THE PSEUDOAUTOMORPHISM GROUP OF $\mathbb{P}^3$ BLOWN-UP AT 8 VERY GENERAL POINTS

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Abstract. We prove that the pseudoautomorphism group of a blow-up of $\mathbb{P}^3$ at 8 very general points is trivial. We also establish the injectivity of the Coble representation associated to blow-ups of $\mathbb{P}^3$ at $r \geq 8$ general points, answering a question of Dolgachev–Ortland.

1. Introduction

It is commonly expected that for r large enough, blow-ups of the projective space $\mathbb{P}^n$ at r very general points have no symmetries. Coble [Cob16] famously showed that for $\mathbb{P}^2$, no automorphism arises as soon as $r \geq 9$. In this short note, we focus on $n = 3$ and show the following result.

Theorem 1.1. *Let X denote the blow-up of $\mathbb{P}^3$ at $r = 8$ very general points. The pseudoautomorphism group of X is trivial.*

Proving such a result in dimension greater than 2 is known to be challenging; see e.g. [SX, Section 1.2]. This is due to the lack of understanding of the subgroup of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$ generated by $\mathrm{Aut}(\mathbb{P}^3)$ and standard Cremona transformations, more specifically to issues pointed out in [Dol11, Remark 1]. To circumvent these issues for $r = 8$, we use the nef anticanonical divisor to decompose pseudo-automorphisms into finite sequences of flops; see Section 3.

Triviality of the pseudoautomorphism group of a blow-up of $\mathbb{P}^3$ at $r \geq 9$ very general points remains an open question. Meanwhile for $r \leq 4$, the blow-ups are toric varieties, and for r ranging from 5 to 7, they are Mori Dream Spaces: In all these cases, the pseudoautomorphism groups are essentially known by [DO88].

Along the way, we establish the injectivity of the Coble representation for $r \geq 8$, hereby answering a question of Dolgachev–Ortland; see [DO88, Page 130].

Theorem 1.2. *For $r \geq 8$, the Coble representation $\mathrm{co}_{3,r}$ of the Weyl group $W_{3,r}$ on the moduli space of semistable r-tuples of points in $\mathbb{P}^3$ is injective.*

Our strategy of proof is inspired by [Hir88]; see Sections 4, 5, and 6.

Let us finally mention a consequence of Theorem 1.1 and of [SX].

Corollary 1.3. *Let X be the blow-up of $\mathbb{P}^3$ at eight very general points and consider a Calabi–Yau pair (X, Δ). Then the pair (X, Δ) is not klt and it fails the movable cone conjecture.*

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2. Notations and preliminaries

Throughout this paper, we work over an uncountable algebraically closed field k of arbitrary characteristics. For a smooth projective variety X, we denote by $N^1(X)$, respectively $N^1(X)_{\mathbb{Q}}$ the space of numerical equivalence classes of $\mathbb{Z}$, respectively $\mathbb{Q}$-divisors on X. We define $\mathrm{Psaut}(X)$ to be the group of birational automorphisms of X that are isomorphisms in codimension one. Pulling back $\mathbb{Q}$-Cartier divisor classes by pseudoautomorphisms induces a natural representation

$$\rho : \mathrm{Psaut}(X) \to \mathrm{GL}(N^1(X)/\mathrm{tors}),$$

whose image is denoted by $\mathrm{Psaut}^*(X)$.

For an isomorphism in codimension one $g : X \dashrightarrow Y$ between two normal projective threefolds, we define the *isomorphism open sets* of g to be the maximal Zariski open sets $U \subset X$ and $V \subset Y$ such that the complements $X \setminus U$ and $Y \setminus V$ have pure dimension one and g induces an isomorphism between U and V.

We say that a set of points in $\mathbb{P}^3$ are *linearly independent* if no four of them lie on a common plane. For $r \geq 1$ and p an r-tuple of distinct, linearly independent points in $\mathbb{P}^3$, we denote by X_{p} the blow-up of $\mathbb{P}^3$ at the center p. We denote by $\varepsilon_{\mathrm{p}} : X_{\mathrm{p}} \to \mathbb{P}^3$ the blow-up of p in $\mathbb{P}^3$, by H the class of a hyperplane in $\mathbb{P}^3$ and by $E_1, \dots, E_r$ the exceptional divisors above the points of p.

We define the following lattice: $\mathrm{H}_r = \bigoplus_{i=0}^{r} \mathbb{Z}h_i$ endowed with the symmetric bilinear form

$$(h_i, h_j) = \begin{cases} 2\delta_{ij} & \text{if } ij = 0, \\ -\delta_{ij} & \text{otherwise.} \end{cases}$$

The corresponding quadratic form is hyperbolic. Following [DO88, Bottom of Page 69], we introduce the strict geometric marking

$$\varphi_{\mathrm{p}} : \mathrm{H}_r \xrightarrow{\sim} N^1(X_{\mathrm{p}})/\mathrm{tors}$$

sending h_0 to $\varepsilon_{\mathrm{p}}^* H$ and h_i to E_i. The induced hyperbolic quadratic form on $N^1(X_{\mathrm{p}})/\mathrm{tor}$ is denoted by q_{p}.

It will often be the case that all points of p belong to the same smooth quartic curve in $\mathbb{P}^3$: We reserve the notation C_{p} for the curve in this case.

2.A. **The Weyl group $W_{3,r}$.**

Definition 2.1. For $r \geq 5$, we denote by $W_{3,r}$ the Weyl group associated to the root system $T_{2,4,r-4}$, whose Dynkin diagram is depicted in Figure 1. It comes with a preferred set of involutive generators, which we denote by $\tau_1, \dots, \tau_{r-1}, s$: We set the generators τ_i to correspond to the vertices of the horizontal chain present in the diagram, from left to right; We set s to correspond to the remaining vertex.

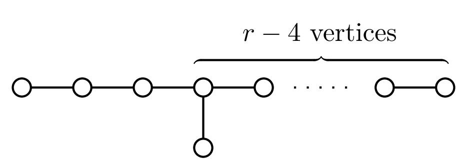

Figure 1. The Dynkin diagram of $T_{2,4,r-4}$

Remark 2.2. We mention a few obvious facts.

(1) $\tau_1, \dots, \tau_{r-1}$ generate a copy of the symmetric group $\mathfrak{S}_r$ in $W_{3,r}$.

(2) For small values of r, we recover known root systems, namely:

$$T_{2,4,1} = A_5, \quad T_{2,4,2} = D_6, \quad T_{2,4,3} = E_7, \quad T_{2,4,4} = \tilde{E}_7.$$

We recall that $\tilde{E}_7$ is the affine root system based on E_7.

We recall a natural action of $W_{3,r}$ on the hyperbolic lattice H_r (see [Dol83, Muk04], [DO88, Page 72, 73], [SX, Section 4.1]).

Proposition 2.3. *There is an injective group morphism*

$$\pi_r : W_{3,r} \hookrightarrow \mathrm{Isom}^+(\mathrm{H}_r)$$

sending the generator τ_i to the hyperbolic reflection relative to $h_i - h_{i+1}$ and s to the hyperbolic reflection relative to $h_0 - h_1 - h_2 - h_3 - h_4$.

Fixing an r-tuple p of distinct, linearly independent points in $\mathbb{P}^3$, we thus obtain a faithful representation of $W_{3,r}$ on the Néron–Severi space of the variety X_{p}, which preserves the quadratic form q_{p}:

$$\varphi_{\mathrm{p}} \circ \pi_r \circ \varphi_{\mathrm{p}}^{-1} : W_{3,r} \hookrightarrow \mathrm{GL}(N^1(X_{\mathrm{p}})/\mathrm{tors}\,; q_{\mathrm{p}}).$$

2.B. **The Coble representation.** The following representation of the Weyl group $W_{3,r}$ was introduced by Kantor, Coble and du Val in [Kan95, Cob16, dV36], and subsequently studied in [Dol83, Koi88, Hir88, DO88, Muk04]. The next proposition is due to [Dol83, Section 7, Page 292]; see also [SX, Remark 4.3].

Proposition 2.4. *Let $r \geq 4$, and let U_r denote the moduli space of r-tuples of distinct, linearly independent points in $\mathbb{P}^3$ There is a representation*

$$\mathrm{co}_{3,r} : W_{3,r} \to \mathrm{Bir}(U_r)$$

sending τ_i to the transposition exchanging the i-th point with the $(i+1)$-th point of the tuple, for $1 \leq i \leq r-1$, and sending s to the birational map

$$\mathrm{co}_{3,r}(s) : [p_i] \mapsto [p_1, p_2, p_3, p_4, \mathrm{cr}(p_5), \mathrm{cr}(p_6), \ldots, \mathrm{cr}(p_r)],$$

where cr *is the standard Cremona transformation of $\mathbb{P}^3$ centered at the four points p_1, p_2, p_3, p_4. We call $\mathrm{co}_{3,r}$ the* Coble representation *of $W_{3,r}$.*

Remark 2.5. To prove that the action of $W_{3,r}$ is generically well-defined on a subset V_r of U_r and preserves V_r, it suffices to check it for each of the $r+1$ standard generators of $W_{3,r}$. As an application, consider the closure of the set

$$V_r := \{\mathrm{p} \in U_r \mid \text{there is a pencil of quadrics containing } \mathrm{p}\}.$$

It is clearly preserved by the action of the symmetric subgroup of $W_{3,r}$. Since the standard Cremona transformation preserves the linear system of quadrics through its four center points, the action by the last generator $s \in W_{3,r}$ also preserves the closure of V_r. So the action of $W_{3,r}$ by $\mathrm{co}_{3,r}$ is generically well-defined on that particular V_r and preserves it.

3. Pseudoautomorphisms preserve the quadratic form q_{p} when $r \leq 8$

The main result of this section is the following proposition.

Proposition 3.1. *Let* p *be a very general r-tuple of points in $\mathbb{P}^3$ with $r \leq 8$. The action by* $\mathrm{Psaut}^*(X_{\mathrm{p}})$ *preserves q_{p}.*

Before proving Proposition 3.1, we prove a lemma.

Lemma 3.2. *Let* p *be a very general r-tuple of points in $\mathbb{P}^3$ with $r \leq 8$. For any finite sequence of flops $\alpha : X_{\mathrm{p}} \dashrightarrow Y$, the very general member S of the linear system $\left|-\frac{1}{2}K_{X_{\mathrm{p}}}\right|$ is such, that the restriction $\alpha|_S : S \dashrightarrow \alpha_*(S)$ is an isomorphism.*

Proof of Lemma 3.2. We argue by induction of the minimal number n of flops needed to factorize α. For $n = 0$, we have an isomorphism, and the claim clearly holds. Assume that the claim is known for any finite sequence of n flops, and let α be a sequence of $n+1$ flops. We decompose $\alpha = \alpha' \circ \varphi$, where $\alpha' : X_{\mathrm{p}} \dashrightarrow Y'$ is a sequence of n flops and $\varphi : Y' \dashrightarrow Y$ is a single flop. Let S be a smooth, very general member of the linear system $\left|-\frac{1}{2}K_{X_{\mathrm{p}}}\right|$ such that

- S is smooth;
- if $r \leq 7$, S is a del Pezzo surface, and if $r = 8$, S contains no (-2)-curve (this condition is very general by [LO16, Lemma 5, Proof of Lemma 6]);
- the restriction $\alpha'|_S : S \dashrightarrow \alpha'_*(S)$ is an isomorphism.

In particular, the surface $S' := \alpha'_*(S)$ is smooth, iromorphic to S and contained in the smooth locus of Y'.

By definition, the flop φ is an isomorphism outside of a finite union of K-trivial smooth rational curves. Let C be such a curve in Y': Then $S' \cdot C = 0$ holds. If C is contained in S', then the adjunction yields that $-K_{S'} \cdot C = 0$. If $r \leq 7$, this contradicts the fact that S' is a del Pezzo surface, and if $r = 8$, it contradicts the fact that S contains no (-2)-curve by [LO16, Lemma 4]. So C is disjoint from S', and thus S' is contained in the isomorphism open set of the flop φ. □

We can now prove Proposition 3.1.

Proof of Proposition 3.1. One notices that for any divisor $D \in N^1(X_{\mathrm{p}})$,

$$q_{\mathrm{p}}(D) = D^2 \cdot \left(-\frac{1}{2}K_{X_{\mathrm{p}}}\right).$$

Let $g \in \mathrm{Psaut}(X_{\mathrm{p}})$. By [SX, Corollary 3.4], one can decompose g as a finite sequence of flops. By Lemma 3.2, we conclude that

$$q_{\mathrm{p}}(D) = (D|_S)^2 = \left[(g^*D)|_{g_*^{-1}S}\right]^2 = (g^*D)^2 \cdot \left(-\frac{1}{2}K_{X_{\mathrm{p}}}\right) = q_{\mathrm{p}}(g^*D),$$

for a very general $S \in \left|-\frac{1}{2}K_{X_{\mathrm{p}}}\right|$ and for $D \in N^1(X_{\mathrm{p}})$. □

We conclude this subsection with a consequence of Lemma 3.2.

Corollary 3.3. *Let* p *be a very general 8-tuple of points in $\mathbb{P}^3$. For any $g \in$* $\mathrm{Psaut}(X_{\mathrm{p}})$*, the isomorphism open sets of g both contain the curve C_{p}*

Proof. Since $r = 8$, the half-anticanonical linear system $\left|-\frac{1}{2}K_{X_{\mathrm{p}}}\right|$ is a pencil. Its base locus is C_{p}. By [SX, Corollary 3.4], the pseudoautomorphism g is a finite sequence of flops, so Lemma 3.2 applies and we can find a very general member S of $\left|-\frac{1}{2}K_{X_{\mathrm{p}}}\right|$ that is contained in the isomorphism open sets of g. So is $C_{\mathrm{p}} \subset S$. □

4. Restricting the action to the quartic curve C_{p}

In this section, we work with $r \geq 8$ and p an r-tuple of points of $\mathbb{P}^3$ that is very general among r-tuples supported on a pencil of quadrics. In the notation of Remark 2.5, p represents a very general point in $V_r \subset U_r$. For $r = 8$, we have $V_8 = U_8$, thus the condition is fulfilled by a very general r-tuple. For $r \geq 9$, it means that the support of p is contained in a very general quartic curve $C_{\mathrm{p}} \subset \mathbb{P}^3$. This curve is uniquely determined by p. As a smooth genus one curve, C_{p} is moreover very general in moduli, and does not have complex multiplication.

The following lemma generalizes [Hir88, 2.4. Lemme] from blow-ups of $\mathbb{P}^2$ in nine or more points to blow-ups of $\mathbb{P}^3$ in eight or more points. It should also relate to [LO16, Lemma 5].

Lemma 4.1. *Let $r \geq 8$ and* p *be an r-tuple of points that is very general among those contained in a common quartic curve C_{p} in $\mathbb{P}^3$. The restriction map*

$$\mathrm{tr} : \mathrm{Pic}(X_{\mathrm{p}}) \to \mathrm{Pic}(C_{\mathrm{p}})$$

is injective.

Proof. We work with divisors modulo linear equivalence. Let us denote by H the class of a hyperplane in $\mathbb{P}^3$. Let D be a divisor on X_{p} with $\mathrm{tr}(D) \sim 0$ and write

$$D \equiv 2d_0 H - \sum_{i=1}^{r} d_i E_i.$$

Without loss of generality, we can assume that $d_0 \geq 0$. We write $D = A - B$ with A and B both ample divisors. Note that $\varepsilon_{\mathrm{p}*} A$ and $\varepsilon_{\mathrm{p}*} B$ define sections of the line bundles $\mathcal{O}_{\mathbb{P}^3}(2d_0 + b_0)$, respectively $\mathcal{O}_{\mathbb{P}^3}(b_0)$, passing through the points p_i with multiplicities $d_i + b_i$ respectively b_i, for some integers $b_0, \ldots, b_8 > 0$. In particular, we have

$$0 = \mathrm{tr}(D) = \varepsilon_{\mathrm{p}*}(A - B)|_{C_{\mathrm{p}}} = \sum_{i=1}^{r} d_i \mathrm{p}_i + (2d_0 H)|_{C_{\mathrm{p}}}.$$

Note that $(2H)|_{C_{\mathrm{p}}}$ is linearly equivalent to the sum of any seven of the p_i and of the eighth base point of the net of quadrics passing through them. Since p is very general, this enforces $d_1 = \ldots = d_r = 0$, thus $d_0 = 0$ and $D = 0$, as wished. □

The next lemma derives from a very classical argument; see [Cob16], [Giz81, Proposition 8], [Hir88, 2.3. Proposition].

Lemma 4.2. *Let $r \geq 8$ and* p *be an r-tuple of points that is very general among those contained in a common quartic curve C_{p} in $\mathbb{P}^3$. Let $g \in \mathrm{Psaut}(X_{\mathrm{p}})$ such that the curve C_{p} in X_{p} is contained in the isomorphism open sets of g and that the pullback g^* preserves the quadratic form q_{p}. Then there exist $\sigma \in \{\pm 1\}$ and $L \in \mathrm{Pic}(X_{\mathrm{p}})$ such that*

$$g^* E_i = \sigma E_i + L \text{ for every } 1 \leq i \leq 8 \text{ and}$$
$$g^* \varepsilon_{\mathrm{p}}^* H = \sigma \varepsilon_{\mathrm{p}}^* H + 4L.$$

Proof of Lemma 4.2. Since C_{p} is entirely contained in the isomorphism open sets of g, pulling back by g transforms the exceptional divisors E_i into prime divisors F_i which still satisfy $F_i \cdot C_{\mathrm{p}} = 1$ for all $1 \leq i \leq r$. The unique intersection point of F_i with C_{p} is then the image of p_i by the automorphism $g^{-1}|_{C_{\mathrm{p}}}$ of the smooth curve C_{p} of genus one. Since C_{p} is a very general elliptic curve, we can write

$$g^{-1}|_{C_{\mathrm{p}}} \in \mathrm{Aut}(C_{\mathrm{p}}) = C_{\mathrm{p}} \rtimes \mathbb{Z}/2\mathbb{Z},$$

which acts by translations and inversion with respect to a fixed origin point, say p_1. Let σ be 1 if $g^{-1}|_{C_{\mathrm{p}}}$ is a translation and -1 otherwise. Let $t \in C_{\mathrm{p}}$ be the image of p_1 by $g^{-1}|_{C_{\mathrm{p}}}$. This shows two things:

- that $t - \sigma \mathrm{p}_1 = (F_i - \sigma E_i)|_{C_{\mathrm{p}}}$ in $\mathrm{Pic}(C_{\mathrm{p}})$, which does not depend on i;
- that $4(t - \sigma \mathrm{p}_1) = (g^* \varepsilon_{\mathrm{p}}^* H - \sigma \varepsilon_{\mathrm{p}}^* H)|_{C_{\mathrm{p}}}$.

By Lemma 4.1, we deduce that the divisor $F_i - \sigma E_i$ in $\mathrm{Pic}(X_{\mathrm{p}})$ does not depend on i either. That is the divisor L we are seeking after. Applying Lemma 4.1 to the divisors $4L$ and $g^*\varepsilon_{\mathrm{p}}^*H - \sigma\varepsilon_{\mathrm{p}}^*H$ concludes. □

5. On certain isometries preserving the effective cone

We keep following the argument of [Hir88] and [DO88] for this last statement, as the more hands-on approach of [Koi88] seems harder to generalize.

Lemma 5.1. *Let $r \geq 8$ and p be an r-tuple of points that is very general among those contained in a common quartic curve in $\mathbb{P}^3$. Let ι be an isometry of the lattice $N^1(X_{\mathrm{p}})$ with q_{p}, that preserves the effective cone and the anticanonical class. Assume that there exist $\sigma \in \{\pm 1\}$ and $L \in \mathrm{Pic}(X_{\mathrm{p}})$ such that*

$$\iota(E_i) = \sigma E_i + L \text{ for every } 1 \leq i \leq r \text{ and}$$
$$\iota(\varepsilon_{\mathrm{p}}^*H) = \sigma\varepsilon_{\mathrm{p}}^*H + 4L.$$

Then $L = 0$ and $\sigma = 1$, i.e., ι is the identity.

Proof. Since ι preserves the quadratic form $q := q_{\mathrm{p}}$, we have

$$(*) \qquad q(E_i, L) = \alpha \text{ for every } 1 \leq i \leq r \text{ and } q(\varepsilon_{\mathrm{p}}^*H, L) = 4\alpha,$$

where α denotes the scalar $\frac{-q(L)}{2\sigma}$. Since q is non-degenerate, this and an easy computation imply that L is numerically equivalent to the divisor class: $-\frac{\alpha}{2}K_{X_{\mathrm{p}}}$.

Let us first assume that $r = 8$. Then we have $q\left(-\frac{1}{2}K_{X_{\mathrm{p}}}\right) = 0$. Therefore, $q(L) = 0$, that is $\alpha = 0$, and as a result $L = 0$. Since ι preserves the effective cone, which is non-degenerate, the sum $\iota(E_i) + E_i$ cannot be zero, and thus $\sigma = 1$.

Let us now assume that $r \geq 9$. Using that ι preserves both q and the anticanonical class on the left handside and Identity $(*)$ on the right handside, we note that

$$1 - \sigma = q\left(\iota(E_1) - \sigma E_1, -\frac{1}{2}K_{X_{\mathrm{p}}}\right) = q\left(L, -\frac{1}{2}K_{X_{\mathrm{p}}}\right) = \alpha(8 - r).$$

In particular, if $\sigma = 1$, then $\alpha = 0$ and thus $L = 0$, as wished.

Let us finally assume by contradiction that $\sigma = -1$. Then α is negative. Then also $L = \iota(E_i) + E_i$ is an effective class, and so are its positive multiples, such as the canonical class $K_{X_{\mathrm{p}}}$. However, the curve class $(\varepsilon_{\mathrm{p}}^*H)^2$ is strongly movable, and

$$K_{X_{\mathrm{p}}} \cdot (\varepsilon_{\mathrm{p}}^*H)^2 = -2\, q(\varepsilon_{\mathrm{p}}^*H) = -4 < 0,$$

a contradiction. □

6. Proof of the main theorems

We start with a simple, yet important fact.

Lemma 6.1. *Let X be a blow-up of $\mathbb{P}^3$ at r points, five of which form a general 5-tuple in $\mathbb{P}^3$. Then the representation $\rho : \mathrm{Psaut}(X) \to \mathrm{GL}(N^1(X)/\mathrm{tors})$ is faithful.*

Proof. An application of the negativity lemma (see for instance [GLSW, Lemma 4.2]) shows that $\ker\pi \subset \mathrm{Aut}(X)$. Any element $g \in \ker\pi$ notably fixes the numerical classes of the exceptional divisors $E_1, \ldots, E_r$ of the blow-up map $\varepsilon : X \to \mathbb{P}^3$. Since E_i is not numerically equivalent to any effective divisor other than itself, g descends under ε to an automorphism $\gamma \in \mathrm{PGL}(4, \mathbb{C})$ fixing the r blown-up points and in particular a general 5-tuple of points. Thus, $\ker\pi = \{\mathrm{id}_X\}$. □

We now prove our main theorems.

Proof of Theorem 1.1. Recall that X denotes the blow-up of $\mathbb{P}^3$ at eight very general points. Let $g \in \mathrm{Psaut}(X)$. By Corollary 3.3 and Proposition 3.1, Lemma 4.2 applies to g. Thus and by Proposition 3.1 again, the isometry $\iota := g^*$ satisfies the assumptions of Lemma 5.1, therefore

$$g^* E_i = E_i \text{ for all } 1 \leq i \leq r \text{ and } g^* \varepsilon^* H = \varepsilon^* H,$$

where $\varepsilon = X \to \mathbb{P}^3$ denotes the blow-up map and E_i its exceptional divisors. Hence g belongs to the kernel of the linear representation

$$\rho : \mathrm{Psaut}(X) \to \mathrm{GL}(N^1(X)/\mathrm{tors}),$$

and Lemma 6.1 concludes that g is trivial. $\square$

Proof of Theorem 1.2. Note that [DO88, Theorem 5, Page 99] and Theorem 1.1 immediately imply Theorem 1.2 for $r = 8$.

We now prove the theorem for $r \geq 9$. Let $w \in W_{3,r}$ with $\mathrm{co}_{3,r}(w)$ trivial. Let p be an r-tuple of points that is very general among those supported on a quartic curve C_p in $\mathbb{P}^3$. By Remark 2.5 and [DO88, Page 99, Lemma 2], we see that $\mathrm{co}_{3,r}(w)$ is defined (and in fact trivial) at p and obtain a pseudoautomorphism g of X_p satisfying

$$\varphi_\mathrm{p} \circ \pi_r(w) \circ \varphi_\mathrm{p}^{-1} = g^*.$$

In particular, the pullback g^* preserves the quadratic form q_p. We claim that the curve C_p is contained in the isomorphism open sets of g. Once that claim is established, we can apply Lemmas 4.2 and 5.1 to derive that the pullback g^* is trivial, and conclude by faithfulness of the representation $\varphi_\mathrm{p} \circ \pi_r \circ \varphi_\mathrm{p}^{-1}$ that $w \in W$ is trivial too.

Let us prove the claim. In fact by Remark 2.5, it makes sense to prove more generally that for any element $v \in W_{3,r}$, denoting $\mathrm{q} := \mathrm{co}_{3,r}(v)(\mathrm{p})$, the isomorphism in codimension one

$$g : X_\mathrm{q} \dashrightarrow X_\mathrm{p}$$

induced as in [Muk04, Theorem 1], [DO88, Page 86, Proposition 1] has its isomorphism open sets contain the curves C_q and C_p respectively. We proceed by induction on the minimal number k of occurences of the generator s necessary to write out $v \in W_{3,r}$. If none is needed, then g is the identity and the result holds. Fix $k \geq 1$ and assume that the result holds for $k-1$. Let $v \in W_{3,r}$ be an element optimally written with exactly k occurences of s. Using that $W_{3,r}$ is a Coxeter group, we rewrite $v = us$, where $u \in W_{3,r}$ can be written with strictly fewer occurences of s. Consider the isomorphisms in codimension one induced by v and u, namely

$$g : X_\mathrm{q} \dashrightarrow X_\mathrm{p} \text{ and } h : X_\mathrm{q} \dashrightarrow X_{\mathrm{co}_{3,r}(s)(\mathrm{p})}$$

respectively, and let $c : X_{\mathrm{co}_{3,r}(s)(\mathrm{p})} \dashrightarrow X_\mathrm{p}$ be the lift of the standard Cremona transformation of $\mathbb{P}^3$ centered at the first four points of p. Note that $g = c \circ h$. The isomorphism open sets of c are known to be the complements of the strict transforms of the six lines through $\mathrm{p}_1, \dots, \mathrm{p}_4$: In particular, they contain the two curves C_p and $C_{\mathrm{co}_{3,r}(s)(\mathrm{p})}$. By the induction hypothesis, the isomorphism open sets of h contain $C_{\mathrm{co}_{3,r}(s)(\mathrm{p})}$ and C_q. This shows that g indeed contains C_q and C_p in its respective isomorphism open sets. $\square$

Proof of Corollary 1.3. Let C be the curve that is the base locus of $\left|-\frac{1}{2}K_X\right|$. The divisor $-K_X$ is not semiample [LO16]. Thus, for $m \geq 1$ and for $D \in |-mK_X|$, the curve C is not disjoint from D: But $-K_X \cdot C = 0$, so C is contained in D. This shows that C is in the base-locus of the linear system $|-mK_X|$ too, hence no Calabi-Yau pair (X, Δ) is klt along C.

By [SX, Lemma 7.1], the movable effective cone $\mathrm{Mov}^{\mathrm{e}}(X)$ is not rational polyhedral. By Theorem 1.1, the group $\mathrm{Psaut}(X)$ is trivial, and so is the subgroup $\mathrm{Psaut}(X, \Delta)$ for any pair (X, Δ). The pair (X, Δ) clearly fails the movable cone conjecture. □

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