, \{\frac{1}{10}, \frac{3}{10}\}$	✗
6	2	$\{\frac{1}{6}, \frac{5}{6}\}$	✗
	6	$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$	✓
6, 8	3, 1	$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}, \{\frac{1}{8}, \frac{3}{8}\}$	✗
6, 12	2, 1	$\{\frac{1}{6}, \frac{1}{6}\}, \{\frac{1}{12}, \frac{7}{12}\}$	✗
	3, 1	$\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}, \{\frac{1}{12}, \frac{5}{12}\}$	✗
7	1	$\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$	✓
8	2	$\{\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}\}$	✓
8, 12	1, 1	$\{\frac{1}{8}, \frac{3}{8}\}, \{\frac{1}{12}, \frac{5}{12}\}$	✗
12	2	$\{\frac{1}{12}, \frac{1}{12}, \frac{5}{12}, \frac{5}{12}\}$	✓
15	1	$\{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}\}$	✓
16	1	$\{\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}\}$	✓
20	1	$\{\frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{9}{20}\}$	✓
24	1	$\{\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}\}$	✓

Table 3: Possibilities for k parcels of data $u_m, \alpha_m, \frac{1}{d_m} \mathbf{A}_m$ such that

$$\sum_{m=1}^k S_{\mathbf{A}_m, d_m}(u_m) = 1$$

We also recall a simple fact from the theory of abelian varieties:

Lemma 3.7. *Let A be an abelian variety of dimension n , and $g \in \text{Aut}(A)$ of finite order. Denote by $P(g)$ the characteristic polynomial of $M(g)$. Then $P(g)\overline{P(g)}$ is a product of cyclotomic polynomials.*

Proof. By [6, Prop.1.2.3], the matrix $M(g) \oplus \overline{M(g)}$ belongs to $\text{GL}_{2n}(\mathbb{Q})$. Hence, $P(g)\overline{P(g)}$ is a polynomial over \mathbb{Q} . Since g has finite order, the roots of this polynomial are roots of unity. Remembering that cyclotomic polynomials are the minimal polynomials of roots of unity over \mathbb{Q} , an easy induction shows that there is a product Π of cyclotomic polynomials that has the exact same roots as $P(g)\overline{P(g)}$. But since both cyclotomic polynomials and characteristic polynomials are unitary, it means that $P(g)\overline{P(g)} = \Pi$. \square

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Denote by d the order of the junior element g , by $(e^{2i\pi a_j/d})_{1 \leq j \leq n}$ its ranked vector of eigenvalues, and by $P(g)$ the characteristic polynomial of its matrix $M(g)$. As g itself acts freely in codimension 2 and g is junior, it must be that $d \geq 3$.

By Lemma 3.7, there are positive integers k , $(u_m)_{1 \leq m \leq k}$ ordered increasingly, and $(\alpha_m)_{1 \leq m \leq k}$, such that:

$$\prod_{j=1}^n (X - e^{2i\pi a_j/d})(X - \overline{e^{2i\pi a_j/d}}) = P(g)\overline{P(g)} = \prod_{m=1}^k \Phi_{u_m}^{\alpha_m}. \quad (3)$$

Note that $\Phi_{u_m}(e^{2i\pi a_j/d}) = 0$, or equivalently $\Phi_{u_m}(\overline{e^{2i\pi a_j/d}}) = 0$, if and only if $u_m = \frac{d}{d \wedge a_j}$. We define the following partition of $\llbracket 1, n \rrbracket$

$$\begin{aligned} \text{for } m \in \llbracket 1, k \rrbracket, \quad I_m &:= \{j \in \llbracket 1, n \rrbracket \mid u_m = \frac{d}{d \wedge a_j}\}; \\ \mathbf{A}_m &:= \{\{a_j \mid j \in I_m\}\}, \text{ as a multiset.} \end{aligned}$$

By Identity 3, for $m \in \llbracket 1, k \rrbracket$ we have:

$$\mathbf{A}_m \cup (d - \mathbf{A}_m) = \{\{r \in \llbracket 1, d-1 \rrbracket \mid u_m = \frac{d}{d \wedge r} = 0\}\}^{*\alpha_m} \quad (4)$$

Moreover, since g is junior:

$$1 = \sum_{j=1}^n \frac{a_j}{d} = \sum_{m=1}^k \sum_{j \in I_m} \frac{a_j}{d} = \sum_{m=1}^k \sum_{j \in I_m} \frac{a_j}{u_m(d \wedge a_j)} = \sum_{m=1}^k S_{\mathbf{A}_m, d}(u_m). \quad (5)$$

So, possibly leaving out the data of index 1, if $u_1 = 1$ (which is determined by the multiplicity $\alpha_1 \in \mathbb{N}$, since then $\mathbf{A}_1 = \{\{\mathbf{0}_{\alpha_1}\}\}$ and $S_{\mathbf{A}_1, d}(u_1) = 0$), Lemma 3.6 applies, showing that there are finitely many possibilities for

$$k, (u_m)_{1 \leq m \leq k}, (\alpha_m)_{1 \leq m \leq k}, \left(\frac{1}{d}\mathbf{A}_m\right)_{1 \leq m \leq k}$$

and listing them. We exclude by hand a lot of these possibilities using the assumption that $\langle g \rangle$ acts freely in codimension 2 on A , i.e., that for all $\ell \in \llbracket 1, d-1 \rrbracket$, there must be distinct indices $j_1(\ell), j_2(\ell), j_3(\ell) \in \llbracket 1, n \rrbracket$, such that none of the $\frac{\ell a_{j_i(\ell)}}{d}$ is an integer. What remains then is precisely the list in Table 1. \square

4 Cyclicity of the pointwise stabilizers of loci of codimension 3 and 4

We now know that G is generated by junior elements, which we have classified into twelve different types. However, this is by far insufficient to determine the structure of G . Even locally, for $W \subset A$ a subvariety, the pointwise stabilizer

$$\text{PStab}(W) := \{g \in G \mid \forall w \in W, g(w) = w\}$$

could as well be cyclic and generated by one junior element, as it could be more complicated, e.g., if it contained non-commuting junior elements.

In this section, we show that in fact, if W has codimension 3 or 4 in A , $\text{PStab}(W)$ is trivial or cyclic, generated by one junior element. Let us outline the proof. Subsection 4.A reduces to proving this in the case when W is a point in an abelian variety B of dimension 3 or 4. Up to conjugating the whole group G by a translation, we therefore just work on the case $W = \{0\}$. Assuming $\text{PStab}(W)$ is not trivial, we can then find a junior element $g \in \text{PStab}(W)$, that is of one of the twelve types of Section 3. Subsection 4.B exhibits a correlation between the type of g and the isogeny type (possibly even isomorphism type) of the abelian variety B on which it acts. A corollary is that if $g, h \in \text{PStab}(W)$ are two junior elements, then they should either have the same type, or one is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$ and the other $(\mathbf{1}_{n-3}, j, j, j)$, or one is of type $(\mathbf{1}_{n-4}, i, i, i, i)$ and the other $(\mathbf{1}_{n-4}, \zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5)$. In particular, if $\text{PStab}(W)$ is cyclic, it must indeed be generated by one junior element. The conclusive Subsection 4.C is the most technical. For any given abelian three- or fourfold B of one of the types just defined, we classify all finite subgroups of

$$\text{Aut}(B, 0) := \{f \in \text{Aut}(B) \mid f(0) = 0, \text{ i.e., } T(f) = 0\}$$

that act freely in codimension 2 on B and are generated by junior elements. The main idea is to bound the order of such groups, to scrutinize the finite list arising, and to rule out all but the cyclic case of the list by the assumption on generators.

4.A Reduction to a 3 or 4-dimensional question

Definition 4.1. Let A be an abelian variety. An *abelian subvariety* of A is a closed subvariety of A that is also a subgroup of the abelian group $(A, +)$. A *translated abelian subvariety* of A is the image by a translation of an abelian subvariety of A .

We say that two translated abelian subvarieties B and C of A are *complementary* if one of the following equivalent statements hold:

- (i) $B \cap C$ is non-empty and, for some $p \in B \cap C$, it holds:

$$H^0(\{p\}, T_B) \oplus H^0(\{p\}, T_C) = H^0(\{p\}, T_A).$$

- (ii) The addition map $i : B \times C \rightarrow A$ is an isogeny.

Proof. (i) \Rightarrow (ii): as the translation by (p, p) , respectively by $2p$, is an isomorphism from $B \times C$ to $(B - p) \times (C - p)$, respectively of A , it is enough to prove the statement for $p = 0$. As $\dim(A) = \dim(B \times C)$ and the varieties are regular, we simply check that i is quasi-finite. Since $B \cap C$ is the intersection of two abelian subvarieties of A satisfying:

$$H^0(\{0\}, T_B) \cap H^0(\{0\}, T_C) = \{0\},$$

the set $B \cap -C$ is discrete in A , hence finite. For $a \in \text{Im}(i)$, say $a = i(a_B, a_C)$, we can express the fiber $i^{-1}(a) = \{(b + a_B, -b + a_C) \mid b \in B \cap -C\}$, so it is finite, and i is indeed quasi-finite.

(ii) \Rightarrow (i): fix $c_0 \in C$. The addition i is onto, so let $(p, c) \in B \times C$ be such that $p + c = 2c_0$. Clearly, $p = 2c_0 - c \in B \cap C$, and as i is locally analytically an isomorphism,

$$H^0(\{p\}, T_B) \oplus H^0(\{p\}, T_C) = H^0(\{2p\}, T_A) = H^0(\{p\}, T_A).$$

□

Remark 4.2. If B and C are complementary translated abelian subvarieties of an abelian variety A , and $t \in A$ is any point, then $B + t$ and C are complementary as well. Our notion of complementarity is weaker than the notion defined for abelian subvarieties in [6, p.125].

Let us now state our reduction result. Note that it applies not only in codimension 3 and 4, but in any higher codimension as well.

Proposition 4.3. *Let A be an abelian variety, G be a finite group acting freely in codimension 2 on A . Suppose that the quotient A/G admits a crepant resolution. Let W be a subvariety of codimension m in A such that $\text{PStab}(W) \neq \{1\}$. Then:*

- (1) *For any $t \in W$ there is a translated abelian subvariety B of A which is $\text{PStab}(W)$ -stable, contains t , and is complementary to W in A .*
- (2) *If t and B are as such, then an element $g \in \text{PStab}(W)$ is junior if and only if $g|_B \in \text{Aut}(B, t)$ is a junior element.*
- (3) *The group $\text{PStab}(W) \subset \text{Aut}(B, t)$ is generated by junior elements.*

Proof of Proposition 4.3. Up to conjugating the G -action by the translation by t , we can assume that $t = 0$. Let us establish (1): As G is finite, we can take a G -invariant polarization L on A . We can apply [6, Prop.13.5.1]: there is a unique complementary abelian subvariety $(B, L|_B)$ to $(W, L|_W)$ in (A, L) , and it is $\text{PStab}(W)$ -stable. By Remark 4.2, B and W are complementary in our sense as well.

We now prove (2): let $g \in \text{PStab}(W)$. As g fixes all points of $B \cap W$, its restriction $g|_B$ has a fixed point. As $g(B) = B$, we have:

$$M(g) = \begin{pmatrix} \text{id}_{\dim(W)} & 0 \\ 0 & M(g|_B) \end{pmatrix},$$

and thus g is indeed junior if and only if $g|_B$ is.

We move on to (3). Take a general point $w \in W$ such that $\text{PStab}(w) = \text{PStab}(W)$. Since $\text{PStab}(w)$ is finite, any analytic neighborhood of w in A contains a contractible analytic neighborhood U of w that is $\text{PStab}(w)$ -stable. Up to reducing it even more, we can assume that for any $g \in G \setminus \text{PStab}(w)$, $g(U) \cap U = \emptyset$. So, an analytic neighborhood of $[w] \in A/G$ is biholomorphic to $U/\text{PStab}(w)$. Hence, Proposition 2.7 applies and $\text{PStab}(w)$ is generated by junior elements. \square

4.B The abelian varieties corresponding to the twelve juniors

Let A be an abelian variety of dimension n , G be a finite group acting freely in codimension 2 on A such that A/G has a crepant resolution. By Corollary 2.6, $G \subset \text{Aut}(A)$ must entail a junior element presented in Table 1 (up to its translation part, and up to similarity for its linear part). The fact that, in some coordinates, a given matrix of Table 1 acts as an automorphism on the abelian variety A imposes some restrictions. Using the theory of abelian varieties with complex multiplication, these restrictions are investigated by Proposition 4.6.

Notation 4.4. Let us defined the following quadratic integers.

$$u_7 = \frac{-1 + i\sqrt{7}}{2}, u_8 = i\sqrt{2}, u_{15} = \frac{1 + i\sqrt{15}}{2}, u_{20} = i\sqrt{5}, u_{24} = i\sqrt{6},$$

$$u_{16} = i\sqrt{4 + 2\sqrt{2}}, v_{16} = i\sqrt{4 - 2\sqrt{2}}.$$

For $z \in \mathbb{C} \setminus \mathbb{R}$, we define the elliptic curve $E_z := \mathbb{C}/\mathbb{Z} \oplus z\mathbb{Z}$. If z is a quadratic integer, then we denote by $\mathbb{Z}[z]$ the \mathbb{Z} -algebra that it generates. It holds $\mathbb{Z}[z] = \mathbb{Z} \oplus z\mathbb{Z} \subset \mathbb{C}$. We also define the simple abelian surface $S_{u_{16}, v_{16}} := \mathbb{C}^2/\mathbb{Z}[(u_{16}, v_{16})]$.

Remark 4.5. Note that the simplicity of S_{16} follows from [39, Prop.27].

With these notations, we can state the main result of the subsection.

Proposition 4.6. *Let A be an abelian variety. Suppose that there is a junior element $g \in \text{Aut}(A)$, and that $\langle g \rangle$ acts freely in codimension 2 on A . Denote by W an irreducible component of $\text{Fix}(g) := \{a \in A \mid g(a) = a\}$. Let B be a complementary to W in A . Then the isogeny type of B is entirely determined by the type of the junior element g by Table 4, unless g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$. Moreover, the isomorphism type of a $\langle g \rangle$ -stable complementary B_{st} to W in A is also entirely determined by the type of g , unless g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$ or $(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$.*

type of g	isogeny type of B	isomorphism type of B_{st}
$(\mathbf{1}_{n-3}, j, j, j)$	E_j^3	E_j^3
$(\mathbf{1}_{n-4}, i, i, i, i)$	E_i^4	E_i^4
$(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$	$E \times E_j^3$, for some elliptic curve E	not determined
$(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$	E_j^5	not determined
$(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$	E_j^6	E_j^6
$(\mathbf{1}_{n-3}, \zeta_7, \zeta_7^2, \zeta_7^4)$	$E_{u_7}^3$	$E_{u_7}^3$
$(\mathbf{1}_{n-4}, \zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3)$	$E_{u_8}^4$	$E_{u_8}^4$
$(\mathbf{1}_{n-4}, \zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5)$	E_i^4	E_i^4
$(\mathbf{1}_{n-4}, \zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$	$E_{u_{15}}^4$	$E_{u_{15}}^4$
$(\mathbf{1}_{n-4}, \zeta_{15}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7)$	$S_{u_{16}, v_{16}}^2$	$S_{u_{16}, v_{16}}^2$
$(\mathbf{1}_{n-4}, \zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9)$	$E_{u_{20}}^4$	$E_{u_{20}}^4$
$(\mathbf{1}_{n-4}, \zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11})$	$E_{u_{24}}^4$	$E_{u_{24}}^4$

Table 4: Correspondence between types of junior elements and types of abelian varieties.

Notation 4.7. Let V be a \mathbb{C} -vector space, $f : V \rightarrow V$ be a linear map. We denote by $\text{EVal}(f)$ the set of eigenvalues of f in \mathbb{C} , by $\mathbf{EVal}(f)$ the multiset of eigenvalues of f in \mathbb{C} counted with multiplicities. If $\lambda \in \text{EVal}(f)$, we denote by $E_f(\lambda)$ the eigenspace of f for the eigenvalue λ .

We denote by $Z(\Phi_d) \subset \mathbb{U}_d$ the set of primitive d -th roots of unity in \mathbb{C} .

Let us first carry out an important computation, that makes plain where these special types of abelian varieties come from. Let $k \geq 3$ be an integer. There is a natural action of $\zeta_k \otimes 1$ on the algebra $\mathbb{Z}[\zeta_k] \otimes \mathbb{C}$. We compute its eigenvalues. By definition, $\mathbb{Z}[\zeta_k] \otimes \mathbb{C}$ is the quotient algebra $\mathbb{C}[X]/(\Phi_k)$, multiplication by $\zeta_k \otimes 1$ corresponding to multiplication by the class $X + \Phi_k \mathbb{C}[X]$. So $\xi \in \mathbb{C}$ is an eigenvalue with eigenvector $P + \Phi_k \mathbb{C}[X]$ if and only if $P \notin \mathbb{C}[X]$ and $XP - \xi P \in \Phi_k \mathbb{C}[X]$, or equivalently, ξ is a root of Φ_k and $P \in \frac{\Phi_k}{X-\xi} \mathbb{C}[X]$. Hence the linear decomposition

$$\mathbb{Z}[\zeta_k] \otimes \mathbb{C} = \bigoplus_{\xi \in Z(\Phi_k)} E_{\zeta_k \otimes 1}(\xi) \quad (6)$$

Now, consider a subset S_k of $Z(\Phi_k)$ such that $S_k \overline{S_k} = Z(\Phi_k)$. For example, if we let g be a junior element of one of the twelve types in Table 1, and we assume that g has an eigenvalue of order k , we could set $S_k = S_k(g) = \text{EVal}(g) \cap Z(\Phi_k)$. This defines a \mathbb{Z} -linear inclusion

$$f(S_k) : \mathbb{Z}[\zeta_k] \hookrightarrow \bigoplus_{\xi \in S_k} E_{\zeta_k \otimes 1}(\xi) \simeq \mathbb{C}^{\phi(k)/2} \quad (7)$$

It is worth noting that the \mathbb{Z} -linear inclusion $f(S_k) \oplus \overline{f(S_k)}$ corresponds to the natural inclusion of $\mathbb{Z}[\zeta_k]$ in $\mathbb{Z}[\zeta_k] \otimes \mathbb{C}$ given by Identity 6.

The following lemma is key.

Lemma 4.8. *If $S_k = S_k(g)$ for a junior element g of Table 1, then the corresponding abelian variety $\mathbb{C}^{\phi(k)/2} / \text{Im}(f(S_k))$ is described in Table 5.*

Remark 4.9. For $k = 3, 4, 6, 8, 12$, we have $S_k = \{j\}, \{i\}, \{j\}, \{\zeta_8, \zeta_8^3\}$, and $\{\zeta_{12}, \zeta_{12}^5\}$ respectively, and Lemma 4.8 is [6, Cor.13.3.4, Cor.13.3.6]. In the other cases, the computation relies on the same ideas as [6, Cor.13.3.6], as we will soon see.

k	S_k	$\mathbb{C}^{\phi(k)/2}/\text{Im}(f(S_k))$
3	$\{j\}$	E_j
4	$\{i\}$	E_i
6	$\{\omega\}$	E_j
7	$\{\zeta_7, \zeta_7^2, \zeta_7^4\}$	$E_{u_7}^3$
8	$\{\zeta_8, \zeta_8^3\}$	$E_{u_8}^2$
12	$\{\zeta_{12}, \zeta_{12}^5\}$	E_i^2
15	$\{\zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8\}$	$E_{u_{15}}^4$
16	$\{\zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7\}$	$S_{u_{16}, v_{16}}^2$
20	$\{\zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9\}$	$E_{u_{20}}^4$
24	$\{\zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11}\}$	$E_{u_{24}}^4$

Table 5: Computing $\mathbb{C}^{\phi(k)/2}/\text{Im}(f(S_k))$ for given S_k stemming from a junior element.

To complete the proof Lemma 4.8 for $k = 7, 15, 16, 20, 24$, we use a part of [39, Proof of Thm.3], recalled here without proof.

Lemma 4.10. *Let $K = \mathbb{Q}(\alpha)$ be a totally imaginary quadratic extension of \mathbb{Q} of degree $2m$. Let F be a finite Galois extension of K , of degree $2r$ over \mathbb{Q} . Let $\{\varphi_i\}_{1 \leq i \leq r}$ be morphisms of \mathbb{Q} -algebras defined from F to \mathbb{C} such that:*

$$\text{Hom}_{\mathbb{Q}\text{-alg}}(F, \mathbb{C}) = \text{Vect}_{\mathbb{Q}}(\varphi_1, \overline{\varphi_1}, \dots, \varphi_r, \overline{\varphi_r}).$$

Suppose also that no two of the restrictions $\varphi_i|_K$ are conjugate.

Then we can restrict m of these morphisms, defining $\psi_j = \varphi_{i_j}|_K$ for some distinct i_j with $j \in [1, m]$, such that:

$$\text{Hom}_{\mathbb{Q}\text{-alg}}(K, \mathbb{C}) = \text{Vect}_{\mathbb{Q}}(\psi_1, \overline{\psi_1}, \dots, \psi_m, \overline{\psi_m}).$$

We obtain a \mathbb{Z} -algebra $\Delta := \mathbb{Z}[(\psi_1(\alpha), \dots, \psi_m(\alpha))]$ that is a lattice of rank $2m$ in \mathbb{C}^m . The complex torus $A := (\mathbb{C}^m/\Delta)^{n/m}$ is an abelian variety of CM-type $(F, \{\varphi_i\}_{1 \leq i \leq r})$.

Proof of Lemma 4.8. Let $F = \mathbb{Q}[\zeta_k]$, $r = \frac{\phi(k)}{2}$. Let us define $\{\varphi_i\}_{1 \leq i \leq r}$: composing $f(S_k)$ defined in Identity 7 with the projections on the r eigenspaces, we obtain morphisms of \mathbb{Z} -algebras $f_i : \mathbb{Z}[\zeta_k] \rightarrow \mathbb{C}$, which we tensor by \mathbb{Q} and normalize to define morphisms of \mathbb{Q} -algebras:

$$\varphi_i = \frac{1}{f_i(1)}(f_i \otimes 1) : \mathbb{Q}[\zeta_k] \rightarrow \mathbb{C}.$$

By Identities 6 and 7, the morphisms $\{\varphi_i, \overline{\varphi_i}\}_{1 \leq i \leq r}$ are linearly independent over \mathbb{Q} , whereas the morphisms $\{\varphi_i\}_{1 \leq i \leq r}$ define an embedding of F into the \mathbb{Q} -algebra of linear endomorphisms of the abelian variety $\mathbb{C}^{\phi(k)/2}/\text{Im}(f(S_k))$. In other words, the abelian variety $\mathbb{C}^{\phi(k)/2}/\text{Im}(f(S_k))$ has CM-type $(F, \{\varphi_i\}_{1 \leq i \leq r})$. This is in fact the sole abelian variety with this CM-type, by [27],[39, Prop.17] remembering that $k \in \{7, 15, 16, 20, 24\}$.

Applying Lemma 4.10 with $K = \mathbb{Q}(u_k)$, we get the wished description of the abelian variety $\mathbb{C}^{\phi(k)/2}/\text{Im}(f(S_k))$, by an easy verification involving that:

- $u_7 = \zeta_7 + \zeta_7^2 + \zeta_7^4$,
- $u_{15} = \zeta_{15} + \zeta_{15}^2 + \zeta_{15}^4 + \zeta_{15}^8$,
- $u_{16} = \zeta_{16} + \zeta_{16}^3 + \zeta_{16}^5 + \zeta_{16}^7$ and $v_{16} = \zeta_{16}^3 + \zeta_{16}^5 + \zeta_{16}^9 + \zeta_{16}^{15}$,
- $u_{20} = \zeta_{20} + \zeta_{20}^3 + \zeta_{20}^7 + \zeta_{20}^9$,
- $u_{24} = \zeta_{24} + \zeta_{24}^5 + \zeta_{24}^7 + \zeta_{24}^{11}$.

□

The next result follows almost effortlessly from the ideas of [32, p.333-334].

Lemma 4.11. *Let B be an abelian variety. Suppose that there is an automorphism g of B whose set of eigenvalues is one of the S_k in Table 5. Then B is isomorphic to a power of the abelian variety $\mathbb{C}^{\phi(k)/2}/\text{Im}(f(S_k))$.*

Proof. Let Λ be a lattice in \mathbb{C}^n such that $B = \mathbb{C}^n/\Lambda$. The linear action of g restricting to Λ , it provides it with a $\mathbb{Z}[g]$ -module, i.e., a $\mathbb{Z}[\zeta_k]$ -module structure, since the minimal polynomial of g is Φ_k . As such, Λ is finitely-generated and torsion-free. But by [27], since $k \in \llbracket 3, 20 \rrbracket \cup \{24\}$, the ring of cyclotomic integers $\mathbb{Z}[\zeta_k]$ is a principal ideal domain. So, by the structure theorem for finitely-generated modules over principal ideal domains, $\Lambda \simeq \mathbb{Z}[\zeta_k]^{2n/\phi(k)}$, and the action of g on Λ identifies with the multiplication by ζ_k on $\mathbb{Z}[\zeta_k]^{2n/\phi(k)}$.

The embedding $\Lambda \hookrightarrow H^0(B, T_B) \simeq \mathbb{C}^n$ can be recovered from the action of g on Λ . Indeed, there is an induced action of $g \oplus \bar{g}$ on $\Lambda \otimes \mathbb{C} = H^0(B, T_{B, \mathbb{R}} \otimes \mathbb{C}) \simeq \mathbb{C}^{2n}$. This action splits into two blocks: g is acting on $H^0(B, T_B)$ and \bar{g} is acting on its supplementary conjugate in $H^0(B, T_{B, \mathbb{R}} \otimes \mathbb{C})$. By the requirement on its set of eigenvalues S_k , g has no eigenvalue in common with \bar{g} , and therefore:

$$H^0(B, T_B) = \bigoplus_{\xi \in \mathbf{EVal}(g)} E_{g \oplus \bar{g}}(\xi).$$

Hence, the corresponding embedding $\mathbb{Z}[\zeta_k]^{2n/\phi(k)} \hookrightarrow \mathbb{C}^n$ must similarly be given by:

$$\mathbb{C}^n = \bigoplus_{\xi \in \mathbf{EVal}(g)} E_{\zeta_k \otimes 1}(\xi),$$

where $\zeta_k \otimes 1$ is the action by componentwise multiplication on $\mathbb{Z}[\zeta_k]^{2n/\phi(k)} \otimes \mathbb{C}$. In other words, this embedding is the blockwise embedding $f(S_k)$, repeated on $\frac{2n}{\phi(k)}$ blocks of dimension $\frac{\phi(k)}{2}$ each. So $B \simeq (\mathbb{C}^{\phi(k)/2}/\text{Im}(f(S_k)))^{2n/\phi(k)}$. □

The proof of Proposition 4.6 is now easy.

Proof of Proposition 4.6. By Proposition 4.3, let B_{st} be a $\langle g \rangle$ -stable complement to W in A . For any other complement B to W , since $B \times W$ and $B_{\text{st}} \times W$ are isogenous, B and B_{st} are isogenous. Let us determine the isogeny (and if possible isomorphism) type of B_{st} .

On one hand, if g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$ or $(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$, then $g|_{B_{\text{st}}}$ has eigenvalues of two different orders. By [6, Thm.13.2.8], there are then two $\langle g \rangle$ -stable complementary translated abelian subvarieties B_1 and B_2 in B_{st} , such that all eigenvalues of $g|_{B_1}$ have order $k_1 = 6$, and all eigenvalues of $g|_{B_2}$ have the same order $k_2 < 6$. By definition, B_{st} is isogenous to $B_1 \times B_2$, and thus its isogeny type can be derived from the isomorphism types of B_1 and B_2 , given by Lemma 4.11 if $k_1, k_2 \geq 3$. However, if g is of type $(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$, then $k_2 = 2$ and B_2 can be any elliptic curve, and that is why the isogeny type of B_{st} is not entirely determined in this case.

On the other hand, if g is of any other type, then all eigenvalues of $g|_{B_{\text{st}}}$ are of the same order $k \geq 3$, and Lemma 4.11 determines the isomorphism type of B_{st} . □

4.C Group theoretical treatment of a point's stabilizer in dimension 3 or 4

We can now establish the following proposition.

Proposition 4.12. *Let A be an abelian variety, $G \subset \text{Aut}(A)$ be a finite group acting freely in codimension 2. Suppose that the quotient A/G admits a crepant resolution. Let W be a subvariety of codimension $m \leq 4$ in A such that $\text{PStab}(W) \neq \{1\}$. Then $\text{PStab}(W)$ is a cyclic group generated by one junior element.*

By Proposition 4.3, it reduces to proving the following result.

Proposition 4.13. *Let B be an abelian variety of dimension $m \leq 4$, $F \subset \text{Aut}(B, 0)$ be a finite group acting freely in codimension 2 and fixing $0 \in B$. Suppose that F is generated by junior elements. Then F is a cyclic group generated by one junior element.*

We refer the reader to [36],[17] for standard facts in finite group theory, and in particular Sylow theory and representation theory. Let us just recall a few notations used in the following.

Notation 4.14. We denote by $C_F(H)$, respectively $N_F(H)$, the centralizer, respectively normalizer, of a subset H of a group F , i.e.,

$$\begin{aligned} C_F(H) &:= \{f \in F \mid \forall h \in H, fh = hf\} \\ N_F(H) &:= \{f \in F \mid fH = Hf\} \end{aligned}$$

If H has a single element or is a subgroup of F , then $C_F(H)$ and $N_F(H)$ are subgroups of F .

Notation 4.15. Let F be a finite group, V be a vector space of finite dimension, $\rho : F \rightarrow \text{GL}(V)$ be a group morphism, i.e., a faithful representation of F in V . The character χ of ρ is the map $\chi : f \in F \rightarrow \text{Tr}(\rho(f)) \in \mathbb{C}^*$.

By Schur's lemma, the representation ρ decomposes as a direct sum of irreducible representations:

$$\rho = \rho_1^{\oplus n_1} \oplus \dots \oplus \rho_k^{\oplus n_k},$$

and, accordingly, if χ_i denotes the character of ρ_i , we have $\chi = n_1\chi_1 + \dots + n_k\chi_k$. By orthogonality of the irreducible characters,

$$\langle \chi, \chi \rangle = (n_1^2 + \dots + n_k^2)|F|.$$

We refer to $u = n_1^2 + \dots + n_k^2$ as the *splitting coefficient* of the representation ρ .

We start proving lemmas towards Proposition 4.13. The first lemma classifies all possible finite order elements in $\text{Aut}(B, 0)$ of determinant one acting freely in codimension 2, when B is an abelian fourfold.

Lemma 4.16. *Let B be an abelian fourfold, and $g \in \text{Aut}(B, 0)$ be a finite order element such that $\langle g \rangle$ acts freely in codimension 2 on B . Then the order of g and the matrix of a generator of $\langle g \rangle$ are given in Table 6, together with the restrictions on B , if any.*

Proof. Let ζ be an eigenvalue of g of order u , such that $(\phi(u), u)$ is maximal in \mathbb{N}^2 for the lexicographic order. By Lemma 3.7, Φ_u divides the characteristic polynomial $\chi_{g \oplus \bar{g}}$ in $\mathbb{Q}[X]$, so $\phi(u) \leq 2 \dim B = 8$. Let us discuss cases:

- (1) If $\phi(u) = 1$, then $u = 1$ or 2 . As g acts freely in codimension 2 and has determinant one, $g = \pm \text{id}_B$.
- (2) Suppose that $\phi(u) = 8$. Then g has four distinct eigenvalues of order u , and hence has order u . Listing integers of Euler number 8, $u \in \{15, 16, 20, 24, 30\}$. There is a generator g' of $\langle g \rangle$ of which $e^{2i\pi/u}$ is an eigenvalue. Denote its other eigenvalues by $e^{2i\pi a/u}$, $e^{2i\pi b/u}$, $e^{2i\pi c/u}$, with
 - $a, b, c \in \llbracket 1, u-1 \rrbracket$ coprime to u
 - u divides $1 + a + b + c$
 - and

$$\begin{aligned} \Phi_u(X) &= (X - e^{2i\pi/u})(X - e^{2i\pi(u-1)/u})(X - e^{2i\pi a/u})(X - e^{2i\pi(u-a)/u}) \\ &\quad (X - e^{2i\pi b/u})(X - e^{2i\pi(u-b)/u})(X - e^{2i\pi c/u})(X - e^{2i\pi(u-c)/u}) \end{aligned}$$

We check by hand the solutions to this system and plug them in Table 6. For example, this is how we add $\text{diag}(\zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$.

order of g	a generator of $\langle g \rangle$ up to similarity	restrictions on B	
1	id	B arbitrary	
2	$-\text{id}$		
3	$\text{diag}(j, j^2, j, j^2)$		
4	$\text{diag}(i, -i, i, -i)$		
5	$\text{diag}(\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$		
6	$\text{diag}(\omega, \omega^5, \omega, \omega^5)$		
8	$\text{diag}(\zeta_8, \zeta_8^3, \zeta_8^5, \zeta_8^7)$		
10	$\text{diag}(\zeta_{10}, \zeta_{10}^3, \zeta_{10}^7, \zeta_{10}^9)$		
12	$\text{diag}(\zeta_{12}, \zeta_{12}^5, \zeta_{12}^7, \zeta_{12}^{11})$		
3	$\text{diag}(1, j, j, j)$		$B \sim E \times E_j^3$
6	$\text{diag}(-1, \omega, \omega, \omega)$		
9	$\text{diag}(j^2, \zeta_9, \zeta_9^4, \zeta_9^7)$		$B \simeq E_j^4$
18	$\text{diag}(\omega^5, \zeta_{18}, \zeta_{18}^7, \zeta_{18}^{13})$		
4	$i\text{id}$	$B \simeq E_i^4$	
12	$\text{diag}(\zeta_{12}, \zeta_{12}^5, \zeta_{12}, \zeta_{12}^5)$		
20	$\text{diag}(\zeta_{20}, \zeta_{20}^9, \zeta_{20}^{13}, \zeta_{20}^{17})$		
7	$\text{diag}(1, \zeta_7, \zeta_7^2, \zeta_7^4)$	$B \sim E \times E_{u_7}^3$	
14	$\text{diag}(-1, \zeta_{14}, \zeta_{14}^9, \zeta_{14}^{11})$		
8	$\text{diag}(\zeta_8, \zeta_8^3, \zeta_8, \zeta_8^3)$	$B \simeq E_{u_8}^4$	
24	$\text{diag}(\zeta_{24}, \zeta_{24}^{11}, \zeta_{24}^{17}, \zeta_{24}^{19})$		
15	$\text{diag}(\zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8)$	$B \simeq E_{u_{15}}^4$	
30	$\text{diag}(\zeta_{30}, \zeta_{30}^{17}, \zeta_{30}^{19}, \zeta_{30}^{23})$		
16	$\text{diag}(\zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^7)$	$B \simeq S_{u_{16}, v_{16}}^2$	
	$\text{diag}(\zeta_{16}, \zeta_{16}^7, \zeta_{16}^{11}, \zeta_{16}^{13})$		
20	$\text{diag}(\zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9)$	$B \simeq E_{u_{20}}^4$	
24	$\text{diag}(\zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{24}^{11})$	$B \simeq E_{u_{24}}^4$	

Table 6: Classification of possible elements of g in $\text{Aut}(B, 0)$, with **colored junior elements**.

- (3) Suppose that $\phi(u) = 6$. Then g has three distinct eigenvalues of order u , and one eigenvalue of order v , with $\phi(v) = 1$ or 2 . Since g^u has three trivial eigenvalues and $\langle g \rangle$ acts freely in codimension 2, $g^u = \text{id}_B$, so g has order u and v divides u . Listing the integers of Euler number 6, $u \in \{7, 9, 14, 18\}$. Using that $\chi_{g \oplus \bar{g}} = \Phi_u \Phi_v$ or $\Phi_u \Phi_v^2$, g has determinant 1 and $\langle g \rangle$ acts freely in codimension 2, we work out all possibilities by hand and add them to the table. One example falling in this case is $\text{diag}(1, \zeta_7, \zeta_7^2, \zeta_7^4)$.
- (4) Suppose that $\phi(u) = 4$. Then g has two distinct eigenvalues of order u , and two remaining eigenvalues of respective order $v_1 \leq v_2$. As $\langle g \rangle$ acts freely in codimension 2, g^u , which has two trivial eigenvalues, must be trivial, so g has order u and v_1 and v_2 divide u . Similarly, $g^{\text{lcm}(v_1, v_2)} = \text{id}_B$, so u divides $\text{lcm}(v_1, v_2)$. Listing integers of Euler number 4, $u \in \{5, 8, 10, 12\}$.
- (a) If v_1 divides v_2 , then $v_2 = u$. We investigate all possibilities of determinant 1 satisfying Lemma 3.7 by hand and add them to the table. One of them is $\text{diag}(\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4)$.
- (b) If v_1 does not divide v_2 , then by Lemma 3.7 again, $\phi(v_1) + \phi(v_2) \leq 4$. Listing

possibilities by hand, we see that $(v_1, v_2) \in \{(2, 3), (3, 4), (4, 6)\}$. From the divisibility relations between v_1, v_2 and u , we obtain that $u = 12$, and in fact, $(v_1, v_2) = (3, 4)$ or $(4, 6)$. In particular, g has order 12, so $g^6 = -\text{id}_B$, and so g^3 has four eigenvalues of order 4. But since $v_1 = 3$ or $v_2 = 6$, this can not be the case. Contradiction!

- (5) The last case is when $\phi(u) = 2$, i.e., $u = 3, 4$, or 6 . In that case, each eigenvalue of g has order 1, 2, 3, 4, or 6. As $\langle g \rangle$ acts freely in codimension 2, g has at most one eigenvalue of order 1 or 2.
- (a) Suppose that g has an eigenvalue of order 4. As it has determinant 1, it has an even number of eigenvalues of order 4, so at least two of them. Hence, by freeness in codimension 2, $g^4 = \text{id}_B$, and so $g^2 = -\text{id}_B$, i.e., all eigenvalues of g have order 4. There is a generator of $\langle g \rangle$ similar to either $\text{diag}(i, i, i, i)$, or $\text{diag}(i, -i, i, -i)$.
- (b) Suppose that $u = 3$. Then as $(\phi(v), v) \leq (\phi(u), u)$ for any order v of another eigenvalue of g , the other eigenvalues have order 1, 2, or 3. Hence, there are at least three eigenvalues of order 3, and thus by freeness in codimension 2, $g^3 = \text{id}_B$. So g has order 3 and there is a generator of $\langle g \rangle$ similar to either $\text{diag}(1, j, j, j)$, or $\text{diag}(j, j^2, j, j^2)$.
- (c) Suppose finally that $u = 6$ and g has no eigenvalue of order 4: Then g has order 6, so $g^3 = -\text{id}_B$. All eigenvalues of g thus have order 2 or 6, so g has at least three eigenvalues of order 6. As g has determinant 1, we only have two possibilities: There is a generator of $\langle g \rangle$ similar to $\text{diag}(-1, \omega, \omega, \omega)$, or $\text{diag}(\omega, \omega^5, \omega, \omega^5)$.

This discussion constructs the first two columns of the table. The restrictions on B given in the third column are given by the same arguments as in the proof of Lemmas 4.8, 4.11. \square

Corollary 4.17. *Let B be an abelian fourfold, and let $g, h \in \text{Aut}(B, 0)$ be junior elements such that $\langle g \rangle$ and $\langle h \rangle$ act freely in codimension 2, with $\text{ord}(g) \leq \text{ord}(h)$. Then there are three possibilities:*

- g and h are similar, in particular have the same order;
- g is similar to $\text{diag}(1, j, j, j)$, h is similar to $\text{diag}(-1, \omega, \omega, \omega)$, and B is isogenous to $E \times E_j^3$ for some elliptic curve E ;
- $g = \text{id}_B$, h is similar to $\text{diag}(\zeta_{12}, \zeta_{12}^5, \zeta_{12}, \zeta_{12}^5)$ and B is isomorphic to E_i^4 .

Proof. If g has order 7, then by Lemma 4.16, B is isogenous to $E \times E_{u7}^3$ for some elliptic curve E . By uniqueness in the Poincaré decomposition of B [6, Thm.5.3.7], B is not isogenous to any of the other special abelian varieties appearing in Lemma 4.16. So, by Lemma 4.16 again, h being junior must have order 7. By Proposition 3.1, any junior element k of order 7 acting on a fourfold with $\langle k \rangle$ acting freely in codimension 2 are similar to $\text{diag}(1, \zeta_7, \zeta_7^2, \zeta_7^4)$. So g and h are similar.

The same argument works if g has order 8, 15, 16, 20, 24.

If g has order 3 or 6, then by Lemma 4.16, B is isogenous to $E \times E_j^3$ for some elliptic curve E . By uniqueness in the Poincaré decomposition of B [6, Thm.5.3.7], B is not isogenous to any of the other special abelian varieties appearing in Lemma 4.16. So, by Lemma 4.16 again, h being junior must have order 3 or 6. As we assumed $\text{ord}(g) \leq \text{ord}(h)$, the only strict inequality is when g has order 3 and h has order 6. In this case, by Proposition 3.1, g is similar to $\text{diag}(1, j, j, j)$ and h to $\text{diag}(-1, \omega, \omega, \omega)$.

The same argument works if g has order 4 or 12. \square

We can now prove cyclicity of F when it contains a junior element of order 3.

Proposition 4.18. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements. Suppose that F contains an element similar to $\text{diag}(1, j, j, j)$. Then F is cyclic and generated by one junior element.*

Proof. By Corollary 4.17, B is isogenous to $E \times E_j^3$ for some elliptic curve E , and any junior element in $\text{Aut}(B, 0)$ is similar to $\text{diag}(1, j, j, j)$, or $\text{diag}(-1, \omega, \omega, \omega)$.

Suppose by contradiction that F is not generated by one junior element. Then there are two junior elements $g, h \in F$ such that $\langle g \rangle \not\subseteq \langle h \rangle$ and $\langle h \rangle \not\subseteq \langle g \rangle$. Up to possibly replacing them by their square, we have \tilde{g} and \tilde{h} both similar to $\text{diag}(1, j, j, j)$. Their eigenspaces satisfy $\dim E_{\tilde{g}}(j) \cap E_{\tilde{h}}(j) = 2 \leq \dim E_{\tilde{g}\tilde{h}^{-1}}(1)$. As $\langle \tilde{g}, \tilde{h} \rangle \subset F$ acts freely in codimension 2, $\tilde{g} = \tilde{h}$. Since $\langle \tilde{g} \rangle \subset \langle h \rangle$, $\tilde{g} \neq g$, so $\tilde{g} = g^2$. Similarly, $\tilde{h} = h^2$. Since $g^3 = h^3 = -\text{id}$, it nonetheless yields $g = h$, contradiction. \square

Let us now present our general strategy to prove that F is cyclic. By Lemma 4.16, the prime divisors of $|F|$ are 2, 3, 5, and 7. Hence, $|F| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta$. Since 2^α (respectively 3^β , etc.) is the order of a 2 (respectively 3, etc.)-Sylow subgroup of F , we can rely on Sylow theory to bound $|F|$, as in the following result.

Proposition 4.19. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements, containing no junior element of order 3. Then*

$$|F| \text{ divides } 2^4 \cdot 3 \cdot 5 \cdot 7 = 1680.$$

The proof of this proposition relies on the following two lemmas.

Lemma 4.20. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, containing no junior element of order 3. Let $p = 3, 5$, or 7 divide $|F|$. Then a p -Sylow subgroup S of F is cyclic of order p .*

Proof. As S is a p -group, its center $Z(S)$ is non-trivial. Hence, it contains an element g of order p . Let $h \neq \text{id} \in S$. By Lemma 4.16, F has no element of order p^2 , so h has order p . Since g and h commute, they are codiagonalizable. Let v, w be two non-colinear common eigenvectors of them associated with eigenvalues other than 1. Let $\tilde{g} \in \langle g \rangle$ and $\tilde{h} \in \langle h \rangle$ satisfy $\tilde{g}(v) = \tilde{h}(v) = \zeta_p v$.

If $p = 3$ or 5, Lemma 4.16 shows that $E_g(1) = E_h(1) = \{0\}$, so $\tilde{g}\tilde{h}^{-1}$ can not have 1 as an eigenvalue and be of order p . So it is trivial, i.e., $\tilde{g} = \tilde{h}$, and $h \in \langle g \rangle$.

Suppose $p = 7$. If $\tilde{g}(w) \neq \tilde{h}(w)$, then by Lemma 4.16, $\{\tilde{g}(w), \tilde{h}(w)\} = \{\zeta_7^2 w, \zeta_7^4 w\}$. So $\tilde{g}\tilde{h}^2$ has eigenvalue ζ_7^3 on v , and ζ_7 or ζ_7^3 on w , which in either case contradicts Lemma 4.16. So $\tilde{g}(w) = \tilde{h}(w)$, i.e., $\tilde{g}\tilde{h}^{-1}$ has eigenvalue 1 with multiplicity two. By freeness in codimension 2, $\tilde{g} = \tilde{h}$, hence $h \in \langle g \rangle$. \square

Lemma 4.21. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2. If not trivial, a 2-Sylow subgroup S of F is cyclic or a generalized quaternion group, and its order divides 16.*

Proof. By Lemma 4.16, the element of order 2 in F is unique: it is $-\text{id}_B$. By [36, 5.3.6], S is hence either cyclic or a generalized quaternion group. Moreover, by Lemma 4.16, S has no element of order 32. Hence, the only case where the order of S does not divide 16, is when S is isomorphic to Q_{32} . Let us however show that this is impossible.

Indeed, Q_{32} contains an element h of order 16 and an element s of order 4 such that $shs^{-1} = h^{-1}$ [36, pp.140-141]. However, if $h \in S$ is an element of order 16, it can not be conjugated in S to h^{-1} , because by Lemma 4.16 they have distinct eigenvalues. \square

Proof of Proposition 4.19. It is straightforward from Lemma 4.20 and Lemma 4.21. \square

The following Lemma and Proposition show that if 7 divides $|F|$, i.e., if F contains a junior element of order 7, then F is cyclic generated by one junior element of order 7.

Lemma 4.22. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, containing no junior element of order 3. Suppose that 7 divide $|F|$. Let S be a 7-Sylow subgroup of F . Then there is a normal subgroup N of F such that $F = N \rtimes S$.*

Proof. By Burnside's normal complement theorem [36, 10.1.8], it is enough to show that $N_F(S) = C_F(S)$.

Let g be a generator of S . By Lemma 4.16, if $f \in N_F(S)$, then $fgf^{-1} \in \{g, g^2, g^4\}$, because they are the only elements with the same set of eigenvalues as g . So $f^3 \in C_F(S)$. Let us show by contradiction that $f \in N_F(S)$ can not have order 3. Looking at the action of f on the eigenspaces of g in coordinates diagonalizing g ,

$$f = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & y \\ 0 & x & 0 & 0 \end{pmatrix},$$

with $xyz = \bar{t}$, and so $\chi_f = (X - t)(X^3 - \bar{t})$. But by Lemma 4.16, elements of order 3 in F (which by assumption cannot be junior) have characteristic polynomial $(X^2 + X + 1)^2$, contradiction. So $N_F(S)$ has no element of order 3. To sum up, if $f \in N_F(S)$, then $f^3 \in C_F(S)$ and 3 is prime to the order of f , so $f \in C_F(S)$. \square

Proposition 4.23. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements, containing no junior element of order 3. Suppose that 7 divides $|F|$. Then F is cyclic and generated by one junior element.*

Proof. Let S be a 7-Sylow subgroup of F . By Lemma 4.22, $F = N \rtimes S$, where N is a normal subgroup of F , and by Proposition 4.19, $|N|$ divides 240. A simple GAP program in the appendix checks that a group of order dividing 240 cannot have an automorphism of order 7. So S acts trivially on N , i.e., $F = N \times S$. But F is generated by its junior elements, which all have order 7 by Corollary 4.17. So N is trivial, and $F = S$ is cyclic of order 7. \square

Now we can focus on the case when F contains no junior element of order 3 or 7. We start by showing that, provided F is cyclic, it is generated by one junior element.

Lemma 4.24. *Let F be a cyclic group. If E is a set of generators of F and all elements of E have the same order, then any element of E actually generates F .*

Proof. Suppose $F = \mathbb{Z}_d$ and every element of E has order k dividing d . Then E is actually a subset of $\mathbb{Z}_k \subset \mathbb{Z}_d$, and since E must generate \mathbb{Z}_d , it must be $k = d$. So any element $e \in E$ satisfies $\langle e \rangle = \mathbb{Z}_d = F$. \square

Corollary 4.25. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements, containing no junior element of order 3 or 7. If F is cyclic, then F is generated by one junior element.*

Proof. Assume that F is cyclic. If F contains one junior element of order 8, 15, 16, 20, or 24, then by Corollary 4.17, all junior elements have the same order and we use Lemma 4.24 to conclude.

Else, the junior elements of F each have order 4 or 12. If there are no junior elements of order 12, Lemma 4.24 concludes again. If there is a junior element g of order 12, then a quick computation from Lemma 4.16 shows that g^3 is the only junior element of order 4 in F , and thus the junior elements of order 12 actually generate F too, so we conclude by Lemma 4.24. \square

These versions of Lemma 4.22 for 3- and 5-Sylow subgroups will be useful too.

Lemma 4.26. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements. Suppose that $p \in \{3, 5\}$ divides $|F|$. Let S be a p -Sylow subgroup of F . Then $N_F(S)/C_F(S)$ is isomorphic to a subgroup of $(\mathbb{Z}_p)^\times$.*

Proof. The quotient $N_F(S)/C_F(S)$ acts faithfully by conjugation on S , and therefore embeds in $\text{Aut}(S)$, which by Lemma 4.20 is isomorphic to $(\mathbb{Z}_p)^\times$. \square

Lemma 4.27. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements. Suppose that 5 divides $|F|$. Let S be a 5-Sylow subgroup of F . Then, if $f \in N_F(S)$ is a junior element of order 8, $[f] \in N_F(S)/C_F(S)$ cannot have order 4.*

Proof. Let $f \in N_F(S)$ be a junior element of order 8 such that $[f] \in N_F(S)/C_F(S)$ has order 4, and let g be a generator of S . Looking at the action of f on the eigenspaces of g in coordinates diagonalizing g ,

$$f = \begin{pmatrix} 0 & 0 & 0 & t \\ x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \end{pmatrix},$$

with $xyzt = -1$, and so $\chi_f = X^4 + 1$. By Lemma 4.16, no junior element of order 8 has this characteristic polynomial, contradiction. \square

We finally prove the following two key propositions, which imply Proposition 4.13.

Proposition 4.28. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements, containing no junior element of order 3 or 7. Then a 2-Sylow subgroup of F is either trivial, or cyclic.*

Proposition 4.29. *Let B be an abelian fourfold, and let F be a finite subgroup of $\text{Aut}(B, 0)$ acting freely in codimension 2, generated by junior elements, containing no junior element of order 3 or 7. Suppose that a 2-Sylow subgroup of F is trivial or cyclic. Then F is cyclic.*

Proof of Proposition 4.29. Let us write $|F| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma$ with $\alpha \in \llbracket 0, 4 \rrbracket$, $\beta, \gamma \in \llbracket 0, 1 \rrbracket$. By Lemma 4.20 and by assumption, the Sylow subgroups of F are cyclic, so [36, pp.290-291] applies and F is a semidirect product: $F \simeq (\mathbb{Z}_{5^\gamma} \rtimes \mathbb{Z}_{3^\beta}) \rtimes \mathbb{Z}_{2^\alpha}$. Since 3^β is coprime to $\phi(5^\gamma)$, the group \mathbb{Z}_{5^γ} has no automorphism of order 3, and thus the first semidirect product is direct:

$$F \simeq (\mathbb{Z}_{5^\gamma} \times \mathbb{Z}_{3^\beta}) \rtimes \mathbb{Z}_{2^\alpha}.$$

If $\beta = \gamma = 1$, the group F contains an element of order 15, so by Lemma 4.16, B is isomorphic to $E_{u_{15}}^4$ and all junior elements of F have order 15. However, since $F \simeq \mathbb{Z}_{15} \rtimes \mathbb{Z}_{2^\alpha}$, and since F is generated by its junior elements, we must have $\alpha = 0$, and so $F \simeq \mathbb{Z}_{15}$ is cyclic and generated by one junior element.

If $\beta = \gamma = 0$, then $F \simeq \mathbb{Z}_{2^\alpha}$ is cyclic.

Else, write $p = 3^\beta 5^\gamma$ and $F \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_{2^\alpha}$. Note that $\mathbb{Z}_p \rtimes \mathbb{Z}_{2^{\alpha-1}}$ is a proper subgroup of F containing all elements whose order divides $2^{\alpha-1}p$. As F is generated by its junior elements, their orders cannot all divide $2^{\alpha-1}p$: There is a junior element $g \in F$ of order 2^α or $2^\alpha p$. If g has order $2^\alpha p$, $\langle g \rangle = F$ and so F is cyclic. If g has order 2^α , we can write $F \simeq \langle u \rangle \rtimes \langle g \rangle$, where u is an element of F of order p . The discussion now depends on α and p .

- (1) By Lemma 4.16, if g has order 4, then $g = \text{id}$ commutes with every element of F , so the semidirect product is direct and F is cyclic.
- (2) If $p = 5$ and g has order 8, by Lemma 4.27, g^2 and u commute, so $g^2 u$ has order 20. Since g is junior of order 8, by Lemma 4.16, B is isomorphic to $E_{u_8}^4$. So by Lemma 4.16 again, B has no automorphism of order 20, contradiction.
- (3) If $p = 5$ and g has order 16, by Lemma 4.26, g^4 and u commute, so $g^4 u$ has order 20. But since g is junior of order 16, by Lemma 4.16, B has no automorphism of order 20, contradiction.
- (4) If $p = 3$ and g has order 16, by Lemma 4.26, $g^2 u$ has order 24. But since g is junior of order 16, by Lemma 4.16, B has no automorphism of order 24, contradiction.
- (5) If $p = 3$ and g has order 8, then $F \simeq \mathbb{Z}_3 \rtimes \mathbb{Z}_8$. With GAP, we check in the Appendix that:

- The irreducible representations of F have rank 1 or 2.
- No irreducible character of F takes value j or j^2 , so $F \subset \text{Aut}(B, 0)$ has no irreducible subrepresentation of rank 1.
- The only two irreducible representations of F of rank 2 sending $-\text{id} \in F$ to $-\text{id}$ indeed are complex conjugates, so all elements of $F \subset \text{Aut}(B, 0)$ have characteristic polynomials in $\mathbb{Q}[X]$.

However, $g \in F$ is a junior element of order 8, which by Lemma 4.16 has a non-rational characteristic polynomial, contradiction. □

We prove Proposition 4.28 by contradiction.

Proof of Proposition 4.28. Suppose that 2 divides $|F|$ and that a 2-Sylow subgroup of F is not cyclic. We first show that any junior element in F has order 15, 20 or 24.

By contradiction and by Proposition 3.1, let $g \in F$ be a junior element of order 4, 8, 12, or 16. If g has order 12, then $g^3 \in F$ is a junior element of order 4, and F thus contains a junior element \tilde{g} of order 4, 8, or 16. Let S be a 2-Sylow subgroup containing that junior element. By assumption, S is not cyclic, so by Lemma 4.21, S is isomorphic to Q_8 or to Q_{16} . Clearly, Q_8 and Q_{16} have no element of order 16, and no element of order 4 in their centers, so \tilde{g} has order 8. As Q_8 has no element of order 8, S is isomorphic to Q_{16} . But we easily check with GAP that:

- The irreducible representations of Q_{16} have rank 1 or 2.
- The only irreducible representations of Q_{16} of rank r sending the unique element of order 2 to $-\text{id}_r$ are two complex conjugates representations with $r = 2$, so all elements of $S \subset \text{Aut}(B, 0)$ have characteristic polynomials in $\mathbb{Q}[X]$.

However, $\tilde{g} \in S$ is a junior element of order 8, which by Lemma 4.16 has a non-rational characteristic polynomial, contradiction.

So any junior element in F has order 15, 20 or 24. We also know that:

- F has exactly one element of order 2, by Lemma 4.16.
- A 2-Sylow subgroup of F is isomorphic to Q_8 or Q_{16} , by Lemma 4.21.
- $|F|$ divides 240, by Proposition 4.19.
- F has no element of order 60 or 40, by Lemma 4.16.
- If F has elements of orders $o, o' \in \{15, 20, 24\}$, then $o = o'$, by Lemma 4.16.

We check with GAP that there are only five groups satisfying all these properties, namely the groups indexed $(40, 4), (40, 11), (80, 18), (48, 8)$, and $(48, 27)$ in the `SmallGroup` library. The function `StructureDescription` then shows that they are respectively of the form $\mathbb{Z}_5 \rtimes Q_8, \mathbb{Z}_5 \times Q_8, \mathbb{Z}_5 \rtimes Q_{16}, \mathbb{Z}_3 \rtimes Q_{16}$, and $\mathbb{Z}_3 \times Q_{16}$. Note that only $\mathbb{Z}_5 \times Q_8, \mathbb{Z}_5 \rtimes Q_{16}$ are generated indeed by their elements of orders $(15, 24, \text{ or } 20)$. Checking the irreducible character tables of these two candidates with GAP shows that they have no appropriate four-dimensional representation (see Appendix for programs supporting this discussion.)

This concludes the proof of Proposition 4.28. □

Proof of Proposition 4.13. If F contains a junior element of order 3, then Proposition 4.18 applies and shows that F is cyclic generated by one junior element. If F contains no junior element of order 3, but one of order 7, then Proposition 4.23 applies and shows that F is cyclic generated by one junior element. Finally, if F contains no junior element of order 3 or 7, Proposition 4.28 shows that its 2-Sylow subgroups are cyclic or trivial, Proposition 4.29 deduces that F is cyclic and Corollary 4.25 proves that F is generated by one junior element. □

5 Ruling out junior elements in codimension 4

The aim of this section is to rule out eight out of the twelve types of junior elements presented in Proposition 3.1, namely those which fix pointwise at least one subvariety of codimension 4, but no subvariety of codimension 3.

Proposition 5.1. *Let A be an abelian variety of dimension n , G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X . Then, if $g \in G$ is a junior element, the matrix $M(g)$ cannot have eigenvalue 1 with multiplicity exactly $n - 4$.*

Remark 5.2. Whether the local affine quotients corresponding to these eight types of junior elements admit a crepant resolution is actually settled by toric geometry in [37]. In fact, by [37, Thm.3.1],

$$\begin{aligned} &\mathbb{C}^4/\langle \text{id} \rangle, \quad \mathbb{C}^4/\langle \text{diag}(\omega, \omega, \omega, -1) \rangle, \quad \mathbb{C}^4/\langle \text{diag}(\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3) \rangle, \\ &\mathbb{C}^4/\langle \text{diag}(\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5) \rangle, \quad \mathbb{C}^4/\langle \text{diag}(\zeta_{15}, \zeta_{15}^2, \zeta_{15}^4, \zeta_{15}^8) \rangle \end{aligned}$$

have a crepant Fujiki-Oka resolution, and by [37, Prop.3.9],

$$\mathbb{C}^4/\langle \text{diag}(\zeta_{16}, \zeta_{16}^3, \zeta_{16}^5, \zeta_{16}^9) \rangle, \quad \mathbb{C}^4/\langle \text{diag}(\zeta_{20}, \zeta_{20}^3, \zeta_{20}^7, \zeta_{20}^9) \rangle, \quad \mathbb{C}^4/\langle \text{diag}(\zeta_{24}, \zeta_{24}^5, \zeta_{24}^7, \zeta_{15}^{11}) \rangle$$

admit no toric crepant resolution. They could nevertheless have a non-toric crepant resolution.

In light of this remark, the proof of Proposition 5.1 must crucially involve global arguments.

5.A Ruling out junior elements of order 4,8,12,15,16,20,24

In this subsection, we rule out the seven types of junior elements of order other than 3, 6, 7.

Proposition 5.3. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X . Then any junior element of G has order 3, 6, or 7.*

Remark 5.4. Let A be an abelian variety, G be a group acting freely in codimension 2 on A . As translations in G form a normal subgroup G_0 , we can write:

$$(A/G_0)/(G/G_0) \simeq A/G.$$

Clearly, A/G_0 is isogenous to A and G/G_0 still acts freely in codimension 2 on it, except that it contains no translation. Hence, we can assume without loss of generality that G contains no translation (other than id). In particular, any element of G has the same finite order as its matrix.

Proof of Proposition 5.3. By contradiction, suppose that $g \in G$ is a junior element of order $d \in \{4, 8, 12, 15, 16, 20, 24\}$, of minimal order among the junior elements of G of such orders. Up to conjugating the whole group G by an appropriate translation, we may assume that g fixes $0 \in A$. In particular, g fixes pointwise an abelian subvariety W of A of codimension 4, so Propositions 4.12 and 4.3 show that $\text{PStab}(W) = \langle g \rangle$, and define a $\langle g \rangle$ -stable complementary abelian subvariety B to W in A . The key to the proof is that a well-chosen power g^α of g has strictly more fixed points in B than g , as many distinct eigenvalues as g , but is not a junior element. Indeed, we set α depending on d as follows, and check with Proposition 3.1 that g^α is not junior and has as many distinct eigenvalues as g . As for fixed points, applying [6, Prop.13.2.5(c)] shows that $(g^\alpha)|_B$ has strictly more of them than $g|_B$ in B .

d	4	8	12	15	16	20	24
α	2	2	4	3	2	4	3

Table 7: Definition of a certain $\alpha \in \llbracket 0, d - 1 \rrbracket$ depending on d

Let $\tau \in B$ be a fixed point of g^α that is not fixed by g . Note that $W + \tau$ is pointwise fixed by g^α . By Proposition 4.12, $\text{PStab}(W + \tau) = \langle h \rangle$ for some junior element h .

By Proposition 4.3, there is an $\langle h \rangle$ -stable translated abelian subvariety B' of A containing τ such that B' and $W + \tau$ are complementary. By uniqueness in Poincaré's complete reducibility theorem [6, Thm.5.3.7], the abelian varieties B and $B' - \tau$ are isogenous, hence determined by the order of g and h respectively, by Lemma 4.16.

Let us discuss the special case when $B \simeq E_i^4$, i.e., when junior elements of order both 4 and 12 exist in $\text{Aut}_{\mathbb{Q}}(B, 0) = \text{Aut}_{\mathbb{Q}}(B', 0)$. If g or h has order 4, then by the minimality assumption on g , g has order 4, and by Lemma 4.16, either $g = h$ or $g^3 = h$. So $g \in \langle h \rangle$, and thus $g(\tau) = \tau$, contradiction!

By Corollary 4.17, we can now assume that g and h have the same order $d \in \{8, 12, 15, 16, 20, 24\}$, and similar matrices. Recall that $g^\alpha \in \langle h \rangle$. Since g and h have the same order, it implies $\langle g^\alpha \rangle = \langle h^\alpha \rangle$, i.e., $g^\alpha = h^{u\alpha}$ for some u coprime to $\frac{d}{\alpha}$. Since g and g^α , and h and $h^{u\alpha}$ have the same number of distinct eigenvalues, it follows from $g^\alpha = h^{u\alpha}$ that the eigenspaces of g and h are the same, i.e., g and h commute. We discuss two cases separately.

- (1) If $d = 8$ or 12 , then in appropriate coordinates, we have:

$$\begin{aligned} M(g) &= \text{diag}(\mathbf{1}_{n-4}, \zeta_d, \zeta_d, \zeta_d^m, \zeta_d^m) \\ M(h) &= \text{diag}(\mathbf{1}_{n-4}, \zeta_d^m, \zeta_d^m, \zeta_d, \zeta_d) \end{aligned}$$

for some integer $m \in \llbracket 2, d-1 \rrbracket$ such that $2 + 2m = d$. In particular, $m^2 \equiv 1 \pmod{d}$, so $g = h^m \in \langle h \rangle$, contradiction!

- (2) Else, $d = 15, 16, 20$, or 24 . There is an integer u' coprime to d such that, in appropriate coordinates,

$$\begin{aligned} M(g) &= \text{diag}(\mathbf{1}_{n-4}, \zeta_d, \zeta_d^a, \zeta_d^b, \zeta_d^c) \\ M(h^{u'}) &= \text{diag}(\mathbf{1}_{n-4}, \zeta_d, \zeta_d^{\sigma(a)}, \zeta_d^{\sigma(b)}, \zeta_d^{\sigma(c)}) \end{aligned}$$

for some distinct integers $a, b, c \in \llbracket 2, d-1 \rrbracket$ coprime to d , and permutation σ of $\{a, b, c\}$. If $\sigma = \text{id}$, then $g = h^{u'} \in \langle h \rangle$, contradiction! Nevertheless, let us prove that $\sigma = \text{id}$. Note that

$$(h^{u'-u})^\alpha = (h^{u'}g^{-1})^\alpha (h^{-u}g)^\alpha = \text{diag}(\mathbf{1}_{n-3}, \zeta_d^{(\sigma(a)-a)\alpha}, \zeta_d^{(\sigma(b)-b)\alpha}, \zeta_d^{(\sigma(c)-c)\alpha}),$$

and thus $(h^{u'-u})^\alpha$ fixes a translated abelian variety $W' \supset W + \tau$ of codimension at most 3. By Proposition 4.12, $\text{PStab}(W')$ is trivial, or cyclic and generated by one junior element k of order 3 or 7. In the second case, as $k \in \text{PStab}(W + \tau)$, k restricts to an automorphism of the fourfold B' , which also has h junior of order $d \neq 3, 6, 7$ acting on it. This contradicts Corollary 4.17. Hence, $(h^{u'-u})^\alpha \in \text{PStab}(W') = \{\text{id}\}$, so for any $\ell \in \{a, b, c\}$, $(\sigma(\ell) - \ell)\alpha$ is a multiple of d . However, α was chosen so that g^α and g have the same number of distinct eigenvalues, i.e., $a\alpha, b\alpha, c\alpha$ are distinct modulo d . In particular, $\sigma(\ell)\alpha = \ell\alpha$ modulo d if and only if $\sigma(\ell) = \ell$. So $\sigma = \text{id}$, contradiction!

□

5.B Ruling out junior elements of order 6 with four non-trivial eigenvalues

In this subsection, we conclude the proof of Proposition 5.1 by ruling out the one remaining type of junior element fixing at least one subvariety of codimension 4, but no subvariety of codimension 3. It is the type of junior element of order 6, and matrix similar to $\text{diag}(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$.

Proposition 5.5. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X . Then there is no junior element of G with matrix similar to $\text{diag}(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1)$.*

The proof involves general arguments which we will use later, hence we factor it into a general lemma.

Lemma 5.6. *Let A be an abelian variety of dimension n , G a group acting freely in codimension 2 on A without translations such that A/G has a simply-connected crepant resolution X . Suppose that $g \in G$ fixes $0 \in A$ and has order d . Let W be the abelian subvariety of codimension k in A that g fixes pointwise, and denote by G_W the subgroup of G generated by*

$$G_{\text{gen}} = G_{\text{gen}}^{-1} = \{h \in G \mid \exists \tau \in A \text{ such that } h \in \text{PStab}(W + \tau)\}.$$

Then

- (1) *There is an $M(G_W)$ -stable complementary abelian subvariety B to W , which induces a representation $\rho : G_W \rightarrow \text{Aut}(B, 0)$ by $\rho(h) := M(h)|_B$.*
- (2) *If we denote by pr_W, pr_B the projections induced by the splitting of the tangent space, then, for any $h \in G_W$,*
 - $M(h) = pr_W + \rho(h)pr_B$
 - $pr_W(T(h)) = 0$, i.e., $T(h) \in B$
- (3) *The representation ρ is faithful and takes values in $\text{SL}(H^0(T_B))$.*
- (4) *The abelian subvariety B is in fact G_W -stable.*
- (5) *Every $h \in G_W$ that fixes a point $\tau \in A$ fixes the point $pr_B(\tau) \in B$.*
- (6) *Moreover, if we assume additionally that there is an integer $\alpha \in \llbracket 1, d-1 \rrbracket$ such that $M(g^\alpha)$ is similar to $\text{diag}(\mathbf{1}_{n-k}, -\mathbf{1}_k)$, then, for any $h \in G_W$, h and g^α commute and*
 - *either there is a point $\tau \in A$ such that $h \in \text{PStab}(W + \tau) \cup g^\alpha \text{PStab}(W + \tau)$;*
 - *or there is no such point, and 1 and -1 are eigenvalues of $\rho(h)$.*
- (7) *Same assumption. The translation part $T(h)$ of h is a 2-torsion point of B .*
- (8) *Same assumption. If h has even order and fixes a point in A , all fixed points of h in B are of 2-torsion.*
- (9) *Same assumption. If h is a junior element of order 3, then h fixes a 2-torsion point in B .*

Proof. (1) follows immediately from [6, Prop.13.5.1], since $M(G_W)$ is a finite group of group automorphisms of the abelian variety A , and W is $M(G_W)$ -stable.

(2) is proven by induction on the number of generators used to write $h \in G_W$. First, if $h \in G_W$ is in G_{gen} , there is a point $\tau \in A$ such that $h \in \text{PStab}(W + \tau)$. In particular, for $w \in W$ and $b \in B$;

$$M(h)(w + b) = h(w + \tau) - h(\tau) + M(h)(b) = w + \rho(h)(b),$$

as wished. Moreover, $T(h) = (\text{id} - M(h))(\tau)$, so $pr_W(T(h)) = 0$.

Second, if $h_1, h_2 \in G_W$ satisfy (2), then

$$M(h_1 h_2) = M(h_1)M(h_2) = pr_W + \rho(h_1 h_2)pr_B,$$

since ρ is a group morphism and $pr_W pr_B = pr_B pr_W = 0$. Moreover, $T(h_1 h_2) = T(h_1) + M(h_1)T(h_2)$, and the fact that $pr_W(T(h_1 h_2)) = 0$ easily follows from the induction assumption, notably using $pr_W(\text{id} - M(h_1)) = 0$.

For (3), let $h \in G_W$ and note that $\rho(h) = \text{id}_B$ if and only if $M(h) = pr_W + pr_B = \text{id}_A$, so ρ is faithful since M is. Note that by Proposition 2.7 and Lemma 2.17, M takes values in $\text{SL}(H^0(T_A))$. Hence, by Item 1 of (2), ρ takes values in $\text{SL}(H^0(T_B))$.

Regarding (4) we note that, for $h \in G_W$, $h(B) = M(h)B + T(h) = B + T(h) = B$ by Item 2 of (2).

(5) is clear from Item 1 of (2).

We now prove (6). Note that $\rho(g^\alpha) = -\text{id}_B$ commutes with any element of $\rho(G_W)$, and thus, as ρ is faithful, g^α is in the center of G_W .

Let $h \in G_W$ and assume that there is no point $\tau \in A$ fixed by h or $g^\alpha h$. In other words, neither $T(h)$ is in $\text{Im}(\text{id} - M(h))$, nor $T(g^\alpha h)$ is in $\text{Im}(\text{id} - M(g^\alpha h|_B))$. By Item 2 of (2), $T(h)$ and $T(g^\alpha h)$ belongs to B . Hence, the images $\text{Im}(\text{id}_B - M(h)|_B)$ and $\text{Im}(\text{id}_B - M(g^\alpha h)|_B) = \text{Im}(\text{id} + M(h)|_B)$ must be proper subvarieties of B , so 1 and -1 must be eigenvalues of $\rho(h) = M(h)|_B$.

For (7), we use that h commutes with g^α by (6), that $g(0) = 0$, that $T(h) \in B$ by Item 2 of (2), and that $g^\alpha|_B = -\text{id}_B$. It yields

$$0 = g^\alpha(h(0)) - h(g^\alpha(0)) = g^\alpha(T(h)) - T(h) = -2T(h),$$

so $T(h)$ is of 2-torsion.

For (8), assume that h fixes a point τ in A and has even order. For some β , h^β has order 2, and thus equals g^α : So, every fixed point of h is a fixed point of g^α . The points fixed by g^α are all of the form $w + \tau$, with $w \in W$, and $\tau \in B$ a 2-torsion point. But such a point $w + \tau$ being fixed by h , we have that $W + w + \tau = W + \tau$ is pointwise fixed by h , and in particular, the 2-torsion point $\tau \in B$ is a fixed point of h .

For (9), assume that h is a junior element of order 3. By (5), it fixes a point $\tau \in B$, and a translated abelian subvariety $W' + \tau$, where W' is an abelian subvariety of codimension 3 in A . Let B' be a $\langle h \rangle$ -stable complementary to $W' \cap B$ in B . We write $\tau = w' + b'$, with $w' \in W' \cap B$ and $b' \in B'$: It gives $h(b') = h(\tau - w') = \tau - w' = b'$, i.e., h fixes $b' \in B'$. Moreover, since $h|_{B'} = \text{id}_{B'}$, it holds

$$0 = h(b') - b' = (j - 1)b' + T(h).$$

Multiplying by $2(j^2 - 1)$, we see that $3b'$ is a point of 2-torsion of B' . Since $h(b') = b'$ and $3T(h) = T(h)$, this point $3b'$ is fixed by h . \square

We can now come back to our Proposition.

Proof of Proposition 5.5. By Remark 5.4, we can assume that G contains no translation other than id_A . By contradiction, suppose that there is an element $g \in G$ such that $g(0) = 0$ and, in some coordinates,

$$M(g) = \text{diag}(\mathbf{1}_{n-4}, \omega, \omega, \omega, -1).$$

We import the notations of Lemma 5.6, whose hypotheses are satisfied by g for $k = 4$, $d = 6$, $\alpha = 3$. The proof of the proposition now goes in three steps. First, we show that every element of $\rho(G_W)$ is similar to an element of $\langle \rho(g) \rangle \simeq \langle \text{diag}(\omega, \omega, \omega, -1) \rangle$. Second, we deduce that $G_W = \langle g \rangle$. Third, we use global considerations on fixed loci to derive a contradiction from this description of G_W .

Step 1: By Lemma 5.6 (1) and (4), there is a G_W -stable complementary B to W . As $\rho(g)$ acts on it, B is isogenous to $E \times E_j^3$ for some elliptic curve E . By Proposition 4.12, for any τ in A , the group $\text{PStab}(W + \tau)$ is trivial, or cyclic generated by one junior element k , and by Corollary 4.17, $\rho(k)$ is similar to $\rho(g)$ (if of order 6) or to $\rho(g^2)$ (if of order 3) in $\text{GL}(H^0(T_B))$. By Lemma 5.6 Item 1 of (2), $M(k)$ is therefore similar to $M(g)$ or $M(g^2)$ in $\{\text{id}_W\} \times \text{GL}(H^0(T_B))$. As g^3 commutes with such conjugation matrices, any element of $\langle k \rangle \cup \langle g^3 k \rangle = \text{PStab}(W + \tau) \cup g^3 \text{PStab}(W + \tau)$ is similar to a power of g .

Now, assume that $h \in G_W$ is not similar to a power of g . Then Lemma 5.6 (6) shows that 1 and -1 are eigenvalues of $\rho(h)$. Applying Lemma 5.6 (6) again to h^2 , we see that either h^2 is similar to a power of g , or 1 and -1 are eigenvalues of $\rho(h^2)$.

If 1 and -1 are eigenvalues of $\rho(h^2)$, $\rho(h)$, which has determinant 1, is similar to $\text{diag}(1, -1, i, i)$, or to $\text{diag}(1, -1, -i, -i)$. Moreover, $\rho(h)$ defines an automorphism of

B , and by [6, Thm.13.2.8, Thm.13.3.2], B must thus be isogenous to $S \times E_i^2$ for some abelian surface S . We already know that B is isogenous to $E \times E_j^3$, but this contradicts the uniqueness of the Poincaré decomposition of B up to isogeny [6, Thm.5.3.7].

Hence, h^2 is similar to a power of g , and as 1 is an eigenvalue of multiplicity at least 2 for it, $\rho(h^2) = \text{id}_B$. Hence, $\rho(h)$ is similar to $\text{diag}(1, 1, -1, -1)$.

We just proved that if $h \in G_W$ is not similar to a power of g , then $\rho(h)$ is similar to $\text{diag}(1, 1, -1, -1)$. However, if $\rho(h)$ is similar to $\text{diag}(1, 1, -1, -1)$, then $\rho(hg)$ has ω and $-\omega$ as eigenvalues, and thus is neither similar to a power of g , nor to $\text{diag}(1, 1, -1, -1)$, contradiction. This concludes Step 1.

Step 2: By Step 1 and since ρ is faithful, we know that every element of G_W has order 1, 2, 3, or 6. Moreover, there is exactly one element of order 2, namely g^3 , so $|G_W| = 2 \cdot 3^\beta$ for some $\beta \geq 1$. Let S be a 3-Sylow subgroup of G_W , and $s \in Z(S)$ of order 3. Let $s' \in S \setminus \{\text{id}_A\}$. By Step 1, every element of $\rho(S)$ other than id_B is similar to $\text{diag}(1, j, j, j)$, or to $\text{diag}(1, j^2, j^2, j^2)$, in particular, this is the case of s and s' , and cannot be both the case of ss' and s^2s' , since they commute. Hence, $s' \in \langle s \rangle$. So $S = \langle s \rangle \simeq \mathbb{Z}_3$, and thus $\beta = 1$.

So $G_W \supset \langle g \rangle$ has order 6: Hence $G_W = \langle g \rangle$.

Step 3: By [6, Cor.13.2.4, Prop.13.2.5(c)], the number of fixed points of g and g^3 on B are respectively 4 and 256. Let τ be a point of B fixed by g^3 but not by g .

By Proposition 4.12, there is a junior element h generating the cyclic group $\text{PStab}(W + \tau)$. By Step 2, $\langle h \rangle \subset G_W = \langle g \rangle$. Moreover, as $g^3 \in \text{PStab}(W + \tau) = \langle h \rangle$, we know that h has even order, hence order 6 by Proposition 5.3. So $\langle h \rangle = \langle g \rangle$, and as both g and h are the only junior elements of order 6 in their generated cyclic groups, $g = h$. But h fixes τ and g does not, contradiction. \square

Proof of Proposition 5.1. It is straightforward from Propositions 5.3 and 5.5. \square

6 The isogeny type of A

This section proves the first part of Theorem 1.6, namely the following proposition, inspired by [29, Proof of Lem.3.4].

Proposition 6.1. *Let A be an abelian variety of dimension n , G be a finite group acting freely in codimension 2 on A . Suppose that A/G has a crepant resolution X which is a Calabi-Yau manifold. Then either A is isogenous to E_j^n and G is generated by junior elements of order 3 and 6, or A is isogenous to $E_{u_7}^n$ and G is generated by junior elements of order 7.*

Proof. By the $M(G)$ -equivariant Poincaré's complete reducibility theorem [6, Thm.13.5.2, Prop.13.5.4, and the paragraph before], there are $M(G)$ -stable abelian subvarieties Y_1, \dots, Y_s of A such that:

- (1) For any $i \in \llbracket 1, s \rrbracket$, Y_i is isogenous to a power of a $M(G)$ -stable $M(G)$ -simple abelian subvariety of A . In particular, by [6, Prop.13.5.5], there is a simple abelian subvariety Z_i of Y_i such that Y_i is isogenous to a power of Z_i .
- (2) For each $i \neq j$, the set of $M(G)$ -equivariant homomorphisms satisfies

$$\text{Hom}_{M(G)}(Y_i, Y_j) = \{0\}.$$

- (3) The addition map $Y_1 \times \dots \times Y_s \rightarrow A$ is an $M(G)$ -equivariant isogeny.

We define

$$\begin{aligned} Y_I &= \prod_{i \in I} Y_i, \text{ where } I = \{i \in \llbracket 1, s \rrbracket \mid Z_i \sim E_j\} \\ Y_J &= \prod_{j \in J} Y_j, \text{ where } J = \{j \in \llbracket 1, s \rrbracket \mid Z_j \sim E_{u_7}\} \\ Y_K &= \prod_{k \in K} Y_k, \text{ where } K = \llbracket 1, s \rrbracket \setminus (I \cup J). \end{aligned}$$

The action of $M(G)$ on $Y_I \times Y_J \times Y_K$ is diagonal by (2), and there is a proper surjective finite morphism $A/M(G) \rightarrow Y_I/M(G) \times Y_J/M(G) \times Y_K/M(G)$ induced by the G -equivariant addition by (3). Composing with projections, we get proper surjective morphisms f_I, f_J, f_K from $A/M(G)$ to $Y_I/M(G)$, to $Y_J/M(G)$, and to $Y_K/M(G)$.

Let $g \in G$ be a junior element. By Propositions 5.3 and 5.5, g has order 3, or 7, or 6 and then five or six non-trivial eigenvalues. By Proposition 4.6, A thus contains an abelian subvariety isogenous to E_j^3 , or to $E_{u_7}^3$. Hence, $\dim Y_I + \dim Y_J \geq 3$, so one of the two quotients $Y_I/M(G)$, $Y_J/M(G)$ has positive dimension. Moreover, by Proposition 4.6 again, if g has order 3 or 6, $M(g)$ acts trivially on Y_J and Y_K , and if g as order 7, it acts trivially on Y_I and Y_K . Hence, $M(g)$ acts with determinant 1 on each of the three factors.

But G is generated by its junior elements by Lemma 2.17 and Proposition 2.7. By [20, 42], $Y_I/M(G)$, $Y_J/M(G)$ and $Y_K/M(G)$ are thus normal Gorenstein varieties.

We can now pullback the volume form of $Y_I/M(G)$ if it has positive dimension $y_I = y$, of $Y_J/M(G)$ of dimension $y_J = y$ else, to an $M(G)$ -invariant non-zero global holomorphic y -form on A . Note that the sections of Ω_A are invariant by translations of A , so that we in fact have a G -invariant non-zero global holomorphic y -form on A . It pulls back to X , which is a Calabi-Yau variety. Hence $y = n$, and either $A \sim E_j^n$ or $A \sim E_{u_7}^n$. The order of junior elements generating G is given accordingly by Propositions 4.6, 5.3. \square

7 Junior elements and pointwise stabilizers in codimension 5

In this section, we extend the results of Sections 4 and 5 to codimension $k = 5$. In the first subsection, we exclude the one type of junior element with exactly five non-trivial eigenvalues. In the second subsection, we prove the following result.

Proposition 7.1. *Let A be an abelian variety on which a finite group G acts freely in codimension 2. Suppose that A/G has a crepant resolution X . Let W be a translated abelian subvariety of codimension $k \leq 5$ in A such that $\{1\} \neq \text{PStab}(W) < G$. Then $\text{PStab}(W)$ is a cyclic group, generated by one junior element g of order 3 or 7.*

7.A Ruling out junior elements of order 6 with five non-trivial eigenvalues

Proposition 7.2. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X . Then there is no junior element of G whose matrix is similar to $\text{diag}(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j)$.*

Proof. Suppose by contradiction that there is an element $g \in G$ such that $g(0) = 0$ and, in some coordinates,

$$M(g) = \text{diag}(\mathbf{1}_{n-5}, \omega, \omega, \omega, \omega, j).$$

Then there is an abelian subvariety W of codimension 4 in A which is pointwise fixed by g^3 . By Proposition 4.12, $\text{PStab}(W)$ is cyclic, generated by one junior element h . As $g^3 \in \langle h \rangle$, h has even order. However, by Propositions 5.3 and 5.5, it must have order 3 or 7, contradiction! \square

7.B The pointwise stabilizer for loci of codimension 5

For proving Proposition 7.1, it is enough to establish the following result.

Proposition 7.3. *Let B be an abelian fivefold isogenous to either E_j^5 or $E_{u_7}^5$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\text{Aut}(B, 0)$ generated by junior elements of order p , and such that any subgroup of it acting not freely in codimension 4 is cyclic and generated by one junior element of order p . Then F is itself cyclic.*

Proof of Proposition 7.1 admitting Proposition 7.3. Let W be a translated abelian subvariety of codimension $k \leq 5$ in A such that $\{1\} \neq \text{PStab}(W) < G$. Propositions 4.12, 5.1 show that if $k \leq 4$, then $k = 3$ and $\text{PStab}(W)$ is cyclic, generated by one junior element. By Proposition 3.1, the junior generator thus has order 3 or 7.

So we can assume $k = 5$. Up to conjugating the whole group G by a translation, we can assume that $0 \in W$, and apply Proposition 4.3 to obtain a $\text{PStab}(W)$ -stable complementary abelian fivefold B to W . Let $F = \text{PStab}(W) \subset \text{Aut}(B, 0)$. It is generated by junior elements by Proposition 4.3 (3), which have order 3 or 7 by Propositions 5.3, 5.5, 7.2. Let F' be a non-trivial subgroup of F acting not freely in codimension 4: There is an abelian variety $W' \supsetneq W$ of codimension at most 4 such that $F' \subset \text{PStab}(W')$. By Propositions 4.12, 5.3, 5.5, $\text{PStab}(W')$ is cyclic of prime order, so $F' = \text{PStab}(W')$ is cyclic generated by one junior element of order 3 or 7.

Note that, by uniqueness of the Poincaré decomposition of B [6, Thm.5.3.7], the group $\text{Aut}(B, 0)$ cannot contain both a junior element of order 3 and a junior element of order 7. Hence, if $F = \text{PStab}(W)$ is cyclic, Lemma 4.24 shows that it is generated by one junior element of order $p = 3$ or $p = 7$.

To conclude the proof of Proposition 7.1, we thus show by contradiction that F is not cyclic. If F is not cyclic, there are two junior elements $g, h \in F$ such that $\langle g, h \rangle$ is not cyclic, hence acts freely in codimension 4 on B . Let B_g and B_h be the abelian subvarieties of dimension 3 fixed pointwise by g and h in B . Note that $B_g \sim B_h \sim E_j^3$ if g and h have order $p = 3$, or $B_g \sim B_h \sim E_{u_7}^3$ if g and h have order $p = 7$. Hence, B is accordingly isogenous to E_j^5 or to $E_{u_7}^5$. So the assumptions of Proposition 7.3 are satisfied, whence F is cyclic, contradiction! \square

To establish Proposition 7.3, we start with a lemma.

Lemma 7.4. *Let B be an abelian fivefold isogenous to either E_j^5 or $E_{u_7}^5$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\text{Aut}(B, 0)$ generated by junior elements of order p , and such that any subgroup of it acting not freely in codimension 4 is cyclic and generated by one junior element of order p . Let g be an element of F of prime order q . Then $p = q$.*

Proof. If 1 is an eigenvalue of g , then $\langle g \rangle$ acts not freely in codimension 4, so it is cyclic of order p , and $p = q$.

Suppose that 1 is not an eigenvalue of g . As g has prime order, and by Lemma 3.7, the characteristic polynomial $\chi_{g \oplus \bar{g}}$ is a power of the cyclotomic polynomial Φ_q . Hence, $\deg(\Phi_q) = q - 1$ divides 10, so $q \in \{2, 3, 11\}$. But:

- Since g has determinant 1 and no 1 among its eigenvalues, $q \neq 2$.
- If $q = 11$, since $\Phi_{11} = \chi_g \bar{\chi}_g$, it is reducible over $\mathbb{Q}[j]$ (if $p = 3$) or $\mathbb{Q}[u_7]$ (if $p = 7$). But by [41, Prop.2.4] Φ_{11} is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction!
- If $q = 3$, then $\Phi_3^5 = \chi_g \bar{\chi}_g$, so Φ_3 is reducible over $\mathbb{Q}[j]$ (if $p = 3$) or $\mathbb{Q}[u_7]$ (if $p = 7$). But by [41, Prop.2.4] Φ_3 is irreducible over $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, so $p = q = 3$.

\square

Proof of Proposition 7.3. In the notations of Proposition 7.3, Lemma 7.4 proves that F is a p -group. Hence, there is an element $g \in Z(F)$ of order p . Let $h \in F \setminus \langle g \rangle$ have order p too. Since $\langle g, h \rangle$ is not cyclic, it must act freely in codimension 4, i.e., $E_g(1) \cap E_h(1) = \{0\}$, or equivalently the trivial representation is not a subrepresentation of $\langle g, h \rangle \subset \text{Aut}(B, 0)$. As g and h commute, they are codiagonalizable.

If $p = 7$, this yields that gh has four or five eigenvalues of order 7, and thus the characteristic polynomial $\chi_{gh \oplus \bar{gh}}$ has exactly eight or ten common roots with Φ_7 , which contradicts its rationality (Lemma 3.7).

If $p = 3$, the elements of order p in F are each similar to one of the following:

$$\text{diag}(1, 1, j, j, j), \text{diag}(1, 1, j^2, j^2, j^2), \text{diag}(j, j, j, j, j^2), \text{diag}(j, j^2, j^2, j^2, j^2).$$

Most importantly, $\text{diag}(1, j, j, j^2, j^2)$ is forbidden because it is neither a power of a junior element, nor acting freely in codimension 4. Let χ be the character of the representation

$\langle g, h \rangle \subset \text{Aut}(B, 0)$, and a be the number of elements of $\langle g, h \rangle$ similar to $\text{diag}(1, 1, j, j, j)$. As $\langle g, h \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, it then has $4 - a$ elements similar to $\text{diag}(j, j, j, j, j^2)$. Hence,

$$0 = \langle \chi, \mathbf{1} \rangle = \chi(\text{id}) + a(2 + 3j + 2 + 3j^2) + (4 - a)(4j + j^2 + 4j^2 + j) = -15 + 6a,$$

contradiction!

Hence, $\langle g \rangle$ is the only cyclic subgroup of order p in F , so by [36, 5.3.6], F is cyclic. \square

8 Junior elements and pointwise stabilizers in codimension 6

The goal of this section is to extend the results of Sections 4, 5, 7 to codimension $k = 6$. For the first time in our study of pointwise stabilizers, and for the second time in this paper after Section 6, we need to assume the existence of a Calabi-Yau resolution, and not just a crepant (or even simply-connected crepant) resolution of the singular quotient A/G . Indeed, in dimension 6, products of the two examples of [32] yield non-Calabi-Yau simply-connected crepant resolutions of certain singular quotients A/G .

We start by proving the following partial classification of pointwise stabilizers in codimension 6 in Subsection 8.A.

Proposition 8.1. *Let A be an abelian variety on which a finite group G acts freely in codimension 2. Suppose that A/G has a crepant resolution X which is a Calabi-Yau manifold. Let W be a translated abelian subvariety of codimension $k \leq 6$ in A such that $\{1\} \neq \text{PStab}(W) < G$ contains no junior element of type $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega)$. Then $\text{PStab}(W)$ is one of the following.*

- A cyclic group generated by one junior element of order 3 or 7.
- An abelian group generated by two junior elements g and h of order both 3 or both 7, satisfying $E_g(1) \cap E_h(1) = H^0(W, T_W)$.
- $\text{SL}_2(\mathbb{F}_3)$, and the representation $M : \text{PStab}(W) \hookrightarrow \text{Aut}(A, 0)$ decomposes as $1^{\oplus n-6} \oplus \sigma^{\oplus 3}$, where σ is the unique irreducible 2-dimensional faithful representation of $\text{SL}_2(\mathbb{F}_3)$ over the splitting field $\mathbb{Q}[j]$.

We then use this result to rule out the existence of junior elements with six non-trivial eigenvalues in Subsection 8.B by a mix of local and global arguments, and finally refine Proposition 8.1 in Subsection 8.C to the following result.

Proposition 8.2. *Let A be an abelian variety on which a finite group G acts freely in codimension 2. Suppose that A/G has a crepant resolution X which is a Calabi-Yau manifold. Let W be a translated abelian subvariety of codimension $k \leq 6$ in A such that $\{1\} \neq \text{PStab}(W) < G$. Then $\text{PStab}(W)$ is one of the following.*

- A cyclic group generated by one junior element of order 3 or 7.
- An abelian group generated by two junior elements g and h of order both 3 or both 7, satisfying $E_g(1) \cap E_h(1) = H^0(W, T_W)$.

8.A The pointwise stabilizers for loci of codimension 6

For proving Proposition 8.1, it is enough to establish the following result.

Proposition 8.3. *Let B be an abelian sixfold isogenous to either E_j^6 or $E_{u_7}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\text{Aut}(B, 0)$ generated by junior elements of order p , such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p . Suppose that $\text{wid}_B \notin F$. Then F is one of the following.*

- A cyclic group generated by one junior element of order p .
- An abelian group generated by two junior elements g and h of order p satisfying $E_1(g) \cap E_1(h) = H^0(W, T_W)$.

- $\mathrm{SL}_2(\mathbb{F}_3)$, and the representation $M : \mathrm{PStab}(W) \hookrightarrow \mathrm{Aut}(B, 0)$ decomposes as $\sigma^{\oplus 3}$, where σ is the unique irreducible 2-dimensional faithful representation of $\mathrm{SL}_2(\mathbb{F}_3)$ over the splitting field $\mathbb{Q}[j]$. In this case, $p = 3$.

Proof of Proposition 8.1 admitting Proposition 8.3. Let W be a translated abelian subvariety of codimension $k \leq 6$ in A such that $\{1\} \neq \mathrm{PStab}(W) < G$ contains no junior element of type $\mathrm{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. Proposition 7.1 settles the cases when $k \leq 5$, so we can assume $k = 6$. Up to conjugating the whole group G by a translation, we can assume that $0 \in W$, and apply Proposition 4.3 to obtain a $\mathrm{PStab}(W)$ -stable complementary abelian sixfold B to W . By Proposition 6.1 and as an abelian subvariety of A , B is isogenous to either E_j^6 or $E_{u_7}^6$.

Let $F = \mathrm{PStab}(W) \subset \mathrm{Aut}(B, 0)$. It is generated by junior elements by Proposition 4.3 (3), which have order 3 or 7 by Propositions 5.1, 5.5, 7.2 and since, by assumption, $\mathrm{id}_B \notin F$. Let F' be a subgroup of F acting not freely in codimension 5: then there is an abelian variety $W' \supsetneq W$ of codimension at most 5 such that $F' \subset \mathrm{PStab}(W')$. By Proposition 7.1, $\mathrm{PStab}(W')$ is cyclic of prime order, so $F' = \mathrm{PStab}(W')$ is cyclic generated by one junior element of order 3 or 7.

So Proposition 8.3 applies, and proves Proposition 8.1. \square

To establish Proposition 8.3, we need numerous lemmas.

Lemma 8.4. *Let B be an abelian sixfold isogenous to either E_j^6 or $E_{u_7}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let $g \in \mathrm{Aut}(B, 0)$ be an element of prime order q . Suppose that, in case $\langle g \rangle$ acts non-freely in codimension 5, it is cyclic generated by one junior element of order p . We have $q \in \{2, 3, 7\}$.*

Proof. If 1 is an eigenvalue of g , then g has order $q = p$, as wished.

Suppose that 1 is not an eigenvalue of g . By Lemma 3.7, the characteristic polynomial $\chi_{g \oplus \bar{g}}$ is thus a power of Φ_q , so $q - 1$ divides 12, so $q \in \{2, 3, 5, 7, 13\}$.

- If $q = 13$, then $\Phi_{13} = \chi_g \bar{\chi}_g$. But by [41, Prop.2.4], Φ_{13} is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction.
- If $q = 5$, then $\Phi_5^3 = \chi_g \bar{\chi}_g$. But by [41, Prop.2.4], the cyclotomic polynomial Φ_5 is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction.

\square

Let us describe the 2-, 3-, and 7-Sylow subgroups of F .

Lemma 8.5. *Let B be an abelian sixfold isogenous to either E_j^6 or $E_{u_7}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\mathrm{Aut}(B, 0)$ generated by junior elements of order p , such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p . If 2 divides $|F|$, a 2-Sylow subgroup S of F is isomorphic to Q_8 .*

Proof. Since $-\mathrm{id}_B$ is the unique element of order 2 that can belong to F , by [36, 5.3.6], S is cyclic or a generalized quaternion group. Let us show that S has no element of order 8. By contradiction, let $s \in S$ be of order 8. Since $s^4 = -\mathrm{id}_B$, all eigenvalues of s have order 8, so the characteristic polynomial $\chi_{s \oplus \bar{s}}$ is a power of Φ_8 . Comparing degrees yields $\Phi_8^3 = \chi_s \bar{\chi}_s$. But by [41, Prop.2.4], Φ_8 is irreducible over $\mathbb{Q}[j]$ and $\mathbb{Q}[\zeta_7] \supset \mathbb{Q}[u_7]$, contradiction! So S is isomorphic to $\mathbb{Z}_2, \mathbb{Z}_4$, or Q_8 .

If S is cyclic, then by [36, 10.1.9], there is a normal subgroup N of F such that $F = N \rtimes S$. But all junior elements of F have odd order, so they belong to N and cannot generate F , contradiction! So S is isomorphic to Q_8 . \square

Lemma 8.6. *Let B be an abelian sixfold. Let $g \in \mathrm{Aut}(B, 0)$ be an element of finite order. Then g cannot have order 27, 49, or 63.*

Proof. It is an immediate consequence of Lemma 3.7. \square

Lemma 8.7. *Let B be an abelian sixfold isogenous to either E_j^6 or $E_{u_7}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\text{Aut}(B, 0)$ generated by junior elements of order p , such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p . Let $q = 7$ if $p = 3$, $q = 3$ if $p = 7$. If q divides $|F|$, a q -Sylow subgroup S of F is cyclic and has order 3, 7, or 9.*

Proof. As S is a q -group, there is an element $g \in Z(S)$ of order q . Let $h \in S \setminus \langle g \rangle$ be another element of order q . Because $q \notin \{2, p\}$, g, h can not be powers of junior elements, and so 1 is not an eigenvalue of them. By Lemma 3.7, g and h are similar to

$$\begin{aligned} & \text{diag}(j, j, j, j^2, j^2, j^2) \text{ if } q = 3 \\ & \text{diag}(\zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6) \text{ if } q = 7 \end{aligned}$$

One can then find a non-trivial element of $\langle g, h \rangle$ with 1 as an eigenvalue. But as g and h commute, it has order $q \notin \{2, p\}$, contradiction. So $\langle g \rangle$ is the unique subgroup of order p in S . By [36, 5.3.6], S is thus cyclic, and its order is given by Lemma 8.6. \square

Lemma 8.8. *Let B be an abelian sixfold isogenous to either E_j^6 or $E_{u_7}^6$, and let $p = 3$ in the first case, $p = 7$ in the second case. Let F be a finite subgroup of $\text{Aut}(B, 0)$ generated by junior elements of order p , such that any subgroup of it acting not freely in codimension 5 is cyclic and generated by one junior element of order p . Then a p -Sylow subgroup S of F is either cyclic, or the direct product of two cyclic groups. It can be*

$$\begin{aligned} & \mathbb{Z}_3, \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3, \text{ or } \mathbb{Z}_3 \times \mathbb{Z}_9 \text{ if } p = 3 \\ & \mathbb{Z}_7, \text{ or } \mathbb{Z}_7 \times \mathbb{Z}_7 \text{ if } p = 7 \end{aligned}$$

Proof. Let $g \in Z(S)$ be an element of order p . If $\langle g \rangle$ is the only subgroup of order p in S , then by [36, 5.3.6], S is cyclic. Control on its order follows from Lemma 8.6. Else, let $[h], [k] \in S/\langle g \rangle$ have order p , $[h]$ belonging to the center of this p -group. Let us prove that $\langle [h] \rangle = \langle [k] \rangle$. If it is the case, then by [36, 5.3.6] again, $S/\langle g \rangle$ is cyclic. A fortiori, $S/Z(S)$ is cyclic, so S is abelian, and $S \simeq \langle g \rangle \times C$ for a cyclic group C containing $\langle h \rangle$. Control on the factors' orders follows from Lemma 8.6, and then concludes the proof.

If $p = 7$, then g has an eigenvalue ζ of order 7 with corresponding eigenspace $E_g(\zeta)$ of dimension 1. By Lemma 8.6, h and k have order 7 in S . As g commutes with h and k , we can thus choose $h' \in [h], k' \in [k]$ which both have 1 as an eigenvalue on $E_g(\zeta)$. Hence, the group $\langle h', k' \rangle$ does not act freely in codimension 5 on B , so it is cyclic generated by one junior element, and $\langle h' \rangle = \langle k' \rangle$ as wished.

If $p = 3$, let us show that $\text{jid}_B \in S$. By contradiction, suppose that elements of order 3 in S are all similar to one of the following matrices

$$\text{diag}(1, 1, 1, j, j, j), \text{diag}(1, 1, 1, j^2, j^2, j^2), \text{diag}(j, j, j, j^2, j^2, j^2).$$

Take $s \in S \setminus \langle g \rangle$. As g and s commute, a simple computation shows that one of the products gs, g^2s, gs^2, g^2s^2 will not fall under these three similarity classes, contradiction.

Hence, we can take $g = \text{jid}_B$. A fortunate consequence of that choice, of Lemma 3.7, and of the fact that matrices in S all have determinant 1 is that g has no cubic root in S , i.e., every element of order 9 in S has a class of order 9 in $S/\langle g \rangle$. Hence, h and k above have order 3. Moreover, recall that $hkh^{-1}k^{-1} \in \langle g \rangle = \langle \text{jid}_B \rangle$. If k is conjugated to jk or j^2k , then 1, j , and j^2 each are eigenvalues of k , contradiction! Hence, $hkh^{-1} = k$, i.e., h and k commute. They commute with g as well, and thus we can find some non-trivial elements in $[h]$ and $[k]$ with a common eigenvector of eigenvalue 1. So $\langle [h] \rangle = \langle [k] \rangle$. \square

Proof of Proposition 8.3. We now run (see Appendix) a GAP search through all groups with such 2, 3, and 7-Sylow subgroups, which have at most an element of order 2, and no element of order 63. Among the ninety-four of them, only \mathbb{Z}_7 and $\mathbb{Z}_7 \times \mathbb{Z}_7$ can be generated by their elements of order 7, whereas $\mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3, \text{SL}_2(\mathbb{F}_3), Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$, and $\mathbb{Z}_3 \times (Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ can be generated by their elements of order 3. However, it is easy to check that $Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$, and $\mathbb{Z}_3 \times (Q_8 \rtimes (\mathbb{Z}_7 \rtimes \mathbb{Z}_3))$ have elements of order 28, which by Lemma 3.7 and [41, Prop.2.4] cannot occur in $\text{Aut}_{\mathbb{Q}}(E_j^6, 0)$.

The representation theoretic description is easily obtained from GAP for $\text{SL}_2(\mathbb{F}_3)$, and follows from the condition about freeness in codimension 5 for $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_7 \times \mathbb{Z}_7$. \square

8.B Ruling out junior elements of order 6 with six non-trivial eigenvalues

Proposition 8.9. *Let A be an abelian variety, G a group acting freely in codimension 2 on A such that A/G has a crepant resolution X . Then there is no junior element of G with matrix similar to $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$.*

In order to prove this, we first reduce to a 6-dimensional situation, where a lot of local information is given by Proposition 8.3.

Lemma 8.10. *Let A be an abelian variety, G a group acting freely in codimension 2 on A without translations such that A/G has a crepant resolution X . Suppose that there is an element $g \in G$ such that $g(0) = 0$, and with matrix similar to $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. Then there are complementary $\langle g \rangle$ -stable abelian subvarieties B and W in A such that $g|_B = \omega \text{id}_B$ and $g|_W = \text{id}_W$. For any $\tau \in B$, it holds $\text{PStab}(W + \tau) \subset \text{PStab}(W)$, and if τ is a non-zero 2-torsion point of B , we have $\text{PStab}(W + \tau) \simeq \text{SL}_2(\mathbb{F}_3)$.*

Proof. The existence of W and B follows from [6, Thm.13.2.8]. The fact that $\omega \text{id}_B \in \text{Aut}(B, 0)$ implies that B is isogenous to E_j^6 , by Proposition 4.6. By Schur's lemma, there is an $M(G)$ -stable supplementary S to $H^0(T_W)$ in $H^0(T_A)$ (which is not necessarily $H^0(T_B)$, since $M(G)$ is a larger group than $\text{PStab}(W)$).

Let $\tau \in B$. Let $h \in \text{PStab}(W + \tau)$. The matrices of both g^3 and h split into blocks with respect to the decomposition $H^0(T_A) = H^0(T_W) \oplus S$, so g^3 commutes with h . As the matrices of g and g^3 have the same eigenspaces (with possibly different eigenvalues), the matrices of g and h commute too, and since G contains no translation, g and h commute themselves. In particular, $g(T(h)) = T(h)$. Let us decompose then $T(h) = w + b$ with $w \in W$, $b \in B$:

$$0 = g(T(h)) - T(h) = g(w + b) - w - b = g(b) - b = (\omega - 1)b.$$

As by [6, Cor.13.2.4], ωid_B has exactly one fixed point on B , namely 0, we have $b = 0$, i.e., $T(h) \in W$. But h has a fixed point, so $T(h) \in \text{Im}(\text{id}_A - M(h))$. These two constraints yield $T(h) = 0$, whence $h \in \text{PStab}(W)$.

Suppose now that τ is a non-zero 2-torsion point. As $g^3|_B = -\text{id}_B$, g^3 fixes τ , i.e., $g^3 \in \text{PStab}(W + \tau)$. Since G contains no translation and contains g , no element with matrix similar to $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$ belongs to $\text{PStab}(W + \tau)$. Proposition 8.3 therefore applies to $\text{PStab}(W + \tau)$, implying that it is isomorphic to $\text{SL}_2(\mathbb{F}_3)$ (as it contains the element g^3 of order 2). \square

Remark 8.11. This notably shows that, if G contains a junior element g of type $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$ such that $g(0) = 0$, and W is the maximal abelian subvariety of A fixed by g , then the group G_W defined in Lemma 5.6 coincides with $\text{PStab}(W)$.

This description of the pointwise stabilizers of the translations of W by 2-torsion points yields the following description of the much larger group $\text{PStab}(W)$.

Lemma 8.12. *Let A be an abelian variety, G a group acting freely in codimension 2 on A without translations such that A/G has a crepant resolution X . Suppose that there is an element $g \in G$ such that $g(0) = 0$, and with matrix similar to $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. Let B, W be as in Lemma 8.10. Then there is an element $h \in \text{PStab}(W)$ of prime order p if and only if $p = 2$ or 3 . Moreover, a 2-Sylow subgroup S_2 of $\text{PStab}(W)$ is isomorphic to Q_8 , and a 3-Sylow subgroup S_3 contains an even number of junior elements (of order 3). The group $\text{PStab}(W)$ contains exactly 260 junior elements.*

Proof. The group $\text{PStab}(W)$ contains a unique element g^3 of order 2, so by [36, 5.3.6], its 2-Sylow subgroup S_2 is cyclic or a generalized quaternion group. Moreover, $\text{PStab}(W)$ acts on a complementary abelian variety to W , which is isomorphic to E_j^6 by Proposition 4.6, and the only elements of $\text{PStab}(W)$ with 1 as an eigenvalue are powers of junior elements. Hence, $\text{PStab}(W) \subset \text{SL}_6(\mathbb{Q}[j])$ has no element of order 8, i.e., S_2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$, or Q_8 . But by Lemma 8.10, a copy of $Q_8 \subset \text{SL}_2(\mathbb{F}_3)$ embeds in $\text{PStab}(W)$, and therefore $S_2 \simeq Q_8$.

The group $\text{PStab}(W)$ contains g^2 , which has order 3. Note that g^2 commutes with all elements of $\text{PStab}(W)$, and thus belongs to any 3-Sylow subgroup of it, in particular S_3 . Now, the map $h \in S_3 \mapsto g^2 h^2 \in S_3$ sends a junior element of order 3 to a junior element of order 3, and is a fixed-point-free involution. Hence, S_3 contains an even number of junior elements (of order 3).

We can also count the number of junior elements in $\text{PStab}(W)$ easily: each of them fixes exactly $2^6 - 1$ non-zero 2-torsion points of B , and every non-zero 2-torsion point of B is fixed by exactly 4 junior elements by Lemma 8.4. Since B has $2^{12} - 1$ non-zero 2-torsion points, the number of junior elements in $\text{PStab}(W)$ is $\frac{(2^{12}-1) \cdot 4}{2^6-1} = 260$.

At last, let $h \in \text{PStab}(W)$ have prime order p . Suppose by contradiction that $p \neq 2, 3$. By Lemma 8.4, we have $p = 7$, and since $\text{SL}_6(\mathbb{Q}[j])$ has no junior element of order 7, 1 is not an eigenvalue of h . Hence, all six eigenvalues of h have order 7. Note that h acts by conjugation on the set of junior elements of $\text{PStab}(W)$, whose cardinal, which we just computed, is not divisible by 7. Hence, h commutes with a junior element $k \in \text{PStab}(W)$, so $hk \in \text{PStab}(W)$ has order 21, and three eigenvalues of order 7, three eigenvalues of order 21. By Lemma 3.7, $\Phi_7 \Phi_{21}$ thus divides the characteristic polynomial of $hk \oplus \bar{h}k$, but they have respective degrees $\phi(7) + \phi(21) = 18$ and 12, contradiction! \square

This result has the following consequence.

Corollary 8.13. *Let A be an abelian variety, G a group acting freely in codimension 2 on A without translations such that A/G has a crepant resolution X . Suppose that there is an element $g \in G$ such that $g(0) = 0$, and with matrix similar to $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. Let B, W be as in Lemma 8.10. Then the group $\text{PStab}(W)$ has exactly four 3-Sylow subgroups S, T, U and V . There is no junior element in the intersection $S \cap T$, and thus S contains exactly 65 junior elements of order 3.*

Proof. By Lemma 8.12, there is a positive integer β such that

$$|\text{PStab}(W)| = 8 \cdot 3^\beta.$$

The number n_3 of 3-Sylow subgroups in $\text{PStab}(W)$ is thus either 1, or 4.

Let $\tau \neq 0$ be a 2-torsion point in B . By Lemma 8.10, there are exactly four junior elements s, t, u, v of order 3 of $\text{PStab}(W)$ fixing τ . We can check in the multiplication table of $\text{SL}_2(\mathbb{F}_3)$ that the product of any two distinct elements of $\{s, t, u, v\}$ has order 6. Hence, each 3-Sylow subgroup of $\text{PStab}(W)$ contains at most one element of $\{s, t, u, v\}$. So $n_3 \geq 4$, hence $n_3 = 4$. Denote by S, T, U , and V the four 3-Sylow subgroups of $\text{PStab}(W)$.

Suppose by contradiction that $S \cap T$ contains a junior element h (of order 3). Let $\tau \neq 0$ be a non-zero 2-torsion point in B fixed by h . Again, there are exactly four junior elements s, t, u, v of order 3 in $\text{PStab}(W + \tau)$, and no two of them belong to the same 3-Sylow subgroup of $\text{PStab}(W)$: In particular, t, u, v belong to either U or V , but that is three elements to fit into two 3-Sylow subgroups, contradiction!

Finally, the junior elements of S, T, U, V , partition the set of junior elements of $\text{PStab}(W)$. By the second Sylow theorem, these four partitioning pieces are in bijection, so S has $\frac{260}{4} = 65$ junior elements. \square

Proof of Proposition 8.9. By contradiction, suppose that G contains a junior element g of type $\text{diag}(\mathbf{1}_{n-6}, \omega, \omega, \omega, \omega, \omega, \omega)$. By Remark 5.4, we can assume that G contains no translation other than id_A , and up to conjugating the whole group by a translation, we can assume that $g(0) = 0$. Now, Lemma 8.12 and Corollary 8.13 apply, but since 65 is odd, they contradict one another. \square

8.C Ruling out the pointwise stabilizer $\text{SL}_2(\mathbb{F}_3)$

In this subsection, we prove Proposition 8.2. By Proposition 8.1, it is enough to show the following:

Lemma 8.14. *Let A be an abelian variety on which a finite group G acts freely in codimension 2 without translations. Suppose that A/G has a simply-connected crepant resolution X . Then there is no abelian subvariety W of codimension 6 in A such that $\text{PStab}(W) \simeq \text{SL}_2(\mathbb{F}_3) < G$, with representation $M = \mathbf{1}^{\oplus n-6} \oplus \sigma^{\oplus 3}$ as in Proposition 8.1.*

This result resembles [2, Sec.6.1], although working under a different set of assumptions and in dimension 6.

Proof. We prove it by contradiction, using global arguments. Consider such an abelian subvariety W , and apply Lemma 5.6, defining the group G_W and a G_W -stable complementary B to W . The peculiar features of the representation $\sigma^{\oplus 3} : \mathrm{SL}_2(\mathbb{F}_3) < G_W \rightarrow \mathrm{Aut}(B, 0)$ yield that B is isogenous to E_j^6 . Let $g \in \mathrm{PStab}(W) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ be the unique element of order 2. Recall that $g|_B = -\mathrm{id}_B$.

Step 1: If $h \in G_W$ fixes no point, then h has even order.

Proof. Indeed, by Lemma 5.6 (6), either hg fixes a point τ , or 1 and -1 are eigenvalues of h . Clearly, h has even order in the second case. In the first case, hg actually is in $\mathrm{PStab}(W + \tau)$, and Propositions 8.1, 8.9 yield that $\mathrm{PStab}(W + \tau)$ is isomorphic to \mathbb{Z}_3 , $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathrm{SL}_2(\mathbb{F}_3)$. So either hg has order 3, in which case h has even order 6, or $hg \in \mathrm{PStab}(W + \tau) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ has order 2, 4, or 6. But then, $g \in \mathrm{PStab}(W + \tau)$ since G_W contains no translation. So $h \in \mathrm{PStab}(W + \tau)$ fixes points, contradiction! \square

Step 2: If $h \in G_W$ has prime order p , then $p \in \{2, 3\}$. Moreover, if $p = 3$, h is a junior element or has junior square.

Proof. By Step 1, $p = 2$ if h fixes no point. By Proposition 8.1 in the case $B \sim E_j^6$, $p \in \{2, 3\}$ if h fixes a point.

Hence, in the case when $p = 3$, we have $h \in \mathrm{PStab}(W + \tau)$ for some $\tau \in A$. Apply Proposition 8.1 to $\mathrm{PStab}(W + \tau)$. Note that by Proposition ??, wid_B does not appear in $\rho(G_W)$, and as $g|_B = -\mathrm{id}_B$ does, jid_B does not. In particular, $\mathrm{PStab}(W + \tau)$ can not be Item 2 (i.e., $\mathbb{Z}_3 \times \mathbb{Z}_3$) of Proposition 8.1. In the remaining Items 1 and 3 of that proposition, every order 3 element of $\mathrm{PStab}(W + \tau)$ is junior or has junior square, and so is h . \square

Step 3: A 3-Sylow subgroup S of G_W is isomorphic to \mathbb{Z}_3 , generated by one junior element.

Proof. Let $h \in S$ be a non-trivial element. It has odd order, hence it fixes a point by Step 1, and thus it has order 3 by Proposition 8.1. By Step 2, it is thus junior or a square of a junior element.

Let $s \in Z(S)$ be non-trivial, hence again (the square of) a junior element. Let us show that $h \in \langle s \rangle$. As h and s commute, either they have the same eigenspace for the eigenvalue 1, in which case $h \in \langle s \rangle$ as wished, or $E_{s|_B}(1)$ and $E_{h|_B}(1)$ are in direct sum, in which case $\mathrm{jid}_B \in \langle s|_B, h|_B \rangle$, and so $\mathrm{wid}_B \in \rho(G_W)$, which contradicts Proposition 8.9. Hence, $h \in \langle s \rangle$ and thus $S = \langle s \rangle \simeq \mathbb{Z}_3$. \square

Step 4: If S_2, S_3 are 2 and 3-Sylow subgroups of G_W , then $G_W = S_2 \rtimes S_3$.

Proof. By Step 3, no two elements of S_3 are conjugated in G_W , so $N_{G_W}(S_3) = C_{G_W}(S_3)$, and by Burnside's normal complement theorem [36, 10.1.8], there is a normal subgroup $N \triangleleft G_W$ such that $G_W = N \rtimes S_3$. By Step 2, N is a 2-group, and it is clearly maximal. As it is normal, it is the unique 2-Sylow subgroup of G , so $N = S_2$. \square

Step 5: S_2 has order 2^9 .

Proof. We first count the number of junior elements in G_W . By Lemma 5.6 (9), every junior element in G_W fixes at least one 2-torsion point in B . Since it acts trivially on a 3-dimensional translated abelian subvariety of B , it fixes precisely 2^6 of the 2-torsion points in B . Each 2-torsion point τ in B is besides fixed by the four junior elements of $\mathrm{PStab}(W + \tau) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ (by Proposition 8.1 and since g of order 2 belongs to $\mathrm{PStab}(W + \tau)$). Hence, there are $\frac{2^{12} \times 4}{2^6} = 2^8$ junior elements in G_W .

Now, note that by Step 3, the number n_3 of 3-Sylow subgroups of G_W equals the number of junior elements in G_W . Hence, denoting by S_3 a 3-Sylow subgroup of G_W ,

$$3|S_2| = |G_W| = n_3|N_{G_W}(S_3)| = n_3|C_{G_W}(S_3)| = 2^9 \cdot 3,$$

since it is easily checked that $C_{G_W}(S_3) = \langle g, S_3 \rangle \simeq \mathbb{Z}_6 < \mathrm{SL}_2(\mathbb{F}_3)$. \square

Step 6: Denote by m_2, m_4 the number of elements of order 2 and 4 in S_2 . Then $m_2 = 6 \cdot 61 + 1$ and $m_4 = 144$.

Proof. We first describe the order and trace of elements $h \in S_2$ different from id_A and g . By Lemma 3.7, since $B \sim E_j^6$, and by [41, Prop.2.4], the characteristic polynomial of $\rho(h) = M(h|_B)$ satisfies

$$\chi_{\rho(h)} = (X - 1)^\alpha (X + 1)^\beta \Phi_4(X)^\gamma \Phi_8(X)^\delta,$$

with $\alpha, \beta, \gamma, \delta \geq 0$, β being even because of the determinant and $\alpha + \beta + 2\gamma + 4\delta = 6$ because of the dimension. Hence, α is even too. If $\alpha\beta = 0$, then by Lemma 5.6, there is $\tau \in A$ such that $h \in \mathrm{PStab}(W + \tau) \cup g\mathrm{PStab}(W + \tau)$, so by Proposition 8.1, the only possibility for h other than id and g satisfies $\chi_{\rho(h)} = \Phi_4^3$, hence $\alpha = \beta = 0$. Else, α and β are positive. So, $(\alpha, \beta, \gamma, \delta)$ can be $(0, 0, 3, 0), (2, 2, 1, 0), (2, 4, 0, 0)$, or $(4, 2, 0, 0)$. In particular, h has order 2 or 4, with order 4 if and only if $\mathrm{Tr}(h|_B) = 0$, and order 2 if and only if $\mathrm{Tr}(h|_B) \in \{-2, 2\}$.

Decomposing the representation $\rho|_{S_2}$ into irreducible subrepresentations yields a splitting coefficient $u \in \mathbb{N}$ such that $u|_{S_2} = 72 + 4(m_2 - 1)$, where m_2 is the number of elements of order 2 in S_2 . Denoting by m_4 the number of elements of order 4 in S_2 and using Step 5, we rewrite $(u - 4) \cdot 2^9 + 4m_4 = 64$. So $u \leq 4$.

Note that $h \in G_W$ junior of order 3 acts by conjugation on the set of elements of order 2 of the normal subgroup S_2 , and the only fixed point is the element $g \in C_{G_W}(\langle h \rangle)$. Hence, $m_2 - 1$ is divisible by 3. So u is divisible by 3, and thus $u = 3$, and $m_2 = 6 \cdot 61 + 1$, and $m_4 = 144$. \square

Step 7: But $m_4 \geq 6 \cdot 2^6$, contradiction!

Proof. Let us show that the number of elements of G_W of order 4 fixing a point is exactly $6 \cdot 2^6$. By Lemma 5.6 (8), if $h \in G_W$ has order 4 and fixes a point, then all its 2^6 fixed points in B are 2-torsion points of B . Moreover, by Proposition 8.1, for any $\tau \in B$ of 2-torsion, $\mathrm{PStab}(W + \tau) \simeq \mathrm{SL}_2(\mathbb{F}_3)$ contains exactly six elements of order 4. Hence the count of $\frac{2^{12} \cdot 6}{2^6} = 6 \cdot 2^6$ elements of order 4 fixing a point in G_W . \square

And with this contradiction ends the proof of Lemma 8.14. \square

Remark 8.15. Local information would not have been enough to rule out $\mathrm{SL}_2(\mathbb{F}_3)$. Indeed, considering a simply-connected neighborhood $U \subset \mathbb{C}^6$ of 0, which is stable by the action of $\rho^{\oplus 3} : \mathrm{SL}_2(\mathbb{F}_3) \hookrightarrow \mathrm{SL}_6(\mathbb{Q}[j])$, the quotient $U/\mathrm{SL}_2(\mathbb{F}_3)$ admits a crepant resolution. Let us construct it.

Under the action of $\mathrm{SL}_2(\mathbb{F}_3)$ on \mathbb{C}^6 , exactly four 3-dimensional linear subspaces Z_1, Z_2, Z_3, Z_4 have non-trivial point-wise stabilizers $\langle g_1 \rangle, \langle g_2 \rangle, \langle g_3 \rangle, \langle g_4 \rangle \simeq \mathbb{Z}_3$, where g_1, g_2, g_3, g_4 are the four junior elements of $\mathrm{SL}_2(\mathbb{F}_3)$. Using `Macaulay2`, a quick computation shows that the blow-up:

$$\varepsilon : B := \mathrm{Bl}_{\mathcal{I}_{Z_1} \cap \mathcal{I}_{Z_2} \cap \mathcal{I}_{Z_3} \cap \mathcal{I}_{Z_4}}(\mathbb{C}^6) \rightarrow \mathbb{C}^6$$

is a smooth quasiprojective variety with a four-dimensional central fiber $\varepsilon^{-1}(0)$. In particular, B contains exactly four prime exceptional divisors, one above each Z_i .

By the universal property of the blow-up, the action of $\mathrm{SL}_2(\mathbb{F}_3)$ on \mathbb{C}^6 lifts to an action on B . The lifted automorphism \tilde{g}_i fixes the exceptional divisor $\varepsilon^{-1}(Z_i)$ pointwise: hence, locally, for any $x \in B$, $\mathrm{PStab}(x)$ is generated by pseudoreflections. Hence by Chevalley-Shepherd-Todd theorem, the quotient $X := B/\mathrm{SL}_2(\mathbb{F}_3)$ is smooth.

We are going to prove that the resolution $X \rightarrow \mathbb{C}^6/\mathrm{SL}_2(\mathbb{F}_3)$ is crepant. As $\mathrm{SL}_2(\mathbb{F}_3) \subset \mathrm{GL}_6(\mathbb{C})$ has one conjugacy class of junior elements, by Theorem 2.4, there is exactly one crepant divisor above $\mathbb{C}^6/\mathrm{SL}(2, 3)$: A smooth resolution must contain this crepant divisor, and is thus crepant if and only if it contains exactly one exceptional divisor. This is clearly the case for X , since the action of $Q_8 \subset \mathrm{SL}_2(\mathbb{F}_3)$ on B is transitive on the set of the four prime exceptional divisors in B .

9 Proof of Theorem 1.4

This is now straightforward.

Proof of Theorem 1.4. By Lemma 2.17, there is an element $g \in G$ that admits a fixed point $a \in A$. As $\text{PStab}(a)$ is non-trivial, by Proposition 4.3, it contains a junior element. By Propositions 5.3, 5.5, 7.2, 8.9, this junior element has eigenvalue 1 with multiplicity $\dim(A) - 3$, i.e., it stabilizes a translated abelian subvariety of A of codimension 3. But G acts freely in codimension 3, contradiction. \square

10 Concluding the proof of Theorem 1.6

Let us assemble the parts of the previous sections to prove Theorem 1.6.

Proof of Theorem 1.6. Let A be an abelian variety of dimension n , and let G be a finite group acting freely in codimension 2 on A , such that A/G has a resolution X that is a Calabi-Yau manifold. By Proposition 6.1, either A is isogenous to E_j^n and G is generated by junior elements of order 3 and 6, or A is isogenous to $E_{u_7}^n$ and G is generated by junior elements of order 7. In particular, G is generated by its elements admitting fixed points. Also note that G contains no junior element of order 6 by Propositions 5.5, 7.2, 8.9.

Let us show that for any translated abelian subvariety $W \subset A$, the pointwise stabilizer $\text{PStab}(W)$ is abelian. It is generated by junior elements by Proposition 4.3. Let g, h be two junior elements in $\text{PStab}(W)$. As g and h both fix abelian varieties of codimension 3, their intersection W' has codimension 3, 4, 5, or 6 in A . By Proposition 8.2, $\text{PStab}(W')$ is thus abelian, and therefore g and h commute.

Moreover, any two junior elements g and h in $\text{PStab}(W)$ have the same order (3 if $A \sim E_j^n$, 7 if $A \sim E_{u_7}^n$). Hence, using the structure theorem for finite abelian groups, $\text{PStab}(W)$ is isomorphic to \mathbb{Z}_3^k for some k if $A \sim E_j^n$, to \mathbb{Z}_7^k for some k if $A \sim E_{u_7}^n$. Finally, if $g, h \in \text{PStab}(W)$ are junior elements, then their eigenspaces with eigenvalues other than 1 are in direct sum by Proposition 8.2. An induction using that all junior elements of $\text{PStab}(W)$ are codiagonalizable then yields Item 3 in Theorem 1.6. \square

11 Proof of Theorem 1.5

In this section, we proceed to the proof of Theorem 1.5, which in fact splits into two pieces. The first piece describes a slight generalization of the situation in dimension 3 [32]. It notably gives an alternative proof of [32, Key Claim 2], replacing the discussion on invariant cohomology and topological Euler characteristics inherent to [32, §3] with group theory and a geometric fixed loci argument ruling out the special linear group $\text{SL}_3(\mathbb{F}_2)$.

Theorem 11.1. *Let A be an abelian variety on which a finite group G acts freely in codimension 2 without translations. Suppose that A/G has a resolution X which is a Calabi-Yau manifold. Then, for any two junior elements $g, h \in G$ such that $\langle g \rangle \neq \langle h \rangle$, the intersection of eigenspaces $E_{M(g)}(1) \cap E_{M(h)}(1)$ has codimension $k \neq 3$ in $H^0(A, T_A)$.*

The second piece is rather specific to dimension 4.

Theorem 11.2. *Let A be an abelian variety on which a finite group G acts freely in codimension 2 without translations. Suppose that A/G has a resolution X which is a Calabi-Yau manifold. Then, for any two junior elements $g, h \in G$ such that $\langle g \rangle \neq \langle h \rangle$, the intersection of eigenspaces $E_{M(g)}(1) \cap E_{M(h)}(1)$ has codimension $k \neq 4$ in $H^0(A, T_A)$.*

Let us show how these two results imply Theorem 1.5.

Proof of Theorem 1.5, using Theorems 11.1, 11.2. Suppose by contradiction that A has dimension 4, and that A/G admits a simply-connected crepant resolution X . Then by

[28, Thm, Cor.1], X can not be holomorphic symplectic. Hence, by the smooth Beauville-Bogomolov decomposition theorem, X is a Calabi-Yau fourfold. Up to replacing A by an isogenous variety, we can assume that G contains no translation.

If G entails two junior elements g, h such that $\langle g \rangle \neq \langle h \rangle$, then Theorems 11.1 and 11.2 show that the eigenspaces $E_{M(g)}(1)$ and $E_{M(h)}(1)$ are in direct sum. But they are 3-dimensional subspaces of the 4-dimensional vector space $H^0(T_A)$, contradiction!

So G has all of its junior elements contained in $\langle g \rangle$, and thus by Item 1 in Theorem 1.6, $G = \langle g \rangle$ and g has order 3 or 7, and admits 1 as an eigenvalue of multiplicity one. Up to conjugating the whole group G by a translation, we can assume $g(0) = 0$. Let $E \subset A$ be the elliptic curve containing 0 and fixed pointwise by g , and B be its $\langle g \rangle$ -stable supplementary. Hence, G acts diagonally on $E \times B$ by $\{\text{id}_E\} \times \langle g|_B \rangle$, and the addition map $E \times B \rightarrow A$ is a G -equivariant isogeny by [6, Thm.13.2.8]. The volume form on E thus pulls back to a G -invariant 1-form on A , and thus to a non-zero global holomorphic 1-form on the Calabi-Yau resolution X of A/G , contradiction. \square

11.A Proof of Theorem 11.1

By Theorem 1.6, the proof reduces to the following two cases. The first one is simple.

Proposition 11.3. *Let A be an abelian variety isogenous to E_{j^n} . Let $g, h \in \text{Aut}(A)$ be two junior elements of order 3 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $g = h$.*

Proof. Recall that $M : \text{Aut}(A) \rightarrow \text{Aut}(A, 0)$ which, to any automorphism of A , associates its matrix, induces a representation of $\langle g, h \rangle$. As $\langle g, h \rangle$ contains no translation, M is faithful. Applying Maschke's theorem to the invariant subspace $E_{M(g)}(1) = E_{M(h)}(1)$ in $H^0(T_A)$ yields an $\langle M(g), M(h) \rangle$ -stable supplementary S to it. Let ρ be the faithful representation of $\langle g, h \rangle$ obtained by restricting M to S . By the classification of junior elements in Proposition 3.1, $\rho(g) = \rho(h) = \text{jid}_B$. But ρ is faithful, and thus $g = h$. \square

The second case is the following result.

Proposition 11.4. *Let A be an abelian variety isogenous to $E_{u_7^n}$. Let $g, h \in \text{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $\langle g \rangle = \langle h \rangle$.*

Its proof relies on two lemmas.

Lemma 11.5. *Let A be an abelian variety isogenous to $E_{u_7^n}$. Let $g, h \in \text{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $\langle g, h \rangle$ is isomorphic to \mathbb{Z}_7 or $\text{SL}_3(\mathbb{F}_2)$.*

Proof. By Maschke's theorem, there is an $\langle M(g), M(h) \rangle$ -stable supplementary S to $E_{M(g)}(1) = E_{M(h)}(1)$ in $H^0(T_A)$. Consider the faithful representation ρ of $\langle g, h \rangle$ given by restricting M to S , with character χ .

Let $k \in \langle g, h \rangle$. If k has a fixed point in A , then k is junior of order 7. Else, 1 is an eigenvalue of $\rho(k)$. Since $\rho(k)$ has determinant 1, by Lemma 3.7 and [41, Prop.2.4], the characteristic polynomial of $\rho(k)$ in $\mathbb{Q}[u_7]$ is one of the following:

$$\Phi_1^3, \Phi_1\Phi_2^2, \Phi_1\Phi_3, \Phi_1\Phi_4, \Phi_1\Phi_6,$$

$$X^3 - \bar{u}_7X^2 + u_7X - 1, X^3 - u_7X^2 + \bar{u}_7X - 1.$$

So, possible prime divisors of $|\langle g, h \rangle|$ belong to $\{2, 3, 7\}$.

Let S_2 be a 2-Sylow subgroup of $\langle g, h \rangle$, it inherits the restricted representation $\rho|_{S_2}$ with character $\chi|_{S_2}$, and splitting coefficient v_2 . As S_2 has a non-trivial center, $v_2 \geq 2$, so

$$9 + |S_2| - 1 = \langle \chi|_{S_2}, \chi|_{S_2} \rangle = v_2|S_2| \geq 2|S_2|$$

yielding that $|S_2|$ divides 8. Let S_3, S_7 be 3 and 7-Sylow subgroups of $\langle g, h \rangle$: Similarly, we obtain $|S_3| = 3$ and $|S_7| = 7$. Hence, the order $|\langle g, h \rangle|$ is a divisor of $8 \cdot 3 \cdot 7 = 168$. A GAP search (see Appendix) through all groups of such order, which have no element of

order 12, 14, or 21, and either none or a non-cyclic 2-Sylow subgroup [36, 10.1.9] yields three candidates: \mathbb{Z}_7 , $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $\mathrm{SL}_3(\mathbb{F}_2)$. We exclude the second candidate as it is not generated by its elements of order 7. \square

We exclude $\mathrm{SL}_3(\mathbb{F}_2)$ by a geometric argument.

Lemma 11.6. *Let A be an abelian variety isogenous to $E_{u_7}^3$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) = E_{M(h)}(1)$. Then $\langle g, h \rangle$ cannot be isomorphic to $\mathrm{SL}_3(\mathbb{F}_2)$.*

Proof. The multiplication table of $\mathrm{SL}_3(\mathbb{F}_2)$ shows that

$$C_{\langle g, h \rangle}(\langle g \rangle) = \langle g \rangle \text{ and } N_{\langle g, h \rangle}(\langle g \rangle)/C_{\langle g, h \rangle}(\langle g \rangle) \simeq \mathbb{Z}_3.$$

Take $k \in N_{\langle g, h \rangle}(\langle g \rangle)$ of order 3. Denote by W_1, \dots, W_7 the seven disjoint translated abelian subvarieties of codimension 3 in A that g fixes pointwise. Then

$$k \left(\bigsqcup_{i=1}^7 W_i \right) = \bigsqcup_{i=1}^7 W_i,$$

and since 3 and 7 are coprime, there is some $1 \leq i \leq 7$ such that $k(W_i) = W_i$. Up to conjugating the whole group $\langle g, h \rangle$, we can assume that $0 \in W_i$. We apply Lemma 5.6 (2) to g , noting that $W = W_i$ and $k \in \langle g, h \rangle < G_W$. It shows that for any $w \in W_i$, one has $k(w) = w + T(k)$, and $pr_{W_i}(T(k)) = 0$. As $k(W_i) = W_i$, we obtain $T(k) = 0$, so k has fixed points and order 3. In particular, it is not a power of a junior element, contradiction. \square

Proof of Proposition 11.4. By Lemmas 11.5 and 11.6, we have $\langle g, h \rangle \simeq \mathbb{Z}_7$. But \mathbb{Z}_7 has no proper subgroup, so $\langle g \rangle = \langle h \rangle$. \square

11.B Proof of Theorem 11.2

By Theorem 1.6, the proof reduces to the following two cases.

Proposition 11.7. *Let A be an abelian variety isogenous to $E_{u_7}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points. Then $E_{M(g)}(1) \cap E_{M(h)}(1)$ cannot have codimension 4 in $H^0(T_A)$.*

Proposition 11.8. *Let A be an abelian variety isogenous to E_j^n . Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 3 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points. Then $E_{M(g)}(1) \cap E_{M(h)}(1)$ cannot have codimension 4 in $H^0(T_A)$.*

Both propositions are proved by classifying matrices of elements in $\langle g, h \rangle$, and using representation theory to infer contradictory properties of $\langle g, h \rangle$. We start with one lemma used in the proof of Proposition 11.7.

Lemma 11.9. *Let A be an abelian variety isogenous to $E_{u_7}^n$. Let $g, h \in \mathrm{Aut}(A)$ be two junior elements of order 7 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points, and $E_{M(g)}(1) \cap E_{M(h)}(1)$ has codimension at most 4 in $H^0(T_A)$. Then for every $k \in \langle g, h \rangle$, the trace of $M(k) \oplus \overline{M(k)}$ is at least $2n - 8$, and equals $2n - 7$ if k is junior of order 7.*

Proof. By Maschke's theorem, there is an $\langle M(g), M(h) \rangle$ -stable supplementary S to $E_{M(g)}(1) = E_{M(h)}(1)$ in $H^0(T_A)$. Consider the faithful representation ρ of $\langle g, h \rangle$ given by restricting M to S , with character χ .

Let $k \in \langle g, h \rangle$. If k has a fixed point in A , then k is junior of order 7, and it is clear from Proposition 3.1 that the trace of $M(k) \oplus \overline{M(k)}$ equals $2n - 7$. Else, 1 is an eigenvalue of $\rho(k)$, and we check as in Lemma 11.5 that its characteristic polynomial is one of the following:

$$\begin{aligned} & \Phi_1^4, \Phi_1^2\Phi_2^2, \Phi_1^2\Phi_3, \Phi_1^2\Phi_4, \Phi_1^2\Phi_6, \\ & (X^3 - \overline{u_7}X^2 + u_7X - 1)\Phi_1, (X^3 - u_7X^2 + \overline{u_7}X - 1)\Phi_1. \end{aligned}$$

The consequence is that $\rho(k) \oplus \overline{\rho(k)}$ has non-negative trace, which concludes. \square

From this lemma follows a reduction to codimension 3 that concludes the proof of Proposition 11.7.

Proof of Proposition 11.7. Denote by $\mathbf{1}$ both the trivial representation of $\langle g, h \rangle$ and its character. We have

$$\langle M|_{\langle g, h \rangle}, \mathbf{1} \rangle = \sum_{k \in \langle g, h \rangle} \text{Tr } M(k) = \frac{1}{2} \sum_{k \in \langle g, h \rangle} \text{Tr } M(k) + \text{Tr } \overline{M(k)} > (n-4)|\langle g, h \rangle|,$$

by Lemma 11.9, the inequality being strict since $\langle g, h \rangle$ contains at least one junior element of order 7. Hence, $\mathbf{1}$ has multiplicity at least $n-3$ as a subrepresentation of M , i.e., $E_1(M(g)) \cap E_1(M(h))$ has codimension at most 3 in $H^0(T_A)$. \square

We now prove an auxiliary lemma for Proposition 11.8.

Lemma 11.10. *Let A be an abelian variety isogenous to E_j^n . Let $g, h \in \text{Aut}(A)$ be two junior elements of order 3 such that $\langle g, h \rangle$ contains no translation and no non-junior element fixing points, and $E_1(M(g)) \cap E_1(M(h))$ has codimension 4 in $H^0(A, T_A)$. Then each non-trivial element of $\langle g, h \rangle$ has order 3.*

Proof. By Maschke's theorem, there is an $\langle M(g), M(h) \rangle$ -stable supplementary S to $E_{M(g)}(1) + E_{M(h)}(1)$ in $H^0(T_A)$, and it has dimension 4. Consider the faithful representation ρ of $\langle g, h \rangle$ given by restricting M to S , with character χ .

Let $k \in \langle g, h \rangle$. If k has a fixed point in A , then k is junior of order 3. Else, 1 is an eigenvalue of $\rho(k)$, and since the intersection $E_{\rho(g)}(j) \cap E_{\rho(h)}(j)$ has dimension 2, it must be that 1, j , or j^2 is an eigenvalue of multiplicity 2 of $\rho(k)$. By Lemma 3.7, [41, Prop.2.4], and as $\rho(k)$ has determinant one, the characteristic polynomial of $\rho(k)$ in $\mathbb{Q}[j]$ is one of the following:

$$\Phi_1^4, \Phi_1^2\Phi_2^2, \Phi_1^2\Phi_3, \Phi_1^2\Phi_4, \Phi_1^2\Phi_6, (X-j)^3\Phi_1, (X-j^2)^3\Phi_1.$$

So the order of k is 1, 3, or an even number.

To conclude, it is enough to show that k cannot have order 2. We prove it by contradiction: Suppose that $\rho(k)$ is similar to $\text{diag}(1, 1, -1, -1)$. As the eigenspace $E_{\rho(g)}(j)$ is a hyperplane in S , $\rho(gk)$ has j and $-j$ as eigenvalues. In particular, it is not junior and thus it fixes no point. But its characteristic polynomial should be one of the polynomials listed above, contradiction. \square

Proof of Proposition 11.8. By Lemma 11.10, $\rho(\langle g, h \rangle)$ contains id_S and elements similar to

$$\text{diag}(1, j, j, j), \text{diag}(1, j^2, j^2, j^2), \text{ or } \text{diag}(1, 1, j, j^2). \quad (8)$$

Note in particular that $\text{diag}(j, j, j^2, j^2)$ is not an option.

As $\langle g, h \rangle$ is a 3-group, we can set $k \in Z(\langle g, h \rangle)$ to be an element of order 3. Up to exchanging g and h , we can assume $k \notin \langle g \rangle$. If $\rho(k)$ is similar to $\text{diag}(1, j, j, j)$ or $\text{diag}(1, j^2, j^2, j^2)$, then respectively $\rho(gk)$ or $\rho(g^2k)$ has no 1 as an eigenvalue, which contradicts (8). Else, $\rho(k)$ is similar to $\text{diag}(1, 1, j, j^2)$. As $E_{\rho(g)}(j) \cap E_{\rho(h)}(j)$ has dimension 2, it is the eigenspace for the eigenvalue 1 of $\rho(k)$. Again, either $\rho(gk)$ or $\rho(g^2k)$ has no 1 as an eigenvalue, which contradicts (8). \square

Appendix

Groups of order dividing 240 with an automorphism of order 7

```
order_list := [];
nb_groups_of_order_list := [];

for a in [0..4] do
  for b in [0..1] do
    for c in [0..1] do
      n := (2^a)*(3^b)*(5^c);
      Add(order_list, n);
      Add(nb_groups_of_order_list, NumberSmallGroups(n));
    od;
  od;
od;

have_aut7 := [];

for i in [1..Length(order_list)] do
  n := order_list[i];
  for v in [1..nb_groups_of_order_list[i]] do
    g := SmallGroup(n,v);
    s := SylowSubgroup(g,2);
    if StructureDescription(s) = "Q16" or StructureDescription(s) = "Q8"
    or StructureDescription(s) = "C16" or StructureDescription(s) = "C8"
    or StructureDescription(s) = "C4" or StructureDescription(s) = "C2"
    or StructureDescription(s) = "1" then
      h := AutomorphismGroup(g);
      if Order(h) mod 7 = 0 then
        Add(1,(n,v));
      fi;
    fi;
  od;
od;
```

Representations of $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$

```
for v in [1..NumberSmallGroups(24)] do
  g := SmallGroup(24, v);
  if StructureDescription(g) = "C3□:□C8"
  then Add(groups_checked, v);
      tbl_conjcl := ConjugacyClasses(g);
      nb_conjcl := Size(tbl_conjcl);
#locating the unique element of order 2
#among conjugacy classes of g
      index_2 := 0;
      for j in [1..nb_conjcl] do
        o := Order(Representative(tbl_conjcl[j]));
        if o = 2 then index_2 := j; fi;
      od;
#only keeping irreducible characters sending
#the unique element of order 2 to -id
      T := Irr(g);
      Tbis := [];
      for k in [1..nb_conjcl] do
        if T[k][index_2] + T[k][1] = 0
        then Add(Tbis,T[k]);
        fi;
      od;
```

```

        od;
        Print(Tbis);
        Print("\n\n");
    fi;
od;

```

Proposition 4.28: Five candidates for F

```

order_list := [];
nb_groups_of_order_list := [];

for a in [3..4] do
    for b in [0..1] do
        for c in [0..1] do
            n := (2^a)*(3^b)*(5^c);
            Add(order_list, n);
            Add(nb_groups_of_order_list, NumberSmallGroups(n));
        od;
    od;
od;

right_sylows := [];
right_sylows_and_orders := [];

for i in [1..Length(order_list)] do
    n := order_list[i];
    for v in [1..nb_groups_of_order_list[i]] do
        g := SmallGroup(n,v);
        s := SylowSubgroup(g,2);
        if StructureDescription(s) = "Q16" or StructureDescription(s) = "Q8" then
            Add(right_sylows, [n,v]);
            Add(right_sylows_and_orders, [n,v]);
            tbl_conjcl := ConjugacyClassesByOrbits(g);
            nb_conjcl := Size(tbl_conjcl);
            remove_once_only := 0;
            is_15 := 0;
            is_20 := 0;
            is_24 := 0;
            for i in [1..nb_conjcl] do
                o := Order(Representative(tbl_conjcl[i]));
                if o = 15 then
                    is_15 := 1;
                fi;
                if o = 20 then
                    is_20 := 1;
                fi;
                if o = 24 then
                    is_24 := 1;
                fi;
                s := Size(tbl_conjcl[i]);
                if remove_once_only = 0 and
                    ((o = 2 and s > 1) or (o mod 60 = 0) or (o mod 40 = 0)
                    or (is_20 = 1 and o mod 15 = 0) or (is_24 = 1 and o mod 15 = 0)
                    or (is_15 = 1 and o mod 20 = 0) or (is_24 = 1 and o mod 20 = 0)
                    or (is_15 = 1 and o mod 24 = 0) or (is_20 = 1 and o mod 24 = 0)) then
                    Remove(right_sylows_and_orders);
                    remove_once_only := 1;
                fi;
            od;
        od;
    od;

```

```

        if remove_once_only = 0 and (1 - is_15)*(1 - is_20)*(1 - is_24) = 1 then
            Remove(right_sylows_and_orders);
            remove_once_only := 1;
        fi;
    fi;
od;
od;

```

Proposition 4.28: Two candidates generated by elements of the right order

```

testing := [[48,8],[48,27]];

for i in [1..2] do
    g := SmallGroup(testing[i][1],testing[i][2]);
    Print(StructureDescription(g));
    tbl_conjcl := ConjugacyClasses(g);
    nb_conjcl := Size(tbl_conjcl);
    nb_elts_order_24 := 0;
    for j in [1..nb_conjcl] do
        o := Order(Representative(tbl_conjcl[j]));
        s := Size(tbl_conjcl[j]);
        if o = 24 then
            nb_elts_order_24 := nb_elts_order_24 + s;
        fi;
    od;
    Print("number of elements of order 24:");
    Print(nb_elts_order_24);
    Print(" ");
    Print("\n");
od;

Print("\n\n");

```

```

testing := [[40,4],[40,11],[80,18]];

for i in [1..3] do
    g := SmallGroup(testing[i][1],testing[i][2]);
    Print(StructureDescription(g));
    tbl_conjcl := ConjugacyClasses(g);
    nb_conjcl := Size(tbl_conjcl);
    nb_elts_order_20 := 0;
    for j in [1..nb_conjcl] do
        o := Order(Representative(tbl_conjcl[j]));
        s := Size(tbl_conjcl[j]);
        if o = 20 then
            nb_elts_order_20 := nb_elts_order_20 + s;
        fi;
    od;
    Print("number of elements of order 20:");
    Print(nb_elts_order_20);
    Print(" ");
    Print("\n");
od;

```

Proposition 4.28: None admitting the right representation

```

tables_char_irr := [[],[[[]]];
indices_20 := [[],[[[]]];
testing := [[40,11],[80,18]];

```

```

for i in [1..2] do
  g := SmallGroup(testing[i][1], testing[i][2]);
  tbl_conjcl := ConjugacyClasses(g);
  nb_conjcl := Size(tbl_conjcl);
#locating the unique element of order 2
#among conjugacy classes of g
  index_2 := 0;
  for j in [1..nb_conjcl] do
    o := Order(Representative(tbl_conjcl[j]));
    if o = 2 then
      index_2 := j;
    fi;
  od;
#only keeping irreducible characters sending
#the unique element of order 2 to -id
  T := Irr(g);
  Tbis := [];
  for k in [1..nb_conjcl] do
    if T[k][index_2] + T[k][1] = 0 then
      Add(Tbis, T[k]);
    fi;
  od;
  Add(tables_char_irr[i], Tbis);
  Print(StructureDescription(g));
  Print("possible irreducible representations have characters:");
  Print(tables_char_irr[i]);
  Print("\n\n");
od;

```

Pointwise stabilizers in codimension 6 as in Subsection 8.A

```

order_list := [];
nb_groups_of_order_list := [];

for a in [0,3] do
  for b in [0..3] do
    for c in [0..2] do
      if b <= 1 or c <= 1 then
        n := (2^a)*(3^b)*(7^c);
        Add(order_list, n);
        Add(nb_groups_of_order_list, NumberSmallGroups(n));
      fi;
    od;
  od;
od;

right_sylovs := [];
right_sylovs_and_orders := [];

for i in [1..Length(order_list)] do
  n := order_list[i];
  for v in [1..nb_groups_of_order_list[i]] do
    g := SmallGroup(n,v);
    s := SylowSubgroup(g,2);
    t := SylowSubgroup(g,3);
    u := SylowSubgroup(g,7);
    if (StructureDescription(s) = "1" or StructureDescription(s) = "Q8")
      and (StructureDescription(t) = "1" or StructureDescription(t) = "C3"

```

```

        or StructureDescription(t) = "C9"
        or StructureDescription(t) = "C3_x_C3"
        or StructureDescription(t) = "C3_x_C9"
    and (StructureDescription(u) = "1" or StructureDescription(u) = "C7"
        or StructureDescription(u) = "C7_x_C7")
    then Add(right_sylows, [n,v]);
        Add(right_sylows_and_orders, [n,v]);
#we now remove of the list right_sylows_and_orders candidates with elements
#of inappropriate order 63, or with several elements of order 2
        tbl_conjcl := ConjugacyClassesByOrbits(g);
        nb_conjcl := Size(tbl_conjcl);
        remove_once_only := 0;
        for i in [1..nb_conjcl] do
            o := Order(Representative(tbl_conjcl[i]));
            s := Size(tbl_conjcl[i]);
            if remove_once_only = 0 and
                ((o = 2 and s > 1) or (o mod 63 = 0))
            then Remove(right_sylows_and_orders);
                remove_once_only := 1;
            fi;
        od;
    fi;
od;
od;

describe := [];
for i in [1..Length(right_sylows_and_orders)] do
    g := SmallGroup(right_sylows_and_orders[i][1], right_sylows_and_orders[i][2]);
    Add(describe, StructureDescription(g));
    Print(StructureDescription(g));
    Print("\n\n");
od;

```

Groups of order dividing 168 as in Lemma 11.5

```

order_list := [];
nb_groups_of_order_list := [];

for a in [0..3] do
    for b in [0..1] do
        n := (2^a)*(3^b)*7;
        Add(order_list, n);
        Add(nb_groups_of_order_list, NumberSmallGroups(n));
    od;
od;

right_syLOW := [];
right_syLOW_description := [];
for i in [1..Length(order_list)] do
    n := order_list[i];
    for v in [1..nb_groups_of_order_list[i]] do
        g := SmallGroup(n,v);
        h := SylowSubgroup(g, 2);
        if StructureDescription(h) = "Q8" or StructureDescription(h) = "D8"
        or StructureDescription(h) = "1"
        then Add(right_syLOW, [n, v]);
            Add(right_syLOW_description, StructureDescription(g));
        fi;
    od;
od;

```

```

    od;
od;

right_syLOW_and_orders := [];
right_syLOW_and_orders_description := [];
for element in right_syLOW do
    n := element[1];
    v := element[2];
    g := SmallGroup(n,v);
tbl_conjcl := ConjugacyClassesByOrbits(g);
nb_conjcl := Size(tbl_conjcl);
v_to_discard := 0;
for i in [1..nb_conjcl] do
    o := Order(Representative(tbl_conjcl[i]));
    if (o = 14 or o = 21 or o = 12) and v_to_discard = 0
    then v_to_discard := 1;
    fi;
od;
if v_to_discard = 0
then Add(right_syLOW_and_orders, [n, v]);
    Add(right_syLOW_and_orders_description, StructureDescription(g));
fi;
od;

Print(right_syLOW_and_orders);
Print("\n");
Print(right_syLOW_and_orders_description);

```

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